MATH 516-101 Solutions to Homework Four

This set of homework problems is concerned with Sobolev spaces and weak solutions.

1. Fix $\alpha > 0$ and let $U = B_1(0)$. Show that there exists a constant C, depending on n and α such that

$$\int_U u^2 dx \leq C \int_U |\nabla u|^2$$

provided

$$u \in W^{1,2}(U), \ |\{x \in U | u(x) = 0\}| \geq \alpha$$

Solutions: We prove it by contradiction. Suppose the inequality is not true. Then there exists u_n such that

$$\int_U u_n^2 dx \geq n \int_U |\nabla u_n|^2$$

$$|\{u_n(x) = 0\}| \geq \alpha$$

As in the proof of Poincare ineqaulity, we may assume that

$$\int_U u_n^2 = 1, \int_U |\nabla u_n|^2 \leq \frac{1}{n}$$

Since u_n is bounded in $W^{1,2}$, by embedding theorem, there exists a subsequence of u_n, still denoted by u_n, such that $u_n \to u_0$ in L^2 and $\int_U u_0^2 = 1$. By Fatou's Lemma, we have

$$\int_U |\nabla u_0|^2 \leq \lim_{n \to +\infty} |\nabla u_n|^2 = 0$$

This implies that

$$u_0 \equiv Constant$$

On the other hand

$$\int_U |u_n - u_0|^2 dx \geq \int_{U \cap \{u_n = 0\}} |u_0|^2 dx \geq |Constant|^2 |\{u_n = 0\}| \geq |Constant|^2 \alpha$$

but $\int_U |u_n - u_0|^2 \to 0$ since $u_n \to u_0$ in L^2. Thus the constant must be zero and hence $u_0 = 0$, $u_n \to 0$ in L^2, which contradicts with $\|u_0\|_{L^2} = 1$.

2. (a) Let $n > 4$. Show that the embedding $W^{2,2}(U) \to L^{\frac{2n}{n-4}}(U)$ is not compact; (b) Describe the embedding of $W^{2,p}(U)$ in different dimensions. State if the embedding is continuous or compact.

Solutions: (a) Suppose $B_r(x_0) \subset U$. Let $\eta(x) = 1$ for $|x - x_0| \leq 1$ and $\eta(x) = 1$ for $|x - x_0| \geq 2$. For $\lambda > 0$, we set

$$u_\lambda(x) = \lambda^{\frac{n-4}{4}} \eta(\lambda x)$$

where λ is large. It is easy to see that

$$\int_U u_\lambda^{\frac{2n}{n-4}} = \int \eta^{\frac{2n}{n-4}}$$

$$\int_U |\nabla^2_\lambda u|^2 = \int |\nabla^2 \eta|^2$$

For $\lambda = 2^k$, u_λ is a bounded sequence in $W^{2,2}$, but u_λ contains a subsequence converging in $L^{2n/(n-4)}$. In fact for $k > m$

$$\int_U (u_{2^k} - u_{2^m})^{\frac{2n}{n-4}} = \int_{R^n} (\eta(x) - 2^{(m-k)\frac{n-4}{4}}\eta(2^{m-k}x))^{\frac{2n}{n-4}} \geq \frac{1}{2}\int \eta^{\frac{2n}{n-4}} > 0$$

(b) If $2 - \frac{n}{p} < 0$, i.e., $p < \frac{n}{2}$, then $W^{2,p}$ is compactly embedded into L^q for any $q < \frac{np}{n-2p}$ and continuously embedded into $L^{\frac{np}{n-2p}}$.

If $p = \frac{n}{2}$,then $W^{2,p}$ is compactly embedded into L^q for any $q < +\infty$ and continuously embedded into Orlicz space e^{u^2}.

If $\frac{n}{2} < p < n$, then $W^{2,p}$ is compactly embedded into C^α for any $\alpha < 2 - \frac{n}{p}$ and continuously embedded into $C^{2-\frac{n}{p}}$

If $p = n$, then $W^{2,p}$ is compactly embedded into C^α for any $\alpha < 1$.

If $p > n$, then $W^{2,p}$ is compactly embedded into $C^{1,\alpha}$ for any $\alpha < 1 - \frac{n}{p}$ and continuously embedded into $C^{1,1-\frac{n}{p}}$.

3. Consider the following one-dimensional problem

$$(1) \qquad -a(x)u_{xx} + b(x)u_x + c(x)u = f, 0 < x < L,\ u(0) = u(L) = 0$$

where

$$0 < c_1 \leq a(x) \leq c_2$$

(a) Show that (1) can always be transformed into a self-adjoint form:

$$(2) \qquad -(\tilde{a}u_x)_x + \tilde{c}u = \tilde{f}$$

Solutions: We multiply (1) by some positive function $\beta(x)$

$$-a\beta u_{xx} + b\beta u_x + c\beta u = f\beta$$

$$-(a\beta u_x) + (a\beta)_x u_x + b\beta u_x + c\beta = f\beta$$

Now we choose

$$(a\beta)_x + b\beta = 0$$

$$a\beta = e^{-\int \frac{b}{a}}$$

and

$$\tilde{a} = a\beta, \tilde{c} = c\beta, \tilde{f} = f\beta$$

(b) State the definition of weak solution.

Solution: A weak solution to (2) is a H^1_0 function u such that

$$\int_U \tilde{a}u_x v_x + \tilde{c}uv = \int_U \tilde{f}v,\ \forall v \in H^1_0(U)$$

(c) Find conditions on $\tilde{c}$ such that the existence of a weak solution to (1) exists.

Solution: By Fredholm Alternative, a weak solution exists for all $\tilde{f} \in L^2(U)$ if there exists only trivial solution to

$$-(\tilde{a}u_x)_x + \tilde{c}u = 0$$

So the condition for existence is: the only weak solution to

$$-(\tilde{a}u_x)_x + \tilde{c}u = 0$$

is zero.

4. (a). Assume that U is connected. A function $u \in W^{1,2}(U)$ is a weak solution of the Neumann problem

$$(3) \qquad -\Delta u = f \text{ in } U; \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

if

$$\int_U Du \cdot Dv = \int_U fv, \quad \forall v \in W^{1,2}$$

Suppose that $f \in L^2$. Show that (3) has a weak solution if and only if

$$\int_U f = 0$$

Solution: necessity is easy to show: just let $v \equiv 1$. To prove that it is also sufficient, we either use Lax-Milgram theorem or Fredholm Alternatives.

On existence: Let $H = \{u \in W^{1,2}(U) \mid \int_U u = 0\}$ with inherited $W^{1,2}$-inner product. It is clearly a linear subspace. To be a Hilbert space, we need to show it is closed: In fact this is trivial since $W^{1,2}$ is compactly embedded into L^2.

Let the bilinear form be

$$B[u; v] = \int_U DuDvdx$$

We check that it satisfies the conditions of the Lax-Milgram Theorem on H. Indeed, clearly $B[u; v] \leq \|u\|_{W^{1,2}}\|v\|_{W^{1,2}}$ and

$$B[u; u] = \int_U |\nabla u|^2$$

By Poincare's inequality

$$\int_U (u - \frac{1}{|U|}\int_U u)^2 = \int_U u^2 \leq C\int_U |\nabla U|^2$$

Hence

$$B[u; u] \geq \frac{1}{C+1}\|u\|_{W^{1,2}}$$

By Lax-Milgram theorem, for all $f \in L^2, \int f = 0$, the problem

$$B[u; v] = (f, v), \forall v \in H$$

has a unique solution u.

Now for any $v \in W^{1,2}$, $v - \bar{v} \in H$, where $\bar{v}$ is the average of v. Then we have

$$B[u; v - \bar{v}] = (f, v - \bar{v})$$

which implies

$$\int_U DuDv = \int_U fv$$

Another proof is to use Fredholm Alternatives.

(b). Discuss how to define a weak solution of the Poisson equation with Robin boundary conditions

$$(4) \qquad -\Delta u = f \text{ in } U;\ u + \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

Solution: If everything is smooth, we multiply (4) by v and integrate:

$$\int_U DuDv - \int_{\partial U} v\frac{\partial u}{\partial \nu} = \int_U fv$$

By the BC, we have

$$\int_U DuDv + \int_{\partial U} uv = \int_U fv$$

We now define a weak solution to (4) if for all $v \in W^{1,2}$ there holds

$$\int_U DuDv + \int_{\partial U} uv = \int_U fv$$

Note that the second term makes sense by the trace theorem.

Defining the bilinear form as

$$B[u; v] = \int_U DuDv + \int_{\partial U} uv$$

we can use Lax-Milgram theorem to prove the existence, but a new kind of Poincare inequality is needed!!!

5. Let $u \in W^{1,2}(R^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + u^3 = f \text{ in } R^n$$

where $f \in L^2(R^n)$. Prove that $u \in W^{2,2}(R^n)$.

Hint: mimic the proof of interior regularity but without the cut-off function.

Solutions: We follow the same proof. The problem is how to deal with the second term. Multiplying the equation by

$$v = D_k^{-h}(D_k^h u)$$

which is $W^{1,2}$, thanks to compact support of u. The rest of the proof is similar to the Theorem we proved in class. the only difference is the second term

$$\int u^3 D_k^{-h}(D_k^h u) = -\int D_k^h(u^3) D_k^h u = -\int \frac{u^3(x+h) - u^3(x)}{h}\frac{u(x+h) - u(x)}{h}$$
$$= -\int \frac{(u(x+h) - u(x))^2}{h^2}(u^2(x+h) + u^2(x) + u(x+h)u(x))$$
$$\leq 0$$