MATH 516-101 Homework THREE
Due Date: October 27, 2015

This set of homework problems is concerned with Sobolev spaces.

1. Show that the following Chain Rule holds for $W^{1,p}(U)$ function: Let $u \in W^{1,p}(U)$ for some $1 \leq p < \infty$ and $f \in C^1$ such that $f^{'}$ is bounded. Show that

$$v := f(u) \in W^{1,p}(U), \text{ and } v_{x_j} = f^{'}(u) u_{x_j}$$

Hint: Approximate $W^{1,p}$ functions by C^∞ functions.

2. Show that if $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$, then $u^+ := \max(u, 0) \in W^{1,p}(U), u^- := \min(u, 0) \in W^{1,p}(U)$ and

$$Du^+ = \begin{cases} Du, \text{ if } u > 0 \\ 0, \text{ if } u \leq 0 \end{cases}$$

$$Du^- = \begin{cases} 0 \text{ if } u \geq 0 \\ Du, \text{ if } u < 0 \end{cases}$$

As a consequence, show that $|D|u|| = |Du|$ and hence

$$\|u\|_{W^{1,p}(U)} = \||u|\|_{W^{1,p}(U)}$$

Hint: $u^+ = \lim_{\epsilon \to 0} f_\epsilon(u)$ where

$$f_\epsilon(u) = \begin{cases} \sqrt{u^2 + \epsilon^2} - \epsilon \text{ if } u \geq 0 \\ 0, \text{ if } u \leq 0 \end{cases}$$

Then apply Chain rule and take limits as $\epsilon \to 0$.

3. Verify that if $n > 1$, the function $u = \log \log(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(U), U = B_1(0)$

4. Let $u \in C^\infty(\bar{R}^n_+)$. Extend u to Eu on R^n such that

$$Eu = u, x \in R^n_+; Eu \in C^3(R^n); \|Eu\|_{W^{3,p}} \leq \|u\|_{W^{3,p}}$$

Here $R^n_+ = \{(x^{'}, x_n); x_n > 0\}$.
Hint: for $x_n < 0$ define

$$Eu(x^{'}, x_n) = c_1 u(x^{'}, -x_n) + c_2 u(x^{'}, -\frac{x_n}{2}) + c_3 u(x^{'}, -\frac{x_n}{3})$$

and find the coefficients c_1, c_2 and c_3.

5. Assume that $n = 1$ and $w \in W^{1,p}(R)$ for $1 < p < \infty$. Show that

$$\sup |u| \leq C\|u\|_{W^{1,p}}, \ |u(x) - u(y)| \leq C|x - y|^{1-\frac{1}{p}} \|u\|_{W^{1,p}}$$

6. (Gagliardo-Nirenberg inequality) Let $n \geq 2, 1 < p < n$ and $1 \leq q < r < \frac{np}{n-p}$. For some $\theta \in (0, 1)$ and some constant $C > 0$ we have

$$\|u\|_{L^r(R^n)} \leq C\|u\|^{1-\theta}_{L^q(R^n)} \|\nabla u\|^{\theta}_{L^p(R^n)}, \forall u \in C^\infty_c(R^n)$$

(i) Use scaling to find the θ.
(ii) Prove the inequality.
Hint: Do an interpretation of L^r in terms of L^q and $L^{\frac{np}{n-p}}$ and then apply Sobolev.