Solutions to MATH 516-101 Homework THREE

This set of homework problems is concerned with Sobolev spaces.

1. Show that the following Chain Rule holds for $W^{1,p}(U)$ function: Let $u \in W^{1,p}(U)$ for some $1 \leq p < \infty$ and $f \in C^1$ such that $f^{'}$ is bounded. Show that

$$v := f(u) \in W^{1,p}(U), \text{ and } v_{x_j} = f^{'}(u) u_{x_j}$$

Hint: Approximate $W^{1,p}$ functions by C^∞ functions.

Solution: Let $M = \sup_{x \in R} |f^{'}(s)|$. By density theorem, there are functions $u^m \in C^\infty(U)$ such that $u^m \to u$ in $W^{1,p}(U)$. Hence $u^m \to u, Du^m \to Du$, *a.e.* in U. Let $v^m = f(u^m)$. Since $F, u^m \in C^1$, v^m is also in C^1 and by chain rule for smooth functions, $v^m_{x_i} = f^{'}(u^m) u^m_{x_i}$. Since $|v^m - v| = |f(u^m) - f(u)| \leq M|u^m - u|$, $v^m \in L^p$, we deduce that $v^m \to v$ in $L^p(U)$. Moreover, $f^{'}(u^m) \to f^{'}(u), a.e.$ since $f^{'} \in C^0$ and $u^m \to u, a.e.$ For any tes function $\phi \in C - 0^\infty(U)$ we have

$$\int_U v \phi_{x_i} = \lim_{m\to\infty} \int_U v^m \phi_{x_k} dx = -\lim_{m\to\infty} f^{'}(u^m) u^m_{x_i} dx = -\int_U f^{'}(u) u_{x_k} \phi dx$$

where in the last equality we used Lebesgue's Dominated Convergence Theorem, $|f^{'}(u^m) u^m_{x_i}| \leq 2M|u_{x_i}| a.e.$ and the a.e. convergence of the integrand. Hence v_{x_i} exists and by defintion, $v_{x_i} = f^{'}(u) u_{x_i}$, which is in L^p since $|v_{x_i}| \leq M|u_{x_i}|$.

2. Show that if $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$, then $u^+ := \max(u, 0) \in W^{1,p}(U), u^- := \min(u, 0) \in W^{1,p}(U)$ and

$$Du^+ = \begin{cases} Du, \text{ if } u > 0 \\ 0, \text{ if } u \leq 0 \end{cases}$$

$$Du^- = \begin{cases} 0 \text{ if } u \geq 0 \\ Du, \text{ if } u < 0 \end{cases}$$

As a consequence, show that $|D|u|| = |Du|$ and hence

$$\|u\|_{W^{1,p}(U)} = \||u|\|_{W^{1,p}(U)}$$

Hint: $u^+ = \lim_{\epsilon\to 0} f_\epsilon(u)$ where

$$f_\epsilon(u) = \begin{cases} \sqrt{u^2 + \epsilon^2} - \epsilon \text{ if } u \geq 0 \\ 0, \text{ if } u \leq 0 \end{cases}$$

Then apply Chain rule and take limits as $\epsilon \to 0$.

Solution: We follow the hint. Let f_ϵ be defined as above. We see that $f_\epsilon \in C^1$ and $|f_\epsilon^{'}(s) leq 1$ for all ϵ, s. Moreover we have $u^+(x) = \lim_{\epsilon\to 0+} f_\epsilon(u(x))$ for all x. By Problem One, $f_\epsilon(u) \in W^{1,p}$ and $D_{x_k} f_\epsilon(u) = f_\epsilon^{'}(u) u_{x_k}$ for each k. hence for any test function $\phi \in C_0^\infty$ we have

$$\int_U u^+ \phi_{x_k} = \lim_{\epsilon\to 0} f_\epsilon(u) \phi_{x_k} = -\lim_{\epsilon\to 0} \int_U f_\epsilon^{'}(u) u_{x_k} \phi dx == \int_U \chi_{\{u>0\}} u_{x_k} \phi$$

where $\chi_{\{u>0\}}$ is the characteristic function of the region $u > 0$. We have used Lebesgue's Dominated Convergence Theorem in the first and the third equalities. Thus Du^+ exists and $Du^+ = \chi_{\{u>0\}} Du$ which belongs to L^p. The proofs for the others are similar.

3. Verify that if $n > 1$, the function $u = \log\log(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(U), U = B_1(0)$

Solution: $u_{x_i} = \frac{1}{\log(1+\frac{1}{|x|})}(-\frac{x_i}{|x|^3})$ and hence $|Du| \leq \frac{1}{\log(1+\frac{1}{|x|})}\frac{1}{|x|^2}$. The rest is trivial computation.

4. Let $u \in C^\infty(\bar{R}^n_+)$. Extend u to Eu on R^n such that

$$Eu = u, x \in R^n_+; Eu \in C^2(R^n); \|Eu\|_{W^{3,p}} \leq \|u\|_{W^{3,p}}$$

Here $R^n_+ = \{(x^{'}, x_n); x_n > 0\}$.
Hint: for $x_n < 0$ define

$$Eu(x^{'}, x_n) = c_1 u(x^{'}, -x_n) + c_2 u(x^{'}, -\frac{x_n}{2}) + c_3 u(x^{'}, -\frac{x_n}{3})$$

and find the coefficients c_1, c_2 and c_3.

Solution: Following the hint, let the extension be as above. Now we compute: C^0 implies $\lim_{x_n\to 0+} u(x^{'}, x_n) = \lim_{x_n\to 0-} Eu(x^{'}, x_n)$ and hence

$$c_1 + c_2 + c_3 = 1$$

For C^1 we get

$$c_1(-1) + c_2(-\frac{1}{2}) + c_3(-\frac{1}{3}) = 1$$

For C^2 we get

$$c_1(-1)^2 + c_2(-\frac{1}{2})^2 + c_3(-\frac{1}{3})^2 = 1$$

Solving the above three equations for c_1, c_2, c_3 we obtain

$$c_1 = 6, c_2 = -32, c_3 = 27$$

5. Assume that $n = 1$ and $w \in W^{1,p}(R)$ for $1 < p < \infty$. Show that

$$\sup|u| \leq C\|u\|_{W^{1,p}}, \ |u(x) - u(y)| \leq C|x-y|^{1-\frac{1}{p}}\|u\|_{W^{1,p}}$$

Solution: By density theorem we may assume that $u \in C_0^\infty(R)$. Then we have by Holder's

$$|u(x) - u(y)| = |\int_x^y Du(s)ds| \leq |x-y|^{1-\frac{1}{p}}\|Du\|_{L^p}$$

$$u^p(x) = u^p(x) - u^p(-\infty) = \int_{-\infty}^x Du^p ds = \int_{-\infty}^x pu^{p-1}Duds \leq p\|u\|_{L^p}\|Du\|_{L^p}$$

6. (Gagliardo-Nirenberg inequality) Let $n \geq 2, 1 < p < n$ and $1 \leq q < r < \frac{np}{n-p}$. For some $\theta \in (0,1)$ and some constant $C > 0$ we have

$$\|u\|_{L^r(R^n)} \leq C\|u\|^{1-\theta}_{L^q(R^n)}\|\nabla u\|^\theta_{L^p(R^n)}, \forall u \in C_c^\infty(R^n)$$

(i) Use scaling to find the θ.

(ii) Prove the inequality.
Hint: Do an interpretation of L^r in terms of L^q and $L^{\frac{np}{n-p}}$ and then apply Sobolev.

Solution: (i) letting $u_\lambda(x) = u(\lambda x)$,

$$\|u_\lambda\|_{L^s} = \lambda^{-n/s}\|u\|_{L^s}$$
$$\|\nabla u_\lambda\|_{L^s} = \lambda^{1-\frac{n}{s}}\|\nabla u\|_{L^s}$$

since $\lambda > 0$ is arbitrary the power of λ must match, we get

$$\frac{n}{r} = (1-\theta)\frac{n}{q} + \theta(-1 + \frac{n}{p})$$

and hence

$$\theta = \frac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{n} + \frac{1}{q} - \frac{1}{p}}$$

(ii) For θ determined as above, the Holder inequality implies that

$$\|u\|_{L^r} \leq \|u\|_{L^q}^{1-\theta}\|u\|_{L^{\frac{np}{n-p}}}^{1-\theta}$$

By Sobolev embedding ives the desired estimate.