Solutions to MATH 516-101, 2016-2017, Homework Five

(sketch)

1. Check $\Delta u + c(x) u = 0$, $c(x) = \frac{1}{|x|^2 \log |x|}$, weakly

$c(x) \in L^{\frac{3}{2}}_{loc}$ but $u \notin L^{\infty}$

2. Follow the Moser iteration for the term

$$\int |c(x)| \eta^2 w^2$$

use Hölder

$$\int |c(x)| \eta^2 w^2 \le C \left(\int |c|^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int (\eta w)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}}$$

$$\le \varepsilon_0^{\frac{2}{n}} \| D(\eta w) \|^2_{L^2}$$

which is absorbed by the left hand side term

3. Let $f(x) = |u|^{p-1} u$

$u \in H_0^1(\Omega) \Longrightarrow u \in L^{\frac{2n}{n-2}} \Rightarrow u^p \in L^{\frac{2n}{p(n-2)}}$

If $\frac{2n}{p(n-2)} \geq \frac{n}{2}$, then we're done

If not, $p > \frac{4}{n-2}$, then

$u \in W^{2, q_1}$, $q_1 = \frac{2n}{p(n-2)}$, $\hookrightarrow L^{\frac{q_2}{p}}$

$$\frac{q_2}{p} = \frac{2n}{n - 2q_1}.$$

Define $\frac{q_k}{p} = \frac{2n}{n-2q_{k-1}}$, $q_0 = $

check $\frac{q_k}{q_{k-1}} > 1$

After finite number of steps $\Rightarrow u^p \in L^{\frac{n}{2}}$.

4. Direct Computation.

5. Let $w = A(1+|x|^2)^{-\frac{l-2}{2}} + \varepsilon$. Since $u \to 0$ as $|x| \to +\infty$,

$\exists R_\varepsilon$, s.t. $|u| < \varepsilon$, $\forall\ |x| > R_\varepsilon$

check: for $|x| > R_0$ (a fixed number)

$$\Delta(1+|x|^2)^{-\frac{l-2}{2}} \approx \Delta |x|^{-(l-2)} \approx -|x|^{-l}$$

Let $A = 2R_0^{l-2}$. Consider $\Omega = B_M \setminus B_{R_0}$, $M > R_\varepsilon$

$$\Delta(u-w) \leq 0 \quad \text{in } \Omega$$

$$\Rightarrow \min_{\Omega}(u-w) \leq \min_{\partial B_M}(u-w) + \min_{\partial B_{R_0}}(u-w)$$
$$< 0 + 0$$

Let $\varepsilon \to 0$.

6. Choose $w = Ae^{-|x|} + \varepsilon e^{|x|}$. Choose R_ε s.t.

$|u(x)| < \frac{\varepsilon}{2} e^{|x|}$, $|x| > R_\varepsilon$

check $|x| > R_0$, $\Delta w - V(x) w < 0$. Choose $A = e^{R_0}$.

Consider $\Omega = B_{R_0} \setminus B_M$, $\forall M > R_\varepsilon$. Apply M. P.

to $u - w$ in Ω. Let $\varepsilon \to 0$.