MATH 516-101 Homework One
Due Date: September 29th, 2015

1. This problem concerns the Newtonian potential

$$u(x) = \int_{R^3} \frac{1}{|x-y|} f(y) dy \quad ($$

For the following three parts, pick up only **one part** to finish
a) Show that if $|f(y)| \leq \frac{C}{|y|^\alpha}$ for $\alpha \in (2,3)$. Then $|u(x)| \leq \frac{C}{|x|^{\alpha-2}}$ for $|x| > 1$

b) Show that if $|f(y)| \leq \frac{C}{|y|^3}$, then $|u(x)| \leq \frac{C}{|x|} \log |x|$ for $|x| > 1$

c) Show that if $|f(y)| \leq \frac{C}{|y|^\alpha}$ for $\alpha > 3$, then $|u(x)| \leq \frac{C}{|x|}$ for $|x| > 1$
Hint: For $|x| = R >> 1$, divide the integral into three parts

$$\int_{R^3} (...) dy = \int_{|y-x|<\frac{|x|}{2}} (...) + \int_{\frac{|x|}{2}<|y-x|<2|x|} (...) + \int_{|y-x|>2|x|} (...)$$

and estimate each parts. For example in the region $|y-x| < \frac{|x|}{2}$ we have $|y| > |x| - |x-y| > \frac{|x|}{2}$ and

$$\int_{|y-x|<\frac{|x|}{2}} \frac{1}{|x-y|} |f(y)| dy \leq \int_0^{\frac{|x|}{2}} \frac{r^2}{r} dr \frac{C}{|x|^\alpha} \leq \frac{C}{|x|^{\alpha-2}}$$

2. This problem concerns the Mean-Value-Property (MVP). We say $v \in C^2(\bar{U})$ is *subharmonic*, if

$$-\Delta v \leq 0 \quad \text{in } U$$

a) Prove that for subharmonic functions

$$v(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} v dy, \quad \forall B(x,r) \subset U$$

Hint: use the formula for $\psi'(r)$.
b) Prove that the Maximum Principle holds for subharmonic functions on bounded domains

$$\max_{\bar{U}} v = \max_{\partial U} v$$

c) Let u be harmonic functions in U. Show that u^2 and $|\nabla u|^2$ are subharmonic functions.
d) Let u satisfy

$$-\Delta u = f \text{ in } U, \quad u = g \text{ on } \partial U$$

Show that there exists a generic constant $C = C(n, U)$ such that

$$\max_U u \leq C(\max_U |f| + \max_{\partial U} |g|)$$

Hint; Consider $v(x) = u(x) + \frac{|x|^2}{2n} \max_U |f| - \max_{\partial U} |g| - \frac{\max_U |x|^2}{2n} \max_U |f|$ and show that v is subharmonic and th apply b).

3. This problem concerns Green's function and Green's representation formula.
a) Write the Green's function for the unit ball $B(0,1)$.
b) Use a) and reflection to find the Green's function in half ball $B^+(0,1) = B(0,1) \cap \{x_n > 0\}$.