MATH 516-101 Solutions to Homework Five

This set of homework problems is concerned with Moser's iterations and maximum principles

1. Show that $u = \log|x|$ is in $H^1(B_1)$, where $B_1 = B_1(0) \subset R^3$ and that it is a weak solution of

$$-\Delta u + c(x)u = 0$$

for some $c(x) \in L^{\frac{3}{2}}(B_1)$ but u is not bounded.

Solutions: Direct Computations.

2. Let u be a weak sub-solution of

$$-\sum_{i,j} \partial_{x_j}(a^{ij}\partial_{x_i}u) + \sum_i b^i \partial_{x_i}u + c(x)u = f$$

where $\theta \leq (a^{ij}) \leq C_2 < +\infty, b^i \in L^\infty$. Suppose that $c(x) \in L^{\frac{n}{2}}(B_1), f \in L^q(B_1)$ where $q > \frac{n}{2}$. Show that there exists a generic constant $\epsilon_0 > 0$ such that if $\int_{B_1} |c|^{\frac{n}{2}} dx \leq \epsilon_0$, then

$$\sup_{B_{1/2}} u^+ \leq C(\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$$

Hint: following the Moser's iteration procedure.

Solutions: The proof is similar to what I did in class: we get first

$$\int |D(w\eta)|^2 dx \leq C \int (|D\eta|^2 + \eta^2)w^2 + \int |c|w^2\eta^2$$

Now the last term can be estimated as

$$\int |c|w^2\eta^2 \leq (\int |c|^{\frac{n}{2}})^{\frac{2}{n}} (\int (w\eta)^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq \epsilon_0^{n/2} \|w\eta\|^2_{L^{\frac{2n}{n-2}}}$$

Since

$$\|w\eta\|^2_{L^{2n/(n-2)}} \leq C \int |D(w\eta)|^2 \leq C \int (|D\eta|^2 + \eta^2)w^2 + C\epsilon_0^{n/2} \|w\eta\|^2_{L^{\frac{2n}{n-2}}}$$

we obtain that for ϵ_0 small

$$\|w\eta\|^2_{L^{2n/(n-2)}} \leq C \int (|D\eta|^2 + \eta^2)w^2$$

The rest of the proof then follows.

3. Let u be a smooth solution of $Lu = -\sum_{i,j} a^{ij} u_{x_i x_j} = 0$ in U and a^{ij} are C^1 and uniformly elliptic. Set $v := |Du|^2 + \lambda u^2$. Show that

$$Lv \leq 0 \text{ in } U, \text{ if } \lambda \text{ is large enough}$$

Deduce, by Maximum Principle that

$$\|Du\|_{L^\infty(U)} \leq C\|Du\|_{L^\infty(\partial\Omega)} + C\|u\|_{L^\infty(\partial\Omega)}$$

Solutions: Direct Computations. In fact we write $v = u_k^2 + \lambda u^2$ then

$$v_{ij} = 2u_{ki}u_{kj} + 2u_{kij}u_k + 2\lambda u_i u_j + 2\lambda u u_{ij}$$

Hence

$$Lv \leq -2\theta \sum_{k,i} |u_{ki}|^2 - 2\lambda\theta |\nabla u|^2 - 2a^{ij}u_{kij}u_k$$

By the equation

$$a^{ij}u_{kij} = -a_k^{ij}u_{ij}$$

So

$$Lv \leq -2\theta \sum_{k,i} |u_{ki}|^2 - 2\lambda\theta |\nabla u|^2 + Cu_{ij}u_k$$

$$\leq 0$$

for λ large.

4. Let u be a harmonic function in a punctured ball

$$\Delta u = 0 \ \text{ in } B_1(0) \backslash \{0\}$$

Show that if $u(x) = o(\log |x|)$ when $n = 2$ and $u(x) = o(|x|^{2-n})$ if $n \geq 3$, then u is bounded.

Solutions: We prove that u is bounded from above first. Let $n \geq 3$. Since $u(x) = o(|x|^{2-n})$, for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $|x| < \delta$,

$$u(x) \leq \epsilon |x|^{2-n}$$

Let $v = C + \epsilon |x|^{2-n}$ where $C = \sup_{x \in \partial B_1} u$. Consider the domain $U = B_1 \backslash B_r$ where $r < \delta$. Then the function $u - v$ satisfies

$$\Delta(u - v) = 0 \text{ in } U, u \leq v \text{ on } \partial U$$

By Maximum Principle,

$$\max_U (u - v) = \max_{\partial U} (u - v) \leq 0$$

Hence

$$u \leq v = C + \epsilon |x|^{2-n}, \ \forall r \leq |x| < 1$$

Now letting $r \to 0$ first we obtain

$$u \leq v = C + \epsilon |x|^{2-n}, \ \forall 0 < |x| < 1$$

Then we let $\epsilon \to 0$.

5. Let u be a smooth function satisfying

$$\Delta u - u = f(x), |u| \leq 1, \ \text{ in } R^n$$

where

$$|f(x)| \leq e^{-\frac{1}{2}|x|}$$

Deduce from maximum principle that u actually decays

$$|u(x)| \leq Ce^{-\frac{1}{2}|x|}$$

Hint: Comparing u with the following function

$$C_1 e^{-\frac{1}{2}|x|} + \epsilon e^{\frac{1}{2}|x|}$$

for $|x|$ large, where C_1 is appropriately chosen.

Solutions: The proof is similar to Problem 4. We note that since u is bounded, we have

$$\lim_{|x|\to+\infty} \frac{u(x)}{e^{1/2|x|}} = 0$$

which means that for $\epsilon > 0$ small there exists $R_\epsilon > 0$ such that

$$u(x) \leq \epsilon e^{\frac{1}{2}|x|}, \quad |x| > R_\epsilon$$

Now we let

$$v = C_1 e^{-\frac{1}{2}|x|} + \epsilon e^{\frac{1}{2}|x|}$$

we have

$$\begin{aligned} -\Delta v + v &= C_1\left(\frac{3}{4} + \frac{1}{2|x|}\right)e^{-1/2|x|} + \epsilon\left(\frac{3}{4} - \frac{1}{2|x|}\right)e^{1/2|x|} \\ &\geq \frac{3}{4} C_1 e^{-1/2|x|} \end{aligned}$$

for $|x| > 1$.

Now we let $R > R_\epsilon$ and $U = B_R \backslash B_1$. We compute for $x \in U$

$$L(u-v) = -\Delta(u-v) + u - v \leq -f - \frac{3}{4} C_1 e^{-1/2|x|} \leq 0$$

if C_1 is large. On the other hand, for $x \in \partial U$, we have either $|x| = 1$,

$$u \leq C \leq C_1 e^{-1/2} \leq v$$

if C_1 large enough, or $|x| = R > R_\epsilon$,

$$u(x) \leq \epsilon e^{1/2|x|} \leq v$$

By Maximum Principle we have

$$\max_U (u-v) = \max_{\partial U} (u-v) \leq 0$$

and hence

$$u \leq C_1 e^{-1/2|x|} + \epsilon e^{1/2|x|}, \ 1 \leq |x| \leq R$$

Now letting $R \to +\infty$ first and $\epsilon \to 0$, we obtain the desired conclusion.