MATH 516-101 Homework Four
Due Date: November 12, 2015

This set of homework problems is concerned with Sobolev spaces and weak solutions.

1. Fix $\alpha > 0$ and let $U = B_1(0)$. Show that there exists a constant C, depending on n and α such that

$$\int_U u^2 dx \leq C \int_U |\nabla u|^2$$

provided

$$u \in W^{1,2}(U), \ |\{x \in U | u(x) = 0\}| \geq \alpha$$

2. (a) Let $n > 4$. Show that the embedding $W^{2,2}(U) \to L^{\frac{2n}{n-4}}(U)$ is not compact; (b) Describe the embedding of $W^{2,p}(U)$ in different dimensions. State if the embedding is continuous or compact.

3. Consider the following one-dimensional problem

$$(1) \qquad -a(x)u_{xx} + b(x)u_x + c(x)u = f, 0 < x < L, \ u(0) = u(L) = 0$$

where

$$0 < c_1 \leq a(x) \leq c_2$$

(a) Show that (1) can always be transformed into a self-adjoint form:

$$(2) \qquad -(\tilde{a}u_x)_x + \tilde{c}u = \tilde{f}$$

(b) State the definition of weak solution.
(c) Find conditions on $\tilde{c}$ such that the existence of a weak solution to (1) exists.

4. (a). Assume that U is connected. A function $u \in W^{1,2}(U)$ is a weak solution of the Neumann problem

$$(3) \qquad -\Delta u = f \text{ in } U; \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

if

$$\int_U Du \cdot Dv = \int_U fv, \quad \forall v \in W^{1,2}$$

Suppose that $f \in L^2$. Show that (3) has a weak solution if and only if

$$\int_U f = 0$$

(b). Discuss how to define a weak solution of the Poisson equation with Robin boundary conditions

$$(4) \qquad -\Delta u = f \text{ in } U; \ u + \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

5. Let $u \in W^{1,2}(R^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + u^3 = f \text{ in } R^n$$

where $f \in L^2(R^n)$. Prove that $u \in W^{2,2}(R^n)$.

Hint: mimic the proof of interior regularity but without the cut-off function.