MATH 516-101, 2016-2017 Homework Six
Due Date: December 9, 2016
This homework accounts for extra points.

1. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$a_{ij}u_{ij} + b_i u_i + c(x)u = f \text{ in } \Omega$$

$$u + \alpha(x)\frac{\partial u}{\partial \nu} = \varphi \text{ on } \partial\Omega$$

where $a_{ij}, b_i, c \in L^\infty$ and $f \in C(\bar{\Omega}), \varphi \in C(\partial\Omega)$. Show that if $c(x) \leq 0, 0 \leq \alpha \leq \alpha_0$, then

$$|u(x)| \leq \max_{\partial\Omega} \varphi(x) + C \max_{\Omega} |f|$$

2. (a) Let $\Omega = B_R(0)$ and $f = f(|x|, u)$ with $\frac{\partial f}{\partial r} < 0$. Show that the solution to

$$\Delta u + f(|x|, u) = 0, u > 0 \text{ in } B_R, \ u = 0 \text{ on } \partial B_R$$

is radially symmetric.

(b) Let Ω be a convex domain and $f(x, u) = K(x)g(u) \geq 0$. Assume that $\frac{\partial K}{\partial \nu} < 0$ for $x \in \partial\Omega$. Show that there exists a fixed neighborhood of $\partial\Omega$ (independent of g) such that $|\nabla u| \neq 0$.

(c) Let Ω be any smooth and bounded domain in R^2 and $f(u) \geq 0$. Suppose that u satisfies

$$\Delta u + f(u) = 0, u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega$$

Show that there exists a fixed neighborhood of $\partial\Omega$ (independent of f) such that $|\nabla u| \neq 0$.

If $n \geq 3$, what can you say?

Hint: consider the Kelvin transform of Laplace equation: The Kelvin transform Ku is defined as $Ku = |x|^{2-n}u(\frac{x}{|x|^2})$. Show that if u satisfies

$$\Delta u + f(u) = 0$$

then Ku satisfies

$$\Delta Ku + \frac{1}{|x|^{n+2}} f(|x|^{n-2}Ku) = 0$$

3. Use sub-super solution method to prove the existence of a positive solution to

$$-\Delta u = (1 + |x|^2)^{-l} u^p, p > 1$$

where $l > 1, n \geq 3$, with the following asymptotic behavior

$$\lim_{|x| \to +\infty} u(x) = c > 0 \tag{1}$$

Hint: consider the following functions

$$c + C \int_{R^n} \frac{1}{|x-y|^{n-2}} \frac{1}{(1+|y|^2)^l} dy$$

4. Use Nehari's manifold method (or constraint method) to prove the existence of a solution for the following equation

$$\Delta u + f(u) = 0, u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega$$

where $f(u) = -u + u^p - au^q$, with $1 < q < p < \frac{n+2}{n-2}, a \geq 0$.

5. Use Mountain-Pass Lemma to prove the existence of a positive solution to

$$\Delta u + f(u) = 0, u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega$$

where $f(u) = -u + u^p - au^q$, with $1 < q < p < \frac{n+2}{n-2}, a \geq 0$.

Show that the critical values obtained in Problem 4 are the same as in Problem 5.