Method of Separation of Variables

$$\begin{cases} u_t = \alpha^2 u_{xx}, \quad 0<x<L \\ u(x,0) = f(x) \\ BC_s \end{cases}$$

There will be 4 boundary conditions. $\begin{cases} u(0,t)=u(L,t)=0 \\ u_x(0,t)=u_x(L,t)=0 \\ u_x(0,t)=0=u(L,t) \\ u(0,t)=0=u_x(L,t) \end{cases}$

(1) BC 1.

$$u(0,t) = u(L,t) = 0$$

In this case the EVP becomes

$$\begin{cases} X''+\lambda X=0, \quad 0<x<L \\ X(0)=X(L)=0 \end{cases}$$

The ODE becomes

$$T' + \alpha^2 \lambda T = 0$$

Let us solve EVP:

Case 1. $\lambda < 0$

$$\lambda = -\gamma^2, \qquad X = c_1 e^{-\gamma x} + c_2 e^{\gamma x}$$

$$\left.\begin{aligned} X(0)=0 &\Rightarrow c_1 + c_2 = 0 \\ X(L)=0 &\Rightarrow c_1 e^{-\gamma L} + c_2 e^{\gamma L} = 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = 0$$

Case 2. $\lambda = 0$

$$X = c_1 + c_2 x$$

$$\left.\begin{array}{l} X(0)=0 \Rightarrow c_1 = 0 \\ X(L)=0 \Rightarrow c_1 + c_2 L = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$$

Case 3 $\lambda > 0$.

$\lambda = \beta^2$, $\quad X = c_1 \cos\beta x + c_2 \sin \beta x$

$X(0)=0 \Rightarrow \quad c_1 = 0$

$X(L)=0 \Rightarrow \quad c_2 \sin\beta L = 0$

So $\sin\beta L = 0$, $\Rightarrow \beta L = n\pi$, $n=1,2,\cdots$

so $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n=1,\cdots$; $\quad X_n = \sin\left(\frac{n\pi}{L}x\right)$

For $\lambda = \lambda_n$, $\quad T_n = c\,e^{-\alpha^2\lambda_n t} = c\,e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$

Final solution:

$$u(x,t) = \sum_{n=1}^{+\infty} b_n\, e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

(2) BC2

$$u_x(0,t)=u_x(L,t)=0$$

In this case, EVP becomes

$$\begin{cases} X''+\lambda X=0, & 0<x<L \\ X'(0)=X'(L)=0 \end{cases}$$

Case 1. $\lambda<0$

$\lambda=-\gamma^2$, $X=c_1 e^{-\gamma x}+c_2 e^{\gamma x}$

$$\left.\begin{aligned} X'(0)=0 &\Rightarrow \gamma(-c_1+c_2)=0 \Rightarrow c_1=c_2 \\ X'(L)=0 &\Rightarrow \gamma(-c_1 e^{-\gamma L}+c_2 e^{\gamma L})=0 \Rightarrow c_1 e^{-\gamma L}=c_2 e^{\gamma L} \end{aligned}\right\} \Rightarrow$$

$$c_1=c_2=0$$

Case 2. $\lambda=0$

$X=c_1+c_2 x$

$X'(0)=0 \Rightarrow c_2=0$

$X'(L)=0 \Rightarrow c_2=0$

So $X=c_1$, $\lambda_0=0$, $X_0=1$ (up to a constant)

Case 3. $\lambda>0$

$\lambda=\beta^2$, $X=c_1\cos\beta x+c_2\sin\beta x$

$X'(0)=0 \Rightarrow c_2=0$

$X'(L)=0 \Rightarrow c_1\beta \sin\beta L = 0$

$$\beta L = n\pi \quad \Rightarrow \beta = \frac{n\pi}{L} \quad \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

For $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\quad X_n = \cos\frac{n\pi}{L}x$.

$$T_n = c\, e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

Final solution

$$u(x,t) = b_0 + \sum_{n=1}^{+\infty} b_n\, e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$$

$$b_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad n=1,2,\cdots$$

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(3) BC3

$u_x(0,t) = u(L,t) = 0$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X(L) = 0 \end{cases}$$

Case 1. $\lambda < 0$

$\lambda = -\gamma^2$, $\quad X = c_1 e^{-\gamma x} + c_2 e^{\gamma x}$

$$\left.\begin{array}{l} X'(0)=0 \Rightarrow \gamma(-c_1+c_2)=0 \\ X(L)=0 \Rightarrow c_1 e^{-\gamma L}+c_2 e^{-\gamma L}=0 \end{array}\right\} \Rightarrow c_1=c_2=0$$

Case2. $\lambda=0$

$$X=c_1+c_2 x$$

$$X'(0)=0 \Rightarrow c_2=0$$

$$X(L)=0 \Rightarrow c_1=0$$

Case3 $\lambda>0$

$$\lambda=\beta^2, \quad X=c_1\cos\beta x+c_2\sin\beta x$$

$$X'(0)=0 \Rightarrow c_2=0$$

$$X(L)=0 \Rightarrow c_1\cos\beta L=0 \Rightarrow \beta L=\frac{(2n-1)\pi}{2}, \quad n=1,2,\cdots$$

$$\beta=\frac{(2n-1)\pi}{2L}$$

$$\lambda_n=\left(\frac{(2n-1)\pi}{2L}\right)^2, \qquad X_n=\cos\frac{(2n-1)\pi}{2L}x$$

$$T_n=c\,e^{-\alpha^2\left(\frac{(2n-1)\pi}{2L}\right)^2 t}$$

$$u(x,t)=\sum_{n=1}^{+\infty} b_n\, e^{-\alpha^2\left(\frac{(2n-1)}{2L}\pi\right)^2 t}\cos\left(\frac{(2n-1)\pi}{2L}x\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2n-1)\pi}{2L}x\right)$$

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(4) BC4

$$u(0,t) = 0 = u_x(L,t)$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0\ , \ X'(L) = 0 \end{cases}$$

Case 1. $\lambda < 0$

$\lambda = -\gamma^2$, $\quad X = c_1 e^{-\gamma x} + c_2 e^{\gamma x}$

$$\left.\begin{array}{l} X(0) = 0 \Rightarrow c_1 + c_2 = 0 \\ X'(L) = 0 \Rightarrow \gamma(-c_1 e^{-\gamma L} + c_2 e^{\gamma L}) = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$$

Case 2 $\lambda = 0$

$$X = c_1 + c_2 x$$

$X(0) = 0 \Rightarrow c_1 = 0$

$X'(L) = 0 \Rightarrow c_2 = 0$

Case 3. $\lambda > 0$

$$\lambda = \beta^2, \quad X = c_1 \cos \beta x + c_2 \sin \beta x$$

$$X(0) = 0 \Rightarrow c_1 = 0$$

$$X'(L) = 0 \Rightarrow c_2 \cos \beta L = 0 \Rightarrow \beta L = \frac{(2n-1)\pi}{2}$$

$$\lambda_n = \beta^2 = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad X_n = \sin \frac{(2n-1)\pi}{2L}$$

$$T_n = c e^{-\alpha^2 \left(\frac{(2n-1)}{2L}\pi\right)^2 t}$$

$$u(x,t) = \sum_{n=1}^{+\infty} b_n e^{-\alpha^2 \left(\frac{(2n-1)}{2L}\pi\right)^2 t} \sin\left(\frac{(2n-1)\pi}{2L} x\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{(2n-1)\pi}{2L} x\right).$$