MATH 305: MIDTERM 1: October 15 2010 (M. WARD)

Closed Book and Notes. 50 minutes. Total 50 points

PROBLEM 1: (10 Points) Find all solutions in the complex plane to the following:

$$\text{(i)}\quad (-1+i)=e^{z}\,; \qquad \text{(ii)}\quad (z+1)^6 = z^6\,; \qquad \text{(iii)}\quad \cos(z)=\sin(z)\,.$$

PROBLEM 2: (5 Points) Find all solutions z in the disk $|z|\leq 2$ to $\sin(z^5/10)=0$.

PROBLEM 3: (10 Points) If $u(x,y) = x^3 - 3xy^2$, find an entire function $f(z)$ with $z = x+iy$ such that $\text{Re}(f) = u$ and $f(1+i) = -2+i$. Express your result for f in terms of z.

PROBLEM 4: (10 Points)

Let $f(z) = (x-y)^2 + 2i(x+y)$, where $z = x+iy$. Where is $f(z)$ differentiable in the complex plane? Where is $f(z)$ analytic? Explain your reasoning carefully.

PROBLEM 5: (15 Points) Find the image of the set S under the map $w = f(z)$ for each of the following:

$$\text{(i)}\ S = \left\{ z \;\middle|\; 1/2 \leq |z| \leq 1 \ \text{with}\ 0 \leq \text{Arg}(z) < \pi/4 \right\} \quad \text{and} \quad f(z) = i\text{Log}(2z^2)\,,$$

$$\text{(ii)}\ S = \left\{ z \;\middle|\; |z-1| \geq 1 \ \text{with}\ \text{Re}(z) \geq 0 \right\} \quad \text{and} \quad f(z) = \frac{2-z}{2+z}\,.$$

SOLUTIONS TO MIDTERM I

PROBLEM 1

(i) $(-1+i)=e^{z}$. WE HAVE $z=\log(-1+i)=\ln(\sqrt{2})+i\left(3\pi/4+2K\pi\right)$, $K=0,\pm1,\pm2,\ldots$

NOW $\ln(\sqrt{2})=\frac{1}{2}\ln 2$, SO THAT $\quad z=\frac{1}{2}\ln 2+i\left(3\pi/4+2K\pi\right)$, $K=0,\pm1,\pm2,\ldots$

(ii) $(z+1)^6-z^6=0$ IS A POLYNOMIAL OF DEGREE 5 (NOT 6). IT WILL HAVE 5 ROOTS AND THEY WILL COME IN COMPLEX CONJUGATE PAIRS. SINCE $z\neq0$ WE WRITE $w^6=1$ WITH $w=(z+1)/z$.

NOW $w=e^{i\varphi}\rightarrow e^{6i\varphi}=e^{2\pi iK}\rightarrow 6\varphi=2\pi K\rightarrow\varphi=\pi K/3$, $K=0,1,2,3,4,5$.

THUS $\quad w_K=e^{\pi iK/3}$ FOR $K=1,2,3,4,5$ SINCE FOR $K=0$, $w_0=1$ WHICH IMPLIES $1=(z+1)/z\rightarrow z=z+1\rightarrow 0=1$ IMPOSSIBLE.

NOW $\quad w=(z+1)/z\rightarrow z+1=zw\rightarrow z_K=\dfrac{1}{w_K-1}$, $w_K=e^{\pi iK/3}$, $K=1,\ldots,5$.

(iii) $\sin z=\cos z\rightarrow\dfrac{e^{iz}-e^{-iz}}{i}=\dfrac{e^{iz}+e^{-iz}}{1}\rightarrow e^{iz}(1+i)=e^{-iz}(i-1)$.

THUS $\quad e^{2iz}=\dfrac{i-1}{i+1}=\dfrac{e^{3\pi i/4}}{e^{\pi i/4}}\dfrac{\sqrt{2}}{\sqrt{2}}=e^{\pi i/2}\rightarrow 2iz=\dfrac{\pi i}{2}+2K\pi i$, $K=0,\pm1,\pm2$

THUS $\quad z=\pi/4+K\pi$, $K=0,\pm1,\pm2,\pm3,\ldots$

PROBLEM 2

NOW $\sin\left(z^5/10\right)=0$ GIVES $z^5/10=n\pi$ $\quad n=0,\pm1,\pm2,\ldots$

NOW $\quad z^5=10n\pi\longrightarrow |z|=(10\pi)^{1/5}|n|^{1/5}\leq 2$ ONLY IF $n=0,1,-1$.

- FOR $n=0\longrightarrow z=0$
- FOR $n=1\longrightarrow z^5=10\pi\longrightarrow z_K=(10\pi)^{1/5}e^{2\pi iK/5}$ $\quad K=0,1,2,3,4$
- FOR $n=-1\longrightarrow z^5=-10\pi\longrightarrow z^5=10\pi e^{i\pi}\longrightarrow z_m=(10\pi)^{1/5}e^{i\pi/5}e^{2\pi im/5}$ $\quad m=0,1,2,3,4$.

THUS THERE ARE 11 ROOTS IN $|z|\leq2$ GIVEN BY

$$z=0,\quad z_K=(10\pi)^{1/5}e^{2\pi iK/5}\quad K=0,\ldots,4$$

$$z_m=(10\pi)^{1/5}e^{\pi i/5}e^{2\pi im/5},\quad m=0,\ldots,4.$$

PROBLEM 3 LET $u = x^3 - 3xy^2$.

FIND v SO THAT CR ARE SATISFIED

$$u_x = v_y \longrightarrow v_y = 3x^2 - 3y^2 \longrightarrow v = 3x^2 y - y^3 + h(x).$$

$$u_y = -v_x \longrightarrow -6xy = -[6xy + h'(x)] \rightarrow h'(x) = 0 \text{ so } h = c.$$

THUS $f = u + iv$

$$f = x^3 - 3xy^2 + i[3x^2 y - y^3 + c].$$

NOW $f(1+i) = (1) - 3 + i[3 - 1 + c] = -2 + i(2 + c)$.

SO CHOOSE $c = -1$. $\longrightarrow$ $f = x^3 - 3xy^2 + i(3x^2y - y^3 - 1)$.

THUS IS $f = z^3 - i$.

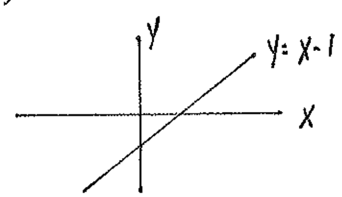
PROBLEM 4 LET $f = (x-y)^2 + 2i(x+y)$

NOW $u = (x-y)^2$ $v = 2x + 2y$.

where do CR hold? $u_x = v_y \longrightarrow 2(x-y) = 2 \longrightarrow x - y = 1$

$u_y = -v_x \longrightarrow -2(x-y) = -2 \longrightarrow x - y = 1$.

NOW SINCE CR HOLD ON THE LINE $y = x - 1$ AND u, v ARE SMOOTH FUNCTIONS, THEN $f(z)$ IS COMPLEX DIFFERENTIABLE ON $y = x - 1$.



HOWEVER, $f(z)$ IS NOT ANALYTIC ANYWHERE SINCE WE DO NOT HAVE DIFFERENTIABILITY IN ANY BALL $|z - z_0| < \rho$ WITH $\rho > 0$ AND z_0 A POINT ON THE LINE $y = x - 1$.

SOLUTION 5

(ii) LET $S = \{ z \mid |z-1| \geq 1 \text{ AND } \mathrm{RE}(z) \geq 0 \}$

AND $w = \dfrac{2-z}{2+z}$. WE SOLVE FOR z: $z(w+1) = 2-2w$.

SO $z = \dfrac{2(1-w)}{w+1}$.

NOW $\mathrm{RE}(z) \geq 0 \rightarrow 2\,\mathrm{RE}\left(\dfrac{(1-w)(\bar{w}+1)}{(w+1)(\bar{w}+1)} \right) = \dfrac{2}{|w+1|^2} \mathrm{RE}\left(1 - w\bar{w} - (w - \bar{w})\right) \geq 0.$

NOW $w - \bar{w} = 2i\,\mathrm{IM}(w)$ SO $\mathrm{RE}(w - \bar{w}) = 0$. THIS YIELDS

$$\frac{2}{|w+1|^2} \mathrm{RE}\left(1 - |w|^2 - 2i\,\mathrm{IM}(w)\right) \geq 0 \rightarrow \frac{2}{|w+1|^2}\left(1 - |w|^2\right) \geq 0 \rightarrow |w| \leq 1.$$

THUS $\mathrm{RE}(z) \geq 0 \longrightarrow |w| \leq 1$.

NOW $|z-1| \geq 1 \longrightarrow \left| \dfrac{2(1-w)}{1+w} - 1 \right| \geq 1 \longrightarrow |2 - 2w - 1 - w| \geq |w+1|$

THIS IS $|1-3w| \geq |w+1| \longrightarrow |3w-1|^2 \geq |w+1|^2$.

NOW LET $w = u + iv \rightarrow \left[(3u-1)^2 + (3v)^2 \right] \geq (u+1)^2 + v^2$.

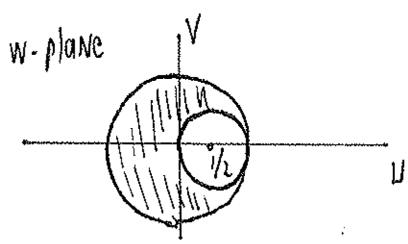
THUS $9u^2 - 6u + 1 + 9v^2 \geq u^2 + 2u + 1 + v^2$

$\longrightarrow 8(u^2+v^2) - 8u \geq 0 \rightarrow u^2 + v^2 - u \geq 0.$

THIS IS $(u^2 - u + 1/4) + v^2 \geq 1/4 \rightarrow (u - 1/2)^2 + v^2 \geq (1/2)^2$.

THUS $|z-1| \geq 1$ MAPS TO $|w - 1/2| \geq 1/2$.

THE IMAGE REGION IS $S' = \{ w \mid |w| \leq 1 \text{ AND } |w - 1/2| \geq 1/2 \}$

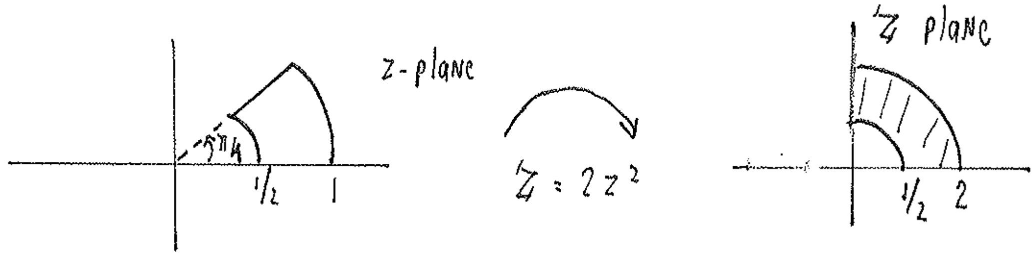


(i) FIND IMAGE OF

$$S = \{ z \mid 1/2 \le |z| \le 1,\ 0 \le \mathrm{ARG}\, z \le \pi/4 \}$$

UNDER $w = i\,\mathrm{LOG}(2z^2)$.

WE DEFINE $Z = 2z^2$, $W = \mathrm{LOG}(Z)$, $w = iW$.

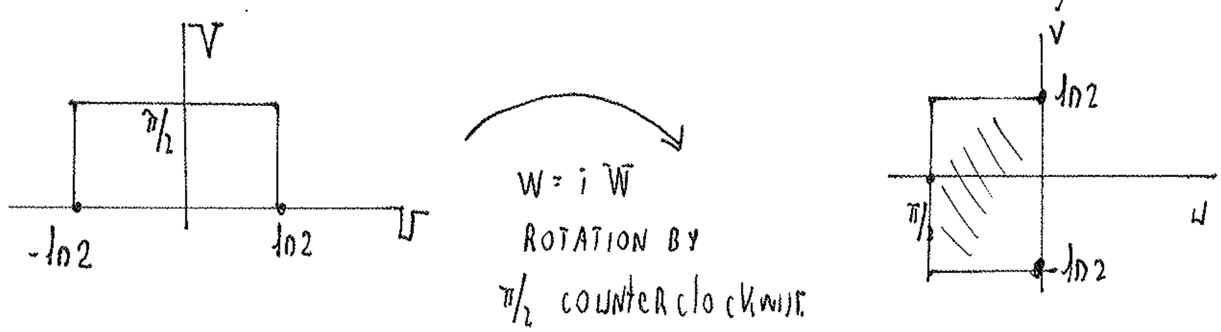


NOW CONSIDER MAP $W = \mathrm{LOG}(Z)$. WE WRITE $Z = re^{i\varphi}$

WITH $\frac{1}{2} \le r \le 2$, $0 \le \varphi \le \pi/2$ $\rightarrow$ $W = \ln r + i\varphi = U + iV$.

NOW IF r IS FIXED IN $1/2 \le r \le 2$ $\longrightarrow$ $U = \ln r$ V IN $(0, \pi/2)$

THUS IN W PLANE WE HAVE A RECTANGLE (NOTE $\ln(1/2) = -\ln 2$)



THE IMAGE REGION IS

$$S' = \{ w \mid -\pi/2 \le \mathrm{RE}(w) \le 0,\ |\mathrm{IM}(w)| \le \ln 2 \}.$$