MATH 516-101 Homework Four 2016-2017
Due Date: November 10, 2016

1. Fix $\alpha > 0, 1 < p < +\infty$ and let $U = B_1(0)$. Show that there exists a constant C, depending on n and α such that

$$\int_U u^p dx \leq C \int_U |\nabla u|^p$$

provided

$$u \in W^{1,p}(U), \ |\{x \in U | u(x) = 0\}| \geq \alpha$$

2. (a) Show that $W^{1,2}(R^N) \subset L^2(R^N)$ is not compact. (b). Let $n > 4$. Show that the embedding $W^{2,2}(U) \to L^{\frac{2n}{n-4}}(U)$ is not compact; (b) Describe the embedding of $W^{2,p}(U)$ in different dimensions. State if the embedding is continuous or compact.

3. Let $U = (-1, 1)$. Show that the dual space of $H^1(U)$ is isomorphic to $H^{-1}(U) + E^*$ where E^* is the two dimensional subspace of $H^1(U)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$.

4. (a). Assume that U is connected. A function $u \in W^{1,2}(U)$ is a weak solution of the Neumann problem

$$-\Delta u = f \text{ in } U; \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \tag{3}$$

if

$$\int_U Du \cdot Dv = \int_U fv, \quad \forall v \in W^{1,2}$$

Suppose that $f \in L^2$. Show that (3) has a weak solution if and only if

$$\int_U f = 0$$

(b). Discuss how to define a weak solution of the Poisson equation with Robin boundary conditions

$$-\Delta u = f \text{ in } U; \ u + \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \tag{4}$$

5. Let $u \in W^{1,2}(R^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + g(u) = f \text{ in } R^n$$

where $f \in L^2(R^n)$, and g is an odd smooth function of u. Prove that $u \in W^{2,2}(R^n)$.

Hint: mimic the proof of interior regularity but without the cut-off function.