MATH 516-101 Homework One
Due Date: September 27th, 2016

1. This problem concerns the Mean-Value-Property (MVP). We say $v \in C^2(\bar{U})$ is *subharmonic*, if

$$-\Delta v \leq 0 \text{ in } U$$

a) Prove that for subharmonic functions

$$v(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} v dy, \quad \forall B(x,r) \subset U$$

Hint: use the formula for $\psi'(r)$.
b) Prove that the Maximum Principle holds for subharmonic functions on bounded domains

$$\max_{\bar{U}} v = \max_{\partial U} v$$

c) Let u be harmonic functions in U. Show that u^2 and $|\nabla u|^2$ are subharmonic functions.
d) Let u satisfy

$$-\Delta u = f \text{ in } U, \quad u = g \text{ on } \partial U$$

Show that there exists a generic constant $C = C(n, U)$ such that

$$\max_U u \leq C(\max_U |f| + \max_{\partial U} |g|)$$

Hint; Consider $v(x) = u(x) + \frac{|x|^2}{2n} \max_U |f| - \max_{\partial U} |g| - \frac{\max_U |x|^2}{2n} \max_U |f|$ and show that v is subharmonic and then apply b).

2. Let $u \in C(\Omega)$. Show that the followings are equivalent
(i) for all $x \in \Omega, B_r(x) \subset\subset \Omega$,

$$u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy$$

(ii) for all $B \subset\subset \Omega$ and for all $h : \bar{B} \to R$ which satisfies

$$-\Delta h = 0, \ x \in B, \quad h \geq u \text{ for } x \in \partial B$$

one has $u(x) \leq h(x)$ for all $x \in \bar{B}$. Here $B = B_r(y)$.
(iii) for all $x \in \Omega, B_r(x) \subset\subset \Omega$,

$$u(x) \leq \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(\sigma) d\sigma$$

(iv) for all $x \in \Omega$ and for all $\phi \in C^2$ such that $u - \phi$ has a local maximum at x, then $-\Delta \phi(x) \leq 0$.

3. The Kelvin transform of a function $u : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\bar{u}(x) = |x|^{2-n} u(\frac{x}{|x|^2})$$

Show that if u satisfies $\Delta u + f(u) = 0$ then $\bar{u}$ satisfies

$$\Delta \bar{u} + \frac{1}{|x|^{n+2}} f(|x|^{n-2} \bar{u}) = 0$$

As a consequence, show that the Yamabe equation $\Delta u + u^{\frac{n+2}{n-2}} = 0$ is invariant under Kelvin transform.

4. This problem concerns Green's function and Green's representation formula.
a) Write the Green's function for the unit ball $B_1(0)$.
b) Use reflection to find the Green's function in half space $\mathbb{R}^n_+ = \{x_n > 0\}$.