MATH 516-101 Homework Five
Due Date: November 26, 2015

This set of homework problems is concerned with Moser's iterations and maximum principles

1. Show that $u = \log |x|$ is in $H^1(B_1)$, where $B_1 = B_1(0) \subset R^3$ and that it is a weak solution of

$$-\Delta u + c(x) u = 0$$

for some $c(x) \in L^{\frac{3}{2}}(B_1)$ but u is not bounded.

2. Let u be a weak sub-solution of

$$-\sum_{i,j} \partial_{x_j}(a^{ij}\partial_{x_i} u) + \sum_i b^i \partial_{x_i} u + c(x) u = f$$

where $\theta \leq (a^{ij}) \leq C_2 < +\infty, b^i \in L^\infty$. Suppose that $c(x) \in L^{\frac{n}{2}}(B_1), f \in L^q(B_1)$ where $q > \frac{n}{2}$. Show that there exists a generic constant $\epsilon_0 > 0$ such that if $\int_{B_1} |c|^{\frac{n}{2}} dx \leq \epsilon_0$, then

$$\sup_{B_{1/2}} u^+ \leq C(\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$$

Hint: following the Moser's iteration procedure.

3. Let u be a smooth solution of $Lu = -\sum_{i,j} a^{ij} u_{x_i x_j} = 0$ in U and a^{ij} are C^1 and uniformly elliptic. Set $v := |Du|^2 + \lambda u^2$. Show that

$$Lv \leq 0 \ \text{ in } U, \quad \text{if } \lambda \text{ is large enough}$$

Deduce, by Maximum Principle that

$$\|Du\|_{L^\infty(U)} \leq C\|Du\|_{L^\infty(\partial\Omega)} + C\|u\|_{L^\infty(\partial\Omega)}$$

4. Let u be a harmonic function in a punctured ball

$$\Delta u = 0 \ \text{ in } B_1(0)\backslash\{0\}$$

Show that if $u(x) = o(\log |x|)$ when $n = 2$ and $u(x) = o(|x|^{2-n})$ if $n \geq 3$, then u is bounded.

5. Let u be a smooth function satisfying

$$\Delta u - u = f(x), |u| \leq 1, \quad \text{in } R^n$$

where

$$|f(x)| \leq e^{-\frac{1}{2}|x|}$$

Deduce from maximum principle that u actually decays

$$|u(x)| \leq C e^{-\frac{1}{2}|x|}$$

Hint: Comparing u with the following function

$$C_1 e^{-\frac{1}{2}|x|} + \epsilon e^{\frac{1}{2}|x|}$$

for $|x|$ large, where C_1 is appropriately chosen.