Solutions to HW1, MATH516-101, 2016-2017

1, a) (7pts) We integrate

$$\int_{B_r(x)} -\Delta v \le 0 \implies -\int_{\partial B_r} \frac{\partial v}{\partial r} \le 0$$

$$\implies \frac{\partial}{\partial r} \int_{|\theta|=1} v(x+r\theta)\,d\theta \ge 0$$

So the function $\psi(r) = \int_{|\theta|=1} v(x+r\theta)\,d\theta \nearrow$

$$\psi(r) \ge \psi(0) = v(x)\cdot|S^{n-1}|$$

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} v\,dy = \frac{1}{|B(x,r)|}\int_0^r \int_{|\theta|=1} v(x+s\theta)\,d\theta\, s^{n-1}ds$$

$$\ge \frac{1}{|B(x,r)|}\int_0^r v(x)\,|S^{n-1}|\, s^{n-1}ds$$

$$= v(x)$$

b) The proof is exactly the same as in the class.

Consider the set

$$U' = \{x \in U \mid v(x) = \max_{\bar{U}} v\}$$

Show that U' is relatively closed and relatively open.

c). $\Delta u^2 = 2u\Delta u + 2|\nabla u|^2 \ge 0.$

$$\Delta|\nabla u|^2 = u_{x_i x_j} u_{x_i x_j} \ge 0$$

d). Let $v = u(x) + \frac{|x|^2}{2n}\max_U |f| - \max_{\partial U}|g| - \frac{\max_U |x|^2}{2n}\max_U |f|.$

$$\Delta v = \Delta u + \max|f| = -f + \max|f| \ge 0$$

$\implies v$ is subharmonic.

Now we apply b).

2. (i) $\Rightarrow$ (ii). For $B_r(x) \subset B$, $u(x) - h(x) \leq \fint_{\partial B_r} u - \fint_{\partial B_r} h$

40pts

so $u-h$ is subharmonic and $u-h \leq 0$.

By 1b), $u \leq h$ in $\bar{B}$

(ii) $\Rightarrow$ (iii). By Poisson formula $\begin{cases} \Delta h = 0 & \text{in } B_r \\ h = u & \text{on } \partial B_r \end{cases}$

$$h(x) = \fint_{\partial B_r(x)} h(y)\, d\sigma = \fint_{\partial B_r(x)} u\, d\sigma$$

By (ii), $u(x) \leq h(x) = \fint_{\partial B_r} u\, d\sigma$.

(iii) $\Rightarrow$ (i). Trivial by integration

(iii) $\Rightarrow$ (iv). As in the class. If $-\Delta\phi(x) > 0 \Rightarrow -\Delta\phi(x) > 0$

in $B_s(x)$. $\Rightarrow$ $-\int_{B_\rho} \Delta\phi > 0$, $\rho < s \Rightarrow$

$$\psi = \int_{|\theta|} \phi(x+\rho\theta)\, d\theta \ \downarrow \ \Rightarrow$$

$$\fint_{\partial B_\rho} \phi\, d\sigma \leq \phi(x)$$

But $u(x) - \phi(x) \geq \fint_{\partial B_\rho} u - \fint_{\partial B_\rho} \phi \geq u(x) - \fint_{\partial B_\rho} \phi$

$> u(x) - \phi(x)$

contradiction

(iv) $\Rightarrow$ (iii): Suppose (iii) doesn't hold for some r and x.

$$u(x) > \frac{1}{|\partial B_r(x)|} \int_{\partial B_r} u\, d\sigma$$

we may choose c Such that

$$u(x) - \frac{1}{|\partial B_r(x)|}\int_{\partial B_r} u\,d\sigma \geq c r^2$$

Now let U be the unique sol'n to the Dirichlet problem

$$\begin{cases} -\Delta U = 0 & B_r(x) \\ U = u & \text{on } \partial B_r(x) \end{cases}$$

Then $U(x) = \fint_{\partial B_r(x)} u\,d\sigma$

Let $\phi(z) := U + c(r^2 - |z-x|^2)$. Then $u(z) = \phi(z)$ on $\partial B_r(x)$,

$\phi \in C^2$ and $u(x) - \phi(x) = u(x) - U(x) - cr^2 = u(x) - \fint_{\partial B_r(x)} u - cr^2 > 0$.

So

$$\max_{B_r(x)} (u(y) - \phi(y)) > 0$$

and $\exists\, z_0 \in B_r(x)$ s.t. $u(z_0) - \phi(z_0) = \max_{B_r(x)} u(z) - \phi(z)$

By (iv) this implies $-\Delta\phi(z_0) \leq 0$, but $\Delta\phi(z_0) = \Delta U(z_0) - c\Delta(|z_0 - x|^2)$

$$= 0 - 2cn < 0.$$

This is a contradiction.

3.(20pts). This is direct computation.

Anothe method: Use the radial formula

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}\Delta_\theta u$$

4.(10pts). (a) $G(x,y) = \Gamma(|x-y|) - \Gamma(|x|\,|y-x^*|)$, $x^* = \frac{x}{|x|^2}$

(b) $G(x,y) = \Gamma(|x-y|) - \Gamma(|x^* - y|)$, $x^* = (x_1, \cdots, x_{n-1}, -x_n)$