MATH 305: MIDTERM 1: October 19th, 2012 (M. WARD)

Closed Book and Notes. 50 minutes. Total 50 points

PROBLEM 1: (12 Points) Find all solutions in the complex plane to the following:

$$\text{(i)} \quad 2\cos z = i\sin z\,; \qquad\qquad \text{(ii)} \quad (z+1)^5 = z^5\,.$$

For each of these, express your solution in the form $z = a + ib$ where a and b are real.

PROBLEM 2: (18 Points) Establish the validity of each of the following statements. If the statement is true, then provide a short proof. If it is false, carefully explain why.

i) Let $f(z) = x^2 + y^2 + 2ixy$ where $z = x + iy$. Then, $f(z)$ is differentiable along the line $y = 0$ but is nowhere analytic.

ii) Let I_1 and I_2 be real and satisfy the complex-valued equation $e^{-\pi i/6} I_1 + e^{\pi i/3} I_2 = 1 + i$. Then,

$$I_1 = \sqrt{2}\left(\frac{\sin(\pi/12)}{\sin(2\pi/3)}\right).$$

(iii) $\text{Log}\,(z_1/z_2) = \text{Log}(z_1) - \text{Log}(z_2)$ for any $z_1 \neq 0$ and $z_2 \neq 0$.

(iv) $|1/(z+2)| \leq 1/2$ when $|z| \geq 4$.

(v) $|e^{-z^3}| \leq 1$ for all z in $\text{Re}(z) \geq 0$.

vi) $\Phi(x,y) = \frac{(y-1)}{x^2+(y-1)^2}$ is a harmonic function when $(x,y) \neq (0,1)$.

PROBLEM 3: (8 Points)

Determine where $f(z) = \text{Log}(1 - z^3)$ is analytic in the complex plane. Here $\text{Log}(\xi)$ denotes the principal branch of $\log \xi$.

PROBLEM 4: (12 Points) Find the image of the set S under the map $w = f(z)$ for each of the following:

i) $S = \{z \mid \text{Im}(z) + \text{Re}(z) \geq 1/2\}$ and $f(z) = 1/z$

ii) $S = \{z \mid 0 \leq \text{Re}(z) \leq 1 \text{ and } 0 \leq \text{Im}(z) \leq \pi/2\}$ and $f(z) = i\,(e^z)^3 + 2i$.

MATH 305 MIDTERM I

SOLUTION 1

(i) SOLVE $2\cos z = i\sin z$.

SO $2\left(\frac{e^{iz}+e^{-iz}}{2}\right) = i\frac{\left(e^{iz}-e^{-iz}\right)}{2i} \rightarrow 2\left(e^{iz}+e^{-iz}\right) = e^{iz}-e^{-iz}$.

SO $e^{iz} = -3e^{-iz} \rightarrow e^{2iz} = -3 \rightarrow 2iz = \log(-3) = \ln 3 + i(\pi + 2k\pi)$

SO $z = \frac{-i}{2}\ln 3 + \frac{1}{2}(\pi + 2k\pi)$, $k = 0, \pm 1, \pm 2, \ldots$

(ii) $(z+1)^5 = z^5$. THIS IS A POLYNOMIAL OF degree 4, SO WE HAVE 4 ROOTS.

LET $w = \frac{z+1}{z}$ SO $w^5 = 1 \rightarrow w = e^{2\pi i k/5}$ $k = 0, 1, 2, 3, 4$.

NOW $zw = z+1$ SO $z(w-1) = 1$ SO $z = \frac{1}{w-1}$.

WE NEED $w \neq 1$ SO ELIMINATE $k = 0$ AND TAKE $w_k = e^{2\pi i k/5}$, $1 \le k \le 4$.

THEN $z_k = \frac{1}{w_k - 1} \cdot \frac{\overline{(w_k-1)}}{\overline{(w_k-1)}} = \frac{\bar{w}_k - 1}{w_k\bar{w}_k + 1 - (w_k + \bar{w}_k)}$ $\quad w_k + \bar{w}_k = 2\,\mathrm{Re}(w_k)$

BUT $w_k\bar{w}_k = 1$ SO $z_k = \frac{\bar{w}_k - 1}{2 - 2\,\mathrm{Re}(w_k)} = \frac{-(1-\bar{w}_k)}{2(1-\mathrm{Re}(w_k))}$

IF $w_k = e^{2\pi i k/5}$ THEN $\bar{w}_k = e^{-2\pi i k/5}$ $\quad \mathrm{Re}(w_k) = \cos(2\pi k/5)$

SO $z_k = \frac{-1}{(2 - 2\cos(2\pi k/5))}\left[1 - \left(\cos\left(\frac{2\pi k}{5}\right) - i\sin\left(\frac{2\pi k}{5}\right)\right)\right]$

SO $z_k = \frac{-(1-\cos(2\pi k/5))}{(2 - 2\cos(2\pi k/5))} - \frac{i\sin(2\pi k/5)}{2(1-\cos(2\pi k/5))}$ $\quad k = 1, \ldots, 4$.

SO $z_k = -\frac{1}{2} - \frac{i\sin(2\pi k/5)}{2(1-\cos(2\pi k/5))} = \frac{-1}{2} - \frac{2i\sin(\pi k/5)\cos(\pi k/5)}{2\cdot 2\sin^2(\pi k/5)} = \frac{-1}{2} - \frac{i}{2}\cot(\pi k/5)$.

SOLUTION 2

(i) $f = x^2 + y^2 + 2ixy$. so $u = x^2 + y^2$, $v = 2xy$ (TRUE)

$u_x = 2x$ $\quad v_y = 2x$ $\quad$ so $\quad u_x = v_y$ ALWAYS

$u_y = 2y$ $\quad v_x = 2y$ $\quad$ so $\quad u_y = -v_x$ ONLY WHEN $y = 0$.

THUS CR HOLD ONLY ALONG $y=0$. $\longrightarrow$ f is differentiable on $y=0$.
But f is nowhere analytic since no neighborhood of any point along $y=0$ where f is differentiable

(ii) $$e^{-\pi i/6} I_1 + e^{\pi i/3} I_2 = 1 + i = \sqrt{2}\, e^{\pi i/4}.$$

TO ISOLATE I_1 MULTIPLY BY $e^{-\pi i/3}$ AND TAKE $\mathrm{IM}(\ )$ OF BOTH SIDES.

$$\mathrm{IM}\left(e^{-\pi i/3} e^{-\pi i/6} I_1 + I_2 \right) = \mathrm{IM}\left(\sqrt{2}\, e^{-\pi i/3} e^{\pi i/4} \right)$$

$$\text{so } \mathrm{IM}\left(e^{-\pi i/2} I_1 \right) = \sqrt{2}\ \mathrm{IM}\left(e^{-\pi i/12} \right) \implies \sin(-\pi/2) I_1 = \sqrt{2} \sin(-\pi/12)$$

$$\text{so } I_1 = \sqrt{2} \sin(\pi/12) \qquad \text{(FALSE)}$$

(iii) FALSE $\quad$ LET $z_1 = e^{-3\pi i/4}$, $z_2 = e^{3\pi i/4}$

THEN $\mathrm{LOG}(z_1) = -3\pi i/4$ $\quad \mathrm{LOG}(z_2) = 3\pi i/4 \rightarrow \mathrm{LOG}\, z_1 - \mathrm{LOG}\, z_2 = -3\pi i/2$.

AND $\mathrm{LOG}(z_1/z_2) = \mathrm{LOG}\left(e^{-3\pi i/2} \right) = \pi i/2 \neq \mathrm{LOG}\, z_1 - \mathrm{LOG}\, z_2$

(iv) TRUE $\quad$ NEED TO GET A BOUND $|z+2| \geq \ldots$ SO THAT $\dfrac{1}{|z+2|} \leq \ldots$.

THUS, NEED REVERSE Δ-INEQUALITY:

$$|z_1 + z_2| \geq \big| |z_1| - |z_2| \big|$$

so $|z+2| \geq \big| |z| - 2 \big| \geq |z| - 2$ WHEN $|z| > 2$.

But $|z| \geq 4$, so $|z+2| \geq |z| - 2 \geq 4 - 2 = 2$.

HENCE $\dfrac{1}{|z+2|} \leq \dfrac{1}{2}$ WHEN $|z| \geq 4$.

(v) FALSE

ALL WE NEED TO DO IS FIND A POINT z IN $\mathrm{Re}(z) \geq 0$ FOR WHICH e^{-z^3} IS REAL AND GREATER THAN 1.

THINK GEOMETRICALLY: LET $z = e^{\pi i/3}$. THEN $\mathrm{Re}(z) \geq 0$.

WE GET $z^3 = e^{\pi i} = -1$. SO $e^{-z^3} = e^1$

THUS $|e^{-z^3}| = e^1 > 1$. THIS IS SUFFICIENT TO PROVE STATEMENT FALSE

EXTRA

MORE GENERALLY (YOU DID NOT NEED TO DO THIS)

LET $z = re^{i\varphi}$ SO $z^3 = r^3(\cos 3\varphi + i\sin 3\varphi)$

THEN $|e^{-z^3}| = e^{-r^3\cos(3\varphi)} \leq 1$ ONLY WHEN $\cos(3\varphi) \geq 0$.

THIS GIVES $-\pi/2 \leq 3\varphi \leq \pi/2 \longrightarrow |\varphi| \leq \pi/6$.

SO $|e^{-z^3}| \leq 1$ IN $|\arg z| \leq \pi/6$ (NOT $\pi/2$).

(vi) RECALL $\dfrac{y}{x^2+y^2} = -\mathrm{Im}\left(\dfrac{1}{z}\right) = -\mathrm{Im}\left(\dfrac{x-iy}{(x+iy)(x-iy)}\right) = \dfrac{y}{x^2+y^2}$ ✓

THUS $\Phi = \dfrac{y-1}{x^2+(y-1)^2} = -\mathrm{Im}\left(\dfrac{1}{z-i}\right)$ (i.e. SHIFT y BY 1 $\rightarrow$ SHIFT z BY i)

$f(z) = \dfrac{-1}{z-i}$ IS ANALYTIC EXCEPT AT $z = i$. $\Longrightarrow$ $\mathrm{Im}(f)$ IS HARMONIC.

SOLUTION 3

DETERMINE WHERE $f(z) = \log(1-z^3)$ IS ANALYTIC IN THE COMPLEX PLANE.

SOLUTION

$f(z)$ IS ANALYTIC EXCEPT ON CURVES WHERE

$$\mathrm{IM}(1-z^3) = 0 \quad \text{AND} \quad \mathrm{RE}(1-z^3) \le 0.$$

LET $z = re^{i\varphi}$ SO $\mathrm{IM}(1-z^3) = -r^3 \sin(3\varphi)$

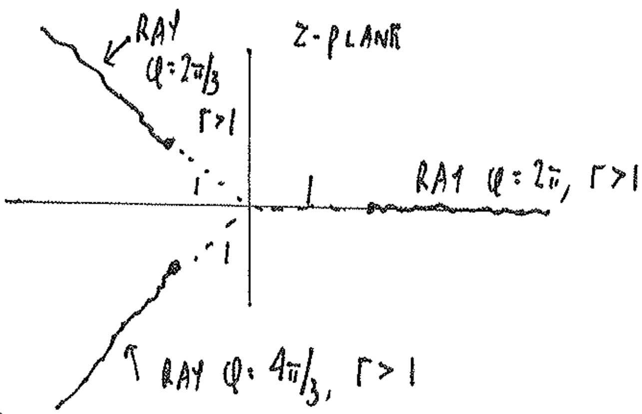
$$\mathrm{RE}(1-z^3) = 1 - r^3 \cos(3\varphi)$$

THEN $\mathrm{IM}(1-z^3) = 0$ WHEN $\sin(3\varphi) = 0 \rightarrow 3\varphi = \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi.$

NOW ON $3\varphi = \pi, 3\pi, 5\pi \longrightarrow \mathrm{RE}(1-z^3) = 1 + r^3 > 0.$

BUT ON $3\varphi = 2\pi, 4\pi, 6\pi \longrightarrow \mathrm{RE}(1-z^3) = 1 - r^3 \le 0$ WHEN $r \ge 1$

THUS $f(z)$ IS ANALYTIC EXCEPT ON RAYS $\varphi = 2\pi/3, 4\pi/3, 2\pi$ WITH $r \ge 1$.



f NOT ANALYTIC ALONG RAYS AS SHOWN.

easy way is as follows

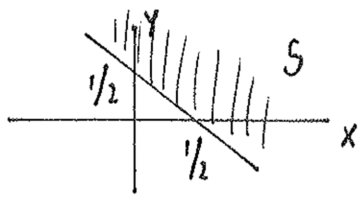
OR EASIER WAY $\log(1-z^3)$ IS ANALYTIC EXCEPT FOR POINTS z FOR WHICH $1-z^3$ IS REAL AND NEGATIVE.

THINK GEOMETRICALLY

- IF $z = r$ REAL THEN $r > 1$ IS A POSSIBILITY
- IF $z = re^{2\pi i/3} \rightarrow z^3 = r^3$ SO $r > 1$ IS POSSIBLE
- IF $z = re^{4\pi i/3} \rightarrow z^3 = r^3$ so $r > 1$ is possible

SOLUTION 4

(i) $S = \{ z \mid \mathrm{IM}(z) + \mathrm{RE}(z) \geq 1/2 \}$, $\quad W = 1/Z$.



NOTICE IF $Z = X + iY$ THEN S IS THE REGION $X + Y \geq 1/2$.

NOW LET $Z = 1/W$ SO $\mathrm{IM}(1/W) + \mathrm{RE}(1/W) \geq 1/2$.

THUS $\mathrm{IM}\left(\bar{W}/|W|^2 \right) + \mathrm{RE}\left(\bar{W}/|W|^2 \right) \geq 1/2$

LET $W = U + iV$, THEN $-V/(U^2+V^2) + U/(U^2+V^2) \geq 1/2$.

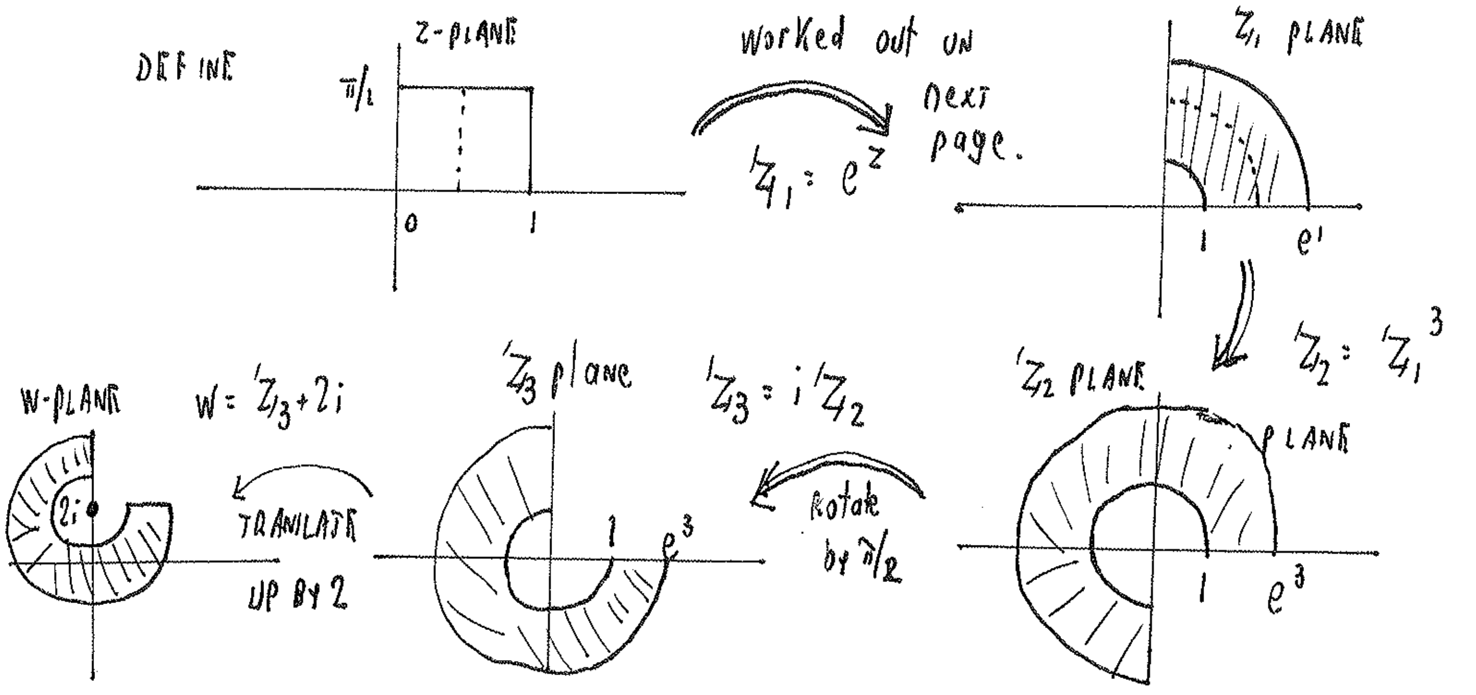
THUS $U^2 + V^2 \leq -2V + 2U \longrightarrow U^2 - 2U + V^2 + 2V \leq 0$.

COMPLETING THE SQUARE, $U^2 - 2U + 1 + (V^2 + 2V + 1) \leq 2$.

SO $(U-1)^2 + (V+1)^2 \leq 2$. CENTER AT $U = 1, V = -1$.

THUS $S' = \{ W \mid |W - (1-i)| \leq \sqrt{2} \}$.

(ii) $S = \{ z \mid 0 \leq \mathrm{RE}(z) \leq 1,\ 0 \leq \mathrm{IM}(z) \leq \pi/2 \}$, $\quad W = f(z) = i(e^z)^3 + 2i$.



THE ONLY ONE THAT NEEDS A CALCULATION IS MAP $Z_1 = e^z$:

NOW $Z_1 = X_1 + i Y_1 = e^x \cos y + i e^x \sin y$.

SO $X_1 = e^x \cos y$, $Y_1 = e^x \sin y$ WITH $0 \le x \le 1$, $0 \le y \le \pi/2$.

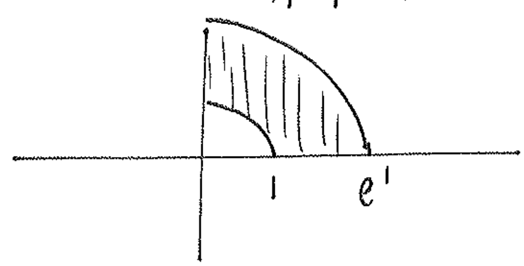
$\Rightarrow$ $X_1 \ge 0$, $Y_1 \ge 0$ SINCE $0 \le y \le \pi/2$.

NOW LINE x FIXED GOES TO QUARTER CIRCLE $X_1^2 + Y_1^2 = e^{2x}$.

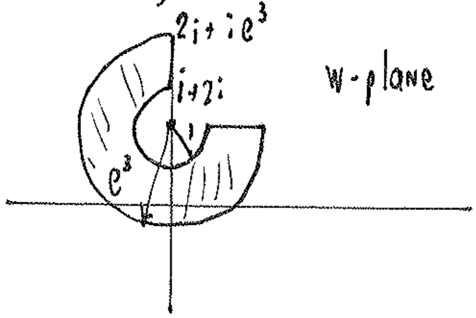
SINCE $0 \le x \le 1$ THE RADIUS OF THE CIRCLE, GIVEN BY e^x, RANGES FROM e^0 TO e^1.

Z_1 plane

THEN THIS GIVES



THE MAPPINGS $Z_2 = Z_1^3$, $Z_3 = i Z_2$, $W = Z_3 + 2i$ ARE THEN EASY TO IMPLEMENT, YIELDING



SO

$$S' = \{ w \mid 1 \le |w - 2i| \le e^3 \text{ WITH } \pi/2 \le \arg(w-2i) \le 2\pi \}$$