MATH 516-101 Homework Six
Due Date: December 14, 2015

1. Assume that u is a smooth solution of

$$Lu = -a^{ij}u_{ij} = f \ \text{ in } U$$

$$\mathbf{u = 0 \ \text{ on } \partial U}$$

In this and next exercise we obtain boundary gradient estimate at $x^0 \in \partial U$. A barrier at x^0 is a C^2 function w such that

$$Lw \geq 1 \text{ in } U \, w(x^0) = 0, w \geq 0 \text{ on } \partial U$$

Show that if w is a barrier at x^0, there exists a constant C such that

$$|Du(x^0)| \leq C|\frac{\partial w}{\partial \nu}(x^0)|$$

Hint: Since $u = 0$ on $\partial\Omega$, $|Du(x^0)| = |\frac{\partial u}{\partial \nu}(x^0)|$.

2. Continue from Problem 1. Suppose that U satisfies exterior ball property at x^0, i.e., there exists $B_R(y) \subset U^c$ and $\bar{B}_R(y) \cap \bar{U} = \{x^0\}$. Find a barrier w of the following type

$$w(x) = \psi(d(x)), \text{ where } d(x) = |x - y| - R$$

Hint: Compute (letting $y = 0$)

$$\psi_i = \psi' \frac{x_i}{|x|}, \psi_{ij} = \psi'' \frac{x_i x_j}{|x|^2} + \psi' \frac{1}{|x|}(\delta_{ij} - \frac{x_i x_j}{|x|^2})$$

Find ψ such that

$$\psi(0) = 0$$

$$\psi' \geq 0$$

$$\psi'' + C\frac{\psi'}{d+R} = -1$$

Solve the above ODE.

Reference: Chapter 2 of Han-Lin's book.

3. (Kelvin transform of Laplace equation) The Kelvin transform Ku is defined as $Ku = |x|^{2-n}u(\frac{x}{|x|^2})$. Show that if u satisfies

$$\Delta u + f(u) = 0$$

then Ku satisfies

$$\Delta Ku + \frac{1}{|x|^{n+2}} f(|x|^{n-2}Ku) = 0$$

4. Use direct method to prove the existence of a smooth solution to

$$\Delta u + \lambda u - u^3 = 0 \text{ in } U, \quad u = 0 \text{ on } \partial U$$

where

$$\lambda > \lambda_{\mathbf{1}}$$

Show all details. Prove the uniqueness of the solution.

Hint: 1. Show that the minimum of the energy

$$J[u] = \frac{1}{2}\int_U |\nabla u|^2 - \frac{\lambda}{2}\int_U u^2 + \frac{1}{4}\int_U u^4$$

is attained in the following Banach space

$$X = H_0^1(U) \cap L^4(U)$$

Use Fatou's Lemma:

$$\int_U \lim_n G(u_n) \leq \lim_n \int_U G(u)$$

where $G(u) = |\nabla u|^2$ or $G(u) = u^4$.

2. Show that the energy is negative by taking a test function $t\phi$ where ϕ is the first eigenfunction.

3. Uniqueness follows from the class work, since $f(u)/u = \lambda - u^2$ is decreasing in u.

5. Use Mountain-Pass Lemma to prove the existence of a positive solution to

$$\epsilon^{\mathbf{2}}\Delta u + u(u - \frac{1}{3})(1-u) = 0 \quad \text{in } U$$

$$0 < u < 1 \quad \text{in } U$$

$$u = 0 \quad \text{on } \partial U$$

where $\epsilon > 0$ is sufficiently small.

Hint: 1. modify the nonlinearity to be zero for $u > 1$ and $u < 0$. 2. Show that the Mountain Pass Lemma is satisfied. To show that $J(e) < 0$. Choose a function $e = 1$ in most of the part of U except a thin part near the boundary. Then for ϵ sufficiently small, $J(e) < 0$. 3. Use Maximum Principle to show that $0 \leq u \leq 1$. In fact if the minimum is negative at some place p then $\Delta u = 0$ in a neighborhood of p. By Maximum Principle for harmonic function this is not possible. 4. Use Strong Maximum Principle to show that $0 < u < 1$.