Solutions to HW#2, Math 516-101, 2016-2017

In this following, I only give sketches of the proof

1a) By Poisson's Formula

$$u(x) = \frac{r^2-|x|^2}{r|S^{n-1}|} \int_{\partial B_r(x)} \frac{u(y)}{|x-y|^n} d\sigma$$

where $(|y|-|x|)^n \leq |x-y|^n \leq (|x|+|y|)^n$

1b) Sending $r \to +\infty$ in 1a) $\Rightarrow$ $u(x) = u(0), \ \forall x$

2. a) Consider $\rho(t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$. Compute its maximum where $\rho'(t)=0$

b)

$$\int_{-\infty}^{+\infty} \Phi(x-x_0,t) f(x) dx - \frac{1}{2}(f(x_0^-)+f(x_0^+))$$

$$= \int_{x_0}^{+\infty} \Phi(x-x_0,t)(f(x)-f(x_0^+)) + \int_{-\infty}^{x_0} \Phi(x-x_0,t)(f(x)-f(x_0^-))$$

$$\leq \int_{x_0}^{x_0+\delta} \Phi(x-x_0,t)\left|(f(x)-f(x_0^+))\right| + \int_{x_0-\delta}^{x_0} \Phi(x-x_0,t)\left|f(x)-f(x_0^-)\right| d$$

$$+ \int_{|x-x_0|>\delta} |\Phi(x-x_0,t)| \, 2\max|f|$$

The first two terms $\to 0$ by continuity.

3 a) Case 1 $c \leq 0$. In this case consider

$$u_\varepsilon(x) = u(x) + \varepsilon e^{\gamma x}$$

Then $L[u_\varepsilon(x)] = L[u] + \varepsilon (a\gamma^2 + b\gamma + \tilde{c}\gamma) e^{\gamma x}$

> 0 if $a\gamma + b > 0$, $\varepsilon > 0$.

Claim $\max_{\Omega_T} u_\varepsilon = \max_{\partial\Omega_T} u_\varepsilon^+$

Suppose on the contrary $\max_{\Omega_T} u_\varepsilon$ has a ~~positive~~ nonnegative maximum at $(x_0, t_0) \in \operatorname{Int}(\Omega_T)$.

Then either $x_0 \in \Omega$, $0 < t < T$, in which case $u_{\varepsilon,xx} \leq 0$, $u_{\varepsilon,x} = 0$, $cu_\varepsilon \leq$

$u_{\varepsilon,t} = 0$. $L[u_\varepsilon] \leq 0$. A contradiction

or $x_0 \in \Omega$, $t = T$, in which case $u_{\varepsilon,xx} \leq 0$, $u_{\varepsilon,x} = 0$, $cu_\varepsilon \leq$

$u_{\varepsilon,t} \geq 0 \Rightarrow L[u_\varepsilon] \leq 0$. A contradiction agai

Finally sending $\varepsilon \to 0+$.

Case 2 $c \geq 0$. In this case consider

$$v = u e^{-c_4 t}$$

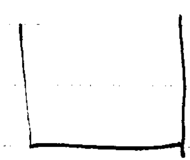
Then $L[v] = a v_{xx} + b v_x + (c - c_4) v - v_t$

where $c - c_4 \leq 0$.

By Case 1 $\max_{\Omega_T} v^+ = \max_{\partial\Omega_T} v^+$

Hence $\max_{\Omega_T} u = \max_{\Omega} e^{c_4 t} v \leq e^{c_4 T} \max v^+ \leq e^{c_4 T} \max_{\partial\Omega_T} u^+$.

4 a). Let $f = 0$, $g = 1$, $U = (-1, 1)$, $T = \frac{1}{2}$.



b). $\frac{dE}{dt} = \int (u_{tt} u_t + u_{xt} u_x) dx = \int (u_{xx} u_t + u_{xt} u_x) = \int (u_x u_t)_x d$

$= 0$

To show that $\int (u_t^2 - u_x^2)\,dx = 0$ for t large, we use d'Alembert's formula to show that

$$u_t^2 - u_x^2 = \left(-f'(x-t) + g(x-t)\right)\left(f'(x+t) + g(x+t)\right)$$

Thus for $|t| > 2M$, where $\operatorname{supp}(f) \cup \operatorname{supp}(g) \subset [-M, M]$

For each fixed x, $|(x+t) - (x-t)| = 2|t| > 4M$

So either $|x+t| > 2M$, in which case $f'(x+t) = 0, g(x+t) = c$

or $|x-t| > 2M$, in which case $f'(x-t) = g(x-t) = 0$

Hence $u_t^2 - u_x^2 \equiv 0$ for $|t| > 2M$.

5 a). $u' = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

u'' doesnot exist

by definiton

b). To show weak derivative, $\forall \phi \in C_0^\infty(U)$, it holds

$$\int_U \left(-\frac{a x_j}{|x|^{a+2}}\right)\phi = -\int_U \frac{1}{|x|^a}\partial_j \phi \qquad (*)$$

Now $\displaystyle\int_U \left(-\frac{a x_j}{|x|^{a+2}}\phi + \frac{1}{|x|^a}\partial_j \phi\right) = \lim_{\varepsilon \to 0} \int_{U \setminus B_\varepsilon(0)} \left(-\frac{a x_j}{|x|^{a+2}}\phi + \frac{1}{|x|^a}\partial_j \phi\right)$

$$= \int_{\partial(U \setminus B_\varepsilon(0))} \partial_j \left(\frac{1}{|x|^a}\phi\right)$$

$$= -\int_{\partial B_\varepsilon(0)} \nu_j \left(\frac{1}{|x|^a}\phi\right)$$

$$= \frac{1}{\varepsilon^a}(\max|\phi|) \cdot \varepsilon^{n-1} \to 0 \quad \text{if } a < n$$

Hence if $a < n-1$, $(*)$ holds.