Solutions to MATH 516-101, Assignment 4, 2016-2017

(Sketch)

1. Prove it by contradiction. $\exists$ a sequence

$$\int_\Omega u_k^2 = 1, \quad \int |\nabla u_k|^2 \leq \frac{1}{k}$$

$$|\{u_k(x) = 0\}| \geq \alpha$$

Let $u_k \to u_0$ weakly in H^1. Then $\int_\Omega u_0^2 = 1$, $\int_\Omega |\nabla u_0|^2 = 0$

Hence $u_0 \equiv$ Constant.

$$\int_\Omega |u_k - u_0|^2 = \int_{\Omega \cap \{u_k = 0\}} |u_k - u_0|^2 + \int_{\Omega \setminus \{u_k = 0\}} |u_k - u_0|^2$$

$$\geq c\,|\{u_k = 0\}| + \int_{\Omega \setminus \{u_k = 0\}} |u_k - u_0|^2$$

Since $\int_{\Omega \cap \{u_k = 0\}} |u_k - u_0|^2 \leq \int_\Omega |u_k - u_0|^2 \to 0$. We obtain a contradiction.

2. (a) Let η be a cut-off in $B_1(0)$. Let $u_k(x) = \eta(x - k e_1)$.

(b) Take η and rescale as

$$\eta_\lambda = \lambda^\beta \eta(\lambda x) \quad \text{such that} \quad \begin{aligned} \int |D^2 \eta_\lambda|^2 &= \int |D^2 \eta|^2 \\ \int \eta_\lambda^{\frac{2n}{n-4}} &= \int \eta_\lambda^{\frac{2n}{n-4}} \end{aligned}$$

(c) $2p < n \Rightarrow W^{2,p} \hookrightarrow L^q$, $q < \frac{2n}{n-4}$, compact; $q = \frac{2n}{n-4}$, continuous

$\frac{n}{2} < p < n \Rightarrow W^{2,p} \hookrightarrow C^{0,\alpha}$, $\alpha < 2 - \frac{n}{p}$, compact; $\alpha = 2 - \frac{n}{p}$, continuous

$p > n \Rightarrow W^{2,p} \hookrightarrow C^{1,\alpha}$ $\alpha < 1 - \frac{n}{p}$, compact; $\alpha = 1 - \frac{n}{p}$, continuous

3. We identify $H^1(U)$ with its continuous representative in $C([0,1])$

- By trace theorem,

$$H_0^1(U) = \{u \in H^1(U) \mid u(0)=0,\ u(1)=0\}$$

The orthogonal complement of $H_0^1(U)$ (in $H^1(U)$) is $u \in (H_0^1(U))^\perp \Leftrightarrow$

$$\langle u, v\rangle = 0,\ \forall v \in H_0^1(0,1) \Leftrightarrow \int_0^1 (uv + u'v')\,dx = 0,\ \forall v \in H_0^1(U)$$

so u is a weak solution of the ODE

$$-u'' + u = 0$$

By regularity, $u \in W^{2,2}$. Hence $u = c_1 e^x + c_2 e^{-x}$. So

$$H^1(0,1) = H_0^1(0,1) \oplus E$$

where E is the two dimensional subspace of H^1 spanned by $\{e^x, e^{-x}\}$.

Thus

$$(H^1(0,1))^* = (H_0^1(0,1))^* \oplus E^*$$

If $f \in (H^1(0,1))^*$. and $u = u_0 + c_1 e^x + c_2 e^{-x}$ where $u_0 \in H_0^1(0,1)$, then

$$(f, u) = (f_0, u_0) + a_1 c_1 + a_2 c_2$$

where $f_0 \in H^{-1}(U)$ is the restriction of f to $H_0^1(U)$ and

$$a_1 = (f, e^x),\quad a_2 = (f, e^{-x})$$

4.(a) The equality is simple. We show existence

Let $\Sigma = \{u \in H^1(U) \mid \int_\Omega u = 0\}$. Show that Σ is a closed subspace of H^1.

We define a bilinear form

$$B[u,v] = \int_\Omega Du\, Dv \qquad \forall u, v \in \Sigma$$

By Poincare, check $B[u,v]$ satisfies the two condition.

By Lax-Milgram, we prove the existence of weak sol'n.

(b) It is easy to see that. if a solution exists then (strong)

$$\int_U Du\cdot Dv + \int_{\partial U} u\cdot v = \int_U fv$$

By trace theorem, $\int_{\partial U} u\cdot v$ makes senses for $u, v \in H^1(U)$.

Now define bilinear

$$B[u, v] = \int_U Du\cdot Dv + \int_{\partial U} u\cdot v$$

and $\quad \Sigma = H_0^1(U)$.

check. $B[u,v]$ is bounded (by trace)

- $B[u,u] = \int_U |Du|^2 + \int_{\partial U} u^2$

Use contradiction argument to show the following new Poincare inequality

$$\int_U u^2 \le C\int_U |Du|^2 + C\int_{\partial U} u^2.$$

Then Lax-Milgram theorem gives the existance

3. $\quad v = D_k^{-h}(D_k^h u)$

$$\int g(u)\, D_k^{-h} D_k^h u = \int D_k^h(g(u))\, D_k^h u$$

$$= \int \frac{g(u(x+he_i)) - g(u(x))}{h} \cdot \frac{u(x+he_i) - u(x)}{h} \geq 0$$

by our assumption.