MATH 516-101 Solutions to Homework Six

1. Assume that u is a smooth solution of

$$Lu = -a^{ij}u_{ij} = f \quad \text{in } U$$

$$\mathbf{u = 0 \quad on \ \partial U}$$

In this and next exercise we obtain boundary gradient estimate at $x^0 \in \partial U$. A barrier at x^0 is a C^2 function w such that

$$Lw \geq 1 \text{ in } U \, w(x^0) = 0, w \geq 0 \text{ on } \partial U$$

Show that if w is a barrier at x^0, there exists a constant C such that

$$|Du(x^0)| \leq C|\frac{\partial w}{\partial \nu}(x^0)|$$

Hint: Since $u = 0$ on $\partial\Omega$, $|Du(x^0)| = |\frac{\partial u}{\partial \nu}(x^0)|$.

Solutions: Consider the following function $u - Cw(x)$. It satisfies

$$L(u - Cw) = f - CLw \leq f - C \leq 0$$

if $C \geq \max_U |f|$. By Maximum Principle, we have

$$\max_U (u - Cw)^+ = \max_{\partial U}(u - Cw)^+ = \max_{\partial U}(-Cw)^+ = 0$$

and hence

$$u \leq w \text{ in } U$$

and so

$$u(x) - u(x_0) = u(x) \leq C(w(x) - w(x_0))$$

Let $x = x_0 - t\nu(x_0)$. Then we obtain

$$u(x_0 - t\nu(x_0)) - u(x_0) \leq C(w(x_0 - t\nu(x_0)) - w(x_0))$$

$$\frac{u(x_0 - t\nu(x_0)) - u(x_0)}{t} \leq C\frac{(w(x_0 - t\nu(x_0)) - w(x_0))}{t}$$

Letting $t \to 0+$, we get

$$-\frac{\partial u}{\partial \nu}(x_0) \leq -C\frac{\partial w}{\partial \nu}(x_0)$$

Similarly we get

$$\frac{\partial u}{\partial \nu}(x_0) \leq -C\frac{\partial w}{\partial \nu}(x_0)$$

Hence

$$|Du(x_0)| = |\frac{\partial u}{\partial \nu}(x_0)| \leq C|\frac{\partial w}{\partial \nu}(x_0)|$$

2. Continue from Problem 1. Suppose that U satisfies exterior ball property at x^0, i.e., there exists $B_R(y) \subset U^c$ and $\bar{B}_R(y) \cap \bar{U} = \{x^0\}$. Find a barrier w of the following type

$$w(x) = \psi(d(x)), \text{ where } d(x) = |x-y| - R$$

Hint: Compute (letting $y = 0$)

$$\psi_i = \psi' \frac{x_i}{|x|}, \psi_{ij} = \psi'' \frac{x_i x_j}{|x|^2} + \psi' \frac{1}{|x|}(\delta_{ij} - \frac{x_i x_j}{|x|^2})$$

Find ψ such that

$$\psi(0) = 0$$

$$\psi' \geq 0$$

$$\psi'' + C\frac{\psi'}{d+R} = -1$$

Solve the above ODE.

Solution: We compute as in the hint:

$$a^{ij}\psi_{ij} = (\psi'' - \frac{\psi'}{|x-y|})\sum_{ij} a^{ij}\frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} + \frac{\psi'}{|x|}\sum_i a_{ii}$$

Assume that

$$\psi'' \leq 0, \psi' \geq 0$$

. Then in order that

$$-a^{ij}\psi_{ij} \geq 1$$

we need

$$\psi'' + C\frac{\psi'}{|x-y|} \leq -1$$

which is

$$\psi'' + C\frac{\psi'}{d+R} \leq -1$$

where C can be chosen to be large.

Now solving the above ODE, we have

$$\psi = \frac{1}{2(C+1)}[R^2 - (d+R)^2] - A[(d+R)^{-C+1} - R^{-C+1}]$$

where A is a free parameter. We need

$$0 \leq \psi'(d) = -\frac{1}{C+1}(d+R) + A(C-1)(d+R)^{-C}$$

which is possible if A is large enough.

3. (Kelvin transform of Laplace equation) The Kelvin transform Ku is defined as $Ku = |x|^{2-n}u(\frac{x}{|x|^2})$. Show that if u satisfies

$$\Delta u + f(u) = 0$$

then Ku satisfies

$$\Delta Ku + \frac{1}{|x|^{n+2}} f(|x|^{n-2}Ku) = 0$$

Solution: Direct computations. One way is to use the following formula

$$\Delta u = u_{rr} + \frac{n-1}{r}u_r + \frac{\Delta_{S^{n-1}}u}{r^2}$$

where

$$Ku(r,\theta) = r^{2-n}u(\frac{1}{r},\theta)$$

4. Use direct method to prove the existence of a smooth solution to

$$\Delta u + \lambda u - u^3 = 0 \text{ in } U, \quad u = 0 \text{ on } \partial U$$

where

$$\lambda > \lambda_{\mathbf{1}}$$

Show all details. Prove the uniqueness of the solution.

Hint: 1. Show that the minimum of the energy

$$J[u] = \frac{1}{2}\int_U |\nabla u|^2 - \frac{\lambda}{2}\int_U u^2 + \frac{1}{4}\int_U u^4$$

is attained in the following Banach space

$$X = H_0^1(U) \cap L^4(U)$$

Use Fatou's Lemma:

$$\int_U \lim_n G(u_n) \leq \lim_n \int_U G(u)$$

where $G(u) = |\nabla u|^2$ or $G(u) = u^4$.

2. Show that the energy is negative by taking a test function $t\phi$ where ϕ is the first eigenfunction.

3. Uniqueness follows from the class work, since $f(u)/u = \lambda - u^2$ is decreasing in u.

Solution: We first prove the existence. Let

$$J[u] = \frac{1}{2}\int_U |\nabla u|^2 - \frac{\lambda}{2}\int_U u^2 + \frac{1}{4}\int_U u^4$$

where

$$u \in X = H_0^1(U) \cap L^4(U)$$

Note that X is a Banach space under the following norm:

$$\|u\| = \|u\|_{H_0^1(U)} + \|u\|_{L^4(U)}$$

Step 1: we prove the existence of a minimizer

$$c = \inf_{u \in X} J[u]$$

In fact, by any test function, it is easy to see that c is bounded. On the other hand, since

$$u^4 - 2\lambda u^2 \geq -C$$

we obtain that

$$|c| \leq C < +\infty$$

Let $\{u_n\}$ be a sequence of minimizing sequence. Then we have

$$\frac{1}{2}\int_U |\nabla u_n|^2 + \frac{1}{8}\int_U u^4 - C \leq \frac{1}{2}\int_U |\nabla u|^2 - \frac{\lambda}{2}\int_U u^2 + \frac{1}{4}\int_U u^4 \leq C$$

which implies that

$$\int_U |\nabla u_n|^2 \leq C, \int_U u_n^4 \leq C$$

Thus u_n contains a subsequence u_{n_k} such that $u_{n_k} \to u_0$ in L^2, by Sobolev embedding.

But by Fatou's Lemma

$$\int_U |\nabla u_0|^2 \leq \lim_{k \to +\infty} \int_U |\nabla u_{n_k}|^2$$

$$\int_U u_0^4 \leq \lim_{k \to +\infty} \int_U u_{n_k}^4$$

Hence

$$u_0 \in X$$

$$J[u_0] \leq \lim_{k \to +\infty} J[u_{n_k}]$$

By the definition, u_0 is a minimizer in X.

Step 2: Let $\phi \in C_0^\infty(U)$. Then

$$J[u_0 + t\phi] \leq J[u_0], \forall t$$

Thus

$$0 = J^{'}[u_0](\phi) = \int_U \nabla u_0 \nabla \phi - \lambda u_0 \phi + u_0^3 \phi$$

Hence u_0 is a weak solution to

$$\Delta u_0 + \lambda u_0 - u_0^3 = 0$$

Step 3: $u_0 > 0$

First of all, let $\epsilon \phi_1$ be a test function, where ϕ_1 is the first positive eigenfunction. Then we have

$$c \leq J[\epsilon \phi_1] \frac{\epsilon^2}{2}(\int_U |\nabla \phi_1|^2 - \lambda \phi_1^2) + O(\epsilon^4) < 0$$

since $\lambda > \lambda_1$. Thus $u_0 \not\equiv 0$.

Next, replacing u_0 by $|u_0|$, we can assume that $u_0 \geq 0$. Since u_0 satisfies

$$-\Delta u_0 + \lambda u_0 = -u_0^4 \leq 0$$

we infer that u_0 is a weak sub-solution, hence by Moser's iteration $u_0 \in L^\infty$. By $W^{2,p}$ theory, $u \in C^{1,\alpha}$. By Schauder, $u_0 \in C^{2,\alpha}$.

Finally by Maximum Principle, $u_0 > 0$.

Step 4: We show uniqueness.

Writing $f(u) = \lambda u - u^3$, then by Picone's identity we get

$$\int_U \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)(u_1 - u_2) - \int_{\partial U} u_1^2 |\nabla \frac{u_2}{u_1}|^2 - \int_{\partial U} u_2^2 |\nabla \frac{u_1}{u_2}|^2 = 0$$

Now note that $f(u)/u = \lambda u - u^2$ is decreasing. All the three terms in the above equality are negative, and hence uniqueness follows.

5. Use Mountain-Pass Lemma to prove the existence of a positive solution to

$$\epsilon^{\mathbf{2}} \Delta u + u(u - \frac{1}{3})(1 - u) = 0 \quad \text{in } U$$

$$0 < u < 1 \quad \text{in } U$$

$$u = 0 \quad \text{on } \partial U$$

where $\epsilon > 0$ is sufficiently small.

Hint: 1. modify the nonlinearity to be zero for $u > 1$ and $u < 0$. 2. Show that the Mountain Pass Lemma is satisfied. To show that $J(e) < 0$. Choose a function $e = 1$ in most of the part of U except a thin part near the boundary. Then for ϵ sufficiently small, $J(e) < 0$. 3. Use Maximum Principle to show that $0 \leq u \leq 1$. In fact if the minimum is negative at some place p then $\Delta u = 0$ in a neighborhood of p. By Maximum Principle for harmonic function this is not possible. 4. Use Strong Maximum Principle to show that $0 < u < 1$.

Solution: Let $f(u) = u(u - 1/3)(1 - u)$. We first modify

$$\bar{f} = \begin{cases} f(u), 0 \leq u \leq 1 \\ 0, u \leq 0 \\ 0, u \geq 1 \end{cases}$$

and $\bar{F}(u) = \int_0^u \bar{f}(s)ds$. Certainly

$$\bar{F}(1) > 0$$

$$\bar{f}'(0) \leq 0$$

Let

$$J[u] = \frac{\epsilon^2}{2} \int_U |\nabla u|^2 - \int_U \bar{F}(U)$$

We now check that J and $\bar{F}$ satisfies the Mountain-Pass Lemma

(1) $J[0] = 0$

(2) $J[u] \geq c\|u\|^2$ for $\|u\| = r << 1$. In fact, for u small, $\bar{F}(u) \sim f^{'}(0)u^2/2 + O(u^3)$ and hence

$$J[u] \geq \frac{\epsilon^2}{2} \int_U |\nabla u|^2$$

(3) There exists $e \in H_0^1$ such that $J[e] < 0$. In fact we choose $e = 1$ when $d(x, \partial U) > \delta > 0$ and $0 \leq e \leq 1$ for $d(x, U) \leq \delta$. Then we compute

$$\begin{aligned} J[e] &= \frac{\epsilon^2}{2} \int_U |\nabla e|^2 - \int_U F(e) \\ &\leq C\epsilon^2 - \int_{d(x,U) \geq \delta} F(1) < 0 \end{aligned}$$

if ϵ is sufficiently small.

(4) $\bar{F}$ satisfies $|\bar{F}(u)| \leq C(1 + |u|^p)$ for $p < \frac{n+2}{n-2}$. This is trivial since $\bar{f}$ is bounded and hence $|\bar{F}(u)| \leq C(1 + |u|)$.

(5) $\bar{F}$ satisfies the (PS) condition: First of all, $J[u_n] \to c$ and $J^{'}[u_n] \to 0$. Then $\frac{\epsilon^2}{2} \int_U |\nabla u_n|^2 \leq J[u_n] + C \leq C$ and hence u_n is bounded in H_0^1. Since $\bar{f}$ is bounded, we have the convergence of $f(u_n)u_n$ and $F(u_n)$.

By Mountain-Pass-Lemma, we obtain the existence of a critical point of $J[u]$ and hence a weak solution to

$$\epsilon^2 \Delta u_0 + \bar{f}(u_0) = 0$$

We then claim that $0 \leq u_0 \leq 1$. In fact, if $\min_U u_0 = u_0(p) < 0$, then $\bar{f}(u_0(x)) = 0$ for $u_0(x) \leq 0$. By Maximum Principle, it is impossible. This implies that $u_0 \geq 0$. Similarly we can prove that $\max u_0 \leq 1$. Finally by Strong Maximum Principle, $0 < u_0 < 1$.