MATH 516-101   Solutions to Homework TWO

1. This problems concerns the Green's representation formula in a ball.

(a) using the Green's function in a ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B_r(0)$.

Solution: From Green's function in a ball we get the following Poisson's formula

$$u(x) = \int_{\partial B_r(0)} \frac{r^2-|x|^2}{|S^{n-1}|r|x-y|^n} u(y)d\sigma(y)$$

Since $r-|x| \le |y-x| \le r+|x|$ for $|y|=r$, we have

$$u(x) \ge \int_{\partial B_r(0)} \frac{r^2-|x|^2}{|S^{n-1}|r(r+|x|)^n} u(y)d\sigma(y) \ge \frac{r^2-|x|^2}{|S^{n-1}|r(r+|x|)^n}\int_{\partial B_r(0)} u(y)d\sigma(y)$$

By the mean-value-property we get

$$u(0) = \frac{1}{|S^{n-1}|r^{n-1}}\int_{\partial B_r(0)} ud\sigma(y)$$

$$u(x) \ge r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0)$$

The other inequality can be proved similarly.

(b) use (a) to prove the following result: let u be a harmonic function in R^n. Suppose that $u \ge 0$. Then $u \equiv Constant$.

Solution

In (a), we let $r \to +\infty$, we obtain that $u(x) = u(0)$ for all x.

2. This problem concerns the heat equation

$$u_t = \Delta u$$

Let

$$\Phi(x-y,t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

(a) Show that $\int_{R^n} \Phi(x-y,t)dy = 1$ for all $t > 0$

Solution

$$\int_{R^n} \Phi(x-y,t)dy = (4\pi t)^{-n/2}\int_{R^n} e^{-\frac{(x-y)^2}{4t}}dy = (4\pi t)^{-n/2}\int_{R^n} e^{-\frac{y^2}{4t}}dy$$

$$= (\pi)^{-n/2}\int_{R^n} e^{-y^2}dy = (\pi)^{-n/2}(\int_R e^{-x^2}dx)^n = 1$$

(b) Show that there exists a generic constant C_n such that

$$\Phi(x-y,t) \le C_n|x-y|^{-n}$$

Hint: maximize the function in t.

Solution: For the function $\Phi(x,t)$ with $x \neq 0$, we know that $\lim_{t\to 0}\Phi(x,t) = 0$ and $\lim_{t\to+\infty}\Phi(x,t) = 0$ and hence the maximum is attained at some $t = t_0$:

$$\frac{\partial \Phi(x,t)}{\partial t} = 0$$

$$\frac{x^2}{4t} = \frac{n}{2}$$

and hence $|\Phi(x,t)| \leq (2n)^{n/2} e^{-n/2} |x|^{-n}$.

(c) Let $f(x)$ be a function such that $f(x_0-)$ and $f(x_0+)$ exists. Show that

$$\lim_{t\to 0} \int_R \Phi(x - x_0, t) f(y) dy = \frac{1}{2}(f(x_0-) + f(x_0+))$$

Solution: The fact that $f(x_0-)$ and $f(x_0+)$ exist implies that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $|x - x_0| < \delta$

$$|f(x) - f(x_0-)| < \epsilon, x_0 - \delta < x < x_0; |f(x) - f(x_0+)| < \epsilon, x_0 < x < x_0 + \delta$$

Thus

$$\int_R \Phi(x - x_0, t) f(y) dy = \int_{|x-x_0|>\delta} \Phi(x - x_0) f(y) dy + \int_{x_0-\delta<x<x_0} \Phi(x - x_0, t) f(y) dy + \int_{x_0<x<x_0+\delta} \Phi(x - x_0, t) f(y) dy$$

The first term on the right hand side can be estimated as

$$|\int_{|x-x_0|>\delta} \Phi(x - x_0, t) f(y) dy| \leq \sup |f| \Phi(\delta, t)$$

For the second term, we have

$$\int_{x_0-\delta<x<x_0} \Phi(x - x_0, t) f(y) dy - \frac{1}{2} f(x_0-) = \int_{x_0-\delta<x<x_0} \Phi(x - x_0)(f(y) - f(x_0-)) dy + f(x_0-) \int_{-\infty}^{x_0-\delta} \Phi(x - x_0, t)$$

Now

$$|\int_{x_0-\delta<x<x_0} \Phi(x - x_0)(f(y) - f(x_0-)) dy| < \epsilon \int_R \Phi(x - x_0, t) dy = \epsilon$$

$$|f(x_0-) \int_{-\infty}^{x_0-\delta} \Phi(x - x_0, t)| \leq C\Phi(\delta, t)$$

Thus

$$|\int_{x_0-\delta<x<x_0} \Phi(x - x_0, t) f(y) dy - \frac{1}{2} f(x_0-)| < \epsilon + C\Phi(\delta, t)$$

Similarly we get

$$|\int_{x_0<x<x_0+\delta} \Phi(x - x_0, t) f(y) dy - \frac{1}{2} f(x_0+)| < \epsilon + C\Phi(\delta, t)$$

The results follows by letting $t \to 0$ first and then let $\epsilon \to$.

3. This problem concerns the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = f(x), u_t(x,0) = g(x)$$

(a) Show that all solutions to the following equation

$$u_{XY} = \frac{\partial^2 u}{\partial X \partial Y} = 0$$

are given by a combination of two functions

$$u = F(X) + G(Y)$$

Solution: Since

$$\partial_X(\partial_Y u) = 0$$

we have $\partial Y u = g(Y)$. Now integrating over Y we get $u = F(X) + G(Y)$.

(b) Show that all solutions to

$$u_{tt} = c^2 u_{xx}$$

are given by

$$u = F(x - ct) + G(x - ct)$$

Hint: let

$$X = x - t, Y = x + t$$

and then use (a)

Solution: Follow the hint

(c) Prove the d'Alembert's formula: all solutions to

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = f(x), u_t(x,0) = g(x)$$

are given by

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Solution: By (b) we have $u(x,t) = F(x - ct) + G(x + ct)$. Now

$$u(x,0) = F(x) + G(x) = f(x)$$

$$u_t(x,0) = c(-F^{'}(x) + G^{'}(x)) = g(x)$$

and hence

$$-F(x) + G(x) = \frac{1}{c}\int_0^x g(s)ds + C$$

So

$$F(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(s)ds - \frac{C}{2}$$

$$G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(s)ds + \frac{C}{2}$$

Substituting into the formula $u(x,t) = F(x-ct) + G(x+ct)$, we obtain the formula.

(d) use (c) to show that Maximum Principle does not hold for wave equation, i.e.,

$$\max_{\bar{U_T}} u(x,t) > \max_{\partial' U_T} u(x,t)$$

Hint: Let $f = 0$ and $g = 1$, $U = (-1,1)$ and choose T large.

Solution

Let $f = 0, g = 1$ for $|x| < \frac{1}{2}$ and $g = 0$ for $|x| \geq \frac{1}{2}$. Then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds = \frac{1}{2c}|(x-ct, x+ct) \cap (-1,1)|$$

Let $T = \frac{1}{2}$. Then

$$\max_{U_T} u > 0 = \max_{\partial'_T} u$$

4. This problem concerns Sobolev space

(a) Let $U = (-1,1)$ and

$$u(x) = |x|$$

What is its weak derivative $u^{'}$? Prove it rigorously.

Solution

We claim that

$$u^{'} = \begin{cases} 1, x > 0 \\ -1, x < 0 \end{cases}$$

In fact let $v = u^{'}$ and let $H(x)$ denote the right hand side function. Then

$$\int_U v\phi = -\int u\phi^{'} = -\int_0^1 x\phi^{'} + \int_{-1}^0 x\phi^{'} dx$$

$$= \int_0^1 \phi - \int_{-1}^0 \phi = \int_{-1}^1 H(x)\phi$$

By the uniqueness of weak derivative, we see that $v = H(x)$.

(b) Does the second order weak derivative $u^{''}$ exist?

Solution: Answer No. Suppose the second order weak derivative exist, called v. Then

$$\int_U v\phi = \int u\phi^{''} = \int_0^1 x\phi^{''} - \int_{-1}^0 \phi^{''} dx$$

$$= -\int_0^1 \phi^{'} + \int_{-1}^0 \phi^{'} = 2\phi(0)$$

Now we choose ϕ_m such that $\phi_m(0) = 1$ and $0 \leq \phi_m \leq 1$ and the support of ϕ_m to be $[-1/m, 1/m]$.

(c) For which integer k and positive $p > 1$, does u belong to $W^{k,p}(U)$?

Solution

For (a) and (b), we have

$$u \in W^{1,p}(U), 1 \leq p \leq +\infty$$
$$u \notin W^{k,p}, k \geq 2$$