MATH 516-101 Homework TWO

Due Date: October 14, 2016, by 5:30pm

1.(a) using the Green's function in a ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B_r(0)$.

(b) use (a) to prove the following result: let u be a harmonic function in R^n. Suppose that $u \ge 0$. Then $u \equiv Constant$

2. This problem concerns the heat equation

$$u_t = \Delta u$$

Let

$$\Phi(x-y,t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

(a) Show that there exists a generic constant C_n such that

$$\Phi(x-y,t) \le C_n |x-y|^{-n}$$

Hint: maximize the function in t.

(b) Let $f(x)$ be a function such that $f(x_0-)$ and $f(x_0+)$ exists. Show that

$$\lim_{t\to 0}\int_R \Phi(x-x_0,t)f(y)dy = \frac{1}{2}(f(x_0-)+f(x_0+))$$

3. Consider the following general parabolic equation

$$L[u] = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u - u_t$$

where

$$0 < C_1 < a(x,t) < C_2, |b(x,t)| \le C_3, c(x,t) \le C_4$$

(a) Show that $L[u] \ge 0$, then

$$\max_{\Omega_T} u \le e^{C_4 T} \max_{\partial' \Omega_T} u^+$$

Here $\Omega_T = (0,L)\times(0,T), \partial\Omega_T = \partial\Omega_T \backslash ((0,L)\times\{T\})$ and $u^+ = \max(u,0)$.

Hint: consider the function $v := ue^{C_4 t}$

(b) Prove the uniqueness of the initial value problem

$$\begin{cases} Lu(x,t) = f(,t), \ \text{in}\Omega_T; \\ u(x,0) = \phi(x), x \in \Omega \\ u(x,t) = g(x,t), \ x \in \partial\Omega, t \in (0,T] \end{cases}$$

4. (a) Use d'Alembert's formula to to show that Maximum Principle does not hold for wave equation, i.e.,

$$u_{tt} = c^2 u_{xx}, 0 < x < L, 0 < t < T$$

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$
$$\max_{\bar{U}_T} u(x,t) > \max_{\partial' U_T} u(x,t)$$

Hint: Let $f = 0$ and $g = 1$, $U = (-1,1)$ and choose T large.

(b) Let u solve the initial value problem for the wave equation in one dimension

$$\begin{cases} u_{tt} = u_{xx} \text{ in } R \times (0, +\infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

where f and g have compact support in R. Let $k(t) = \frac{1}{2}\int u_t(x,t)^2 dt$ be the potential energy and $p(t) = \frac{1}{2}u_x^2(x,t)dx$ be t
potential energy. Show that

(a) $k(t) + p(t)$ is constant in t

(b) $k(t) = p(t)$ for all large enough time t.

5. This problem concerns Sobolev space

(a) Let $U = (-1,1)$ and

$$u(x) = |x|$$

What is its weak derivative $u^{'}$? Prove it rigorously. Does the second order weak derivative $u^{''}$ exist?

(c) Let $U = R^n$ and

$$u(x) = \frac{1}{|x|^a}$$

Find out its weak derivative. Prove it rigorously.