MATH 516-101 (2016-2017) Homework THREE
Due Date: October 27, 2016

1. Consider the following function

$$u(x) = \frac{1}{|x|^{\gamma}}$$

in $\Omega = B_1(0)$. Show that if $\gamma + 1 < n$, the weak derivatives are given by

$$\partial_j u = -\gamma \frac{x_j}{|x|^{\gamma+2}}$$

i.e., you need to show that

$$\int u \partial_j \phi = \int \phi \gamma \frac{x_j}{|x|^{\gamma+2}}$$

As a consequence, show that $u \in W^{1,p}$ if and only if $(\gamma + 1)p < n$.

2. Let $\eta(t) = t$ for $t \leq 0$ and $\eta(t) = 0$ for $t > 1$. Let $f \in W^{k,p}(R^n)$ and $f_k = f\eta(|x| - k)$. Show that $\|f_k - f\|_{W^{k,p}} \to 0$ as $k \to +\infty$. As a consequence show that $W^{k,p}(R^n) = W_0^{k,p}(R^n)$.

3. Let $u \in C^{\infty}(\bar{R}^n_+)$. Extend u to Eu on R^n such that

$$Eu = u, x \in R^n_+; Eu \in C^4(R^n); \|Eu\|_{W^{4,p}} \leq \|u\|_{W^{4,p}}$$

Here $R^n_+ = \{(x', x_n); x_n > 0\}$.

4. (a) If $n = 1$ and $u \in W^{1,1}(\Omega)$ then $u \in L^{\infty}$ and u is continuous. (b) If $n > 1$, find an example of $u \in W^{1,n}(B_1)$ and $u \notin L^{\infty}$.

5. Prove the following Poincare type inequality: Suppose that $\Omega \subset \{a < x_1 < b\}$. Then for $u \in W_0^{1,2}(\Omega)$ it holds that

$$\|u\|_{L^2(\Omega)} \leq 2(b - a)\|\partial_{x_1} u\|_{L^2(\Omega)}$$

6. (Gagliardo-Nirenberg inequality) Let $n \geq 2, 1 < p < n$ and $1 \leq q < r < \frac{np}{n-p}$. For some $\theta \in (0, 1)$ and some constant $C > 0$ we have

$$\|u\|_{L^r(R^n)} \leq C\|u\|^{1-\theta}_{L^q(R^n)} \|\nabla u\|^{\theta}_{L^p(R^n)}, \forall u \in C_c^{\infty}(R^n)$$

(i) Use scaling to find the θ.
(ii) Prove the inequality.
Hint: Do an interpretation of L^r in terms of L^q and $L^{\frac{np}{n-p}}$ and then apply Sobolev.