**The University of British Columbia**
Midterm Examinations - November 2011

**Mathematics 305**

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Closed book examination. No notes, texts, or calculators allowed. Time: 55 minutes

Marks

[20] **1.** For each of the following, calculate the integral $I$ over the given path $C$. You must give justification for your results to receive credit. (Hint: very little calculation is needed to evaluate these).

(i) $I = \int_C \sqrt{z}\,dz$, where $\sqrt{z}$ denotes the principal branch of the square root and $C$ is the straight line from $z = \sqrt{2}$ to $z = -1 + i$.

(ii) $I = \int_C \frac{1}{z^4-(4+3i)}\,dz$, where $C$ is the circle $|z| = 1$ counter-clockwise.

(iii) $I = \int_C \frac{1}{z^4-1}\,dz$, where $C$ is the circle $|z| = 2$ counter-clockwise.

(iv) $I = \int_C \frac{1}{i\sqrt{z}+3i+1}\,dz$, where $\sqrt{z}$ denotes the principal branch of the square root, and $C$ is the circle $|z-8| = 7$ counter-clockwise.

(v) $I = \int_C \mathrm{Log}\left(1 - \frac{1}{z^2}\right)\,dz$, where Log denotes the principal branch of the multi-valued logarithm function and $C$ is the circle $|z - 3i| = 1$ counter-clockwise.

[10] **2.** Let $C$ be the unit circle $|z| = 1$ oriented counter-clockwise.

(i) By using partial fractions, calculate the integral $I$ defined by

$$I = -i \int_C \frac{dz}{z^2 + 4z + 1}\,.$$

(ii) Then, from a direct parametrization of the integral $I$ over the unit circle show that the following identity emerges:

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}\,.$$

Continued on page 2

[10] **3.** Let $C$ be the unit circle $|z| = 1$ oriented counter-clockwise and let $a$ be any real constant with $a > 1$.

(i) Calculate the integral $I$ defined by

$$I = \int_C \frac{\mathrm{Log}(a-z)}{z}\, dz\,,$$

where Log denotes the principal branch of the multi-valued logarithm function (Hint: use the Cauchy integral formula)

(ii) From a direct parametrization of the integral derive the identity

$$\int_0^{2\pi} \ln\left(a^2 + 1 - 2a\cos\theta\right)\, d\theta = 4\pi \ln a\,.$$

[10] **4.** Suppose that $f(z)$ is analytic inside and on a simple closed curve $C$. Assume also that $|f(z) - 1| < 1$ for $z$ on $C$.

(i) Prove that there is no point $z_0$ inside $C$ for which $f(z_0) = 0$.

(i) Using the result in (i) prove that there are no solutions to $3z^3 - 2iz^2 + iz - 7 = 0$ inside the unit disk $|z| \leq 1$.

[50] **Total Marks**

**The End**

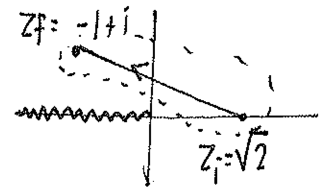
# MATH 305 MIDTERM 2

## PROBLEM 1

(i) $I = \int_C \sqrt{z}\, dz$ WHERE $C$ IS STRAIGHT LINE FROM $z = \sqrt{2}$ TO $z = -1 + i$.

SINCE $\sqrt{z}$ IS ANALYTIC IN A REGION CONTAINING $C$, WE HAVE THE ANTI-DERIVATIVE $F(z) = \frac{2}{3} z^{3/2}$, WITH $z_i = \sqrt{2}$ AND $z_f = \sqrt{2}\, e^{3\pi i/4}$

THUS $I = F(z_f) - F(z_i) = \frac{2}{3}\left[ 2^{3/4} e^{9\pi i/8} - 2^{3/4} \right]$

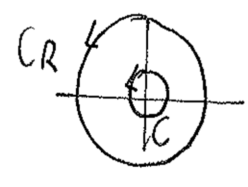



(ii) $I = \int_C \frac{1}{z^4 - (4+3i)}\, dz$ $\quad C: |z| = 1$ counter-clockwise.

THE SINGULAR POINTS SATISFY $z^4 = 4 + 3i$ SO $|z|^4 = |4+3i| = 5 \rightarrow |z| = 5^{1/4} > 1$.

SINCE ALL SINGULAR POINTS ARE <u>OUTSIDE</u> $|z| = 1$, THEN $I = 0$ BY CAUCHY INTEGRAL THEOREM.

(iii) $I = \int_C \frac{1}{z^4 - 1}\, dz$ $\quad C: |z| = 2$ counterclockwise.

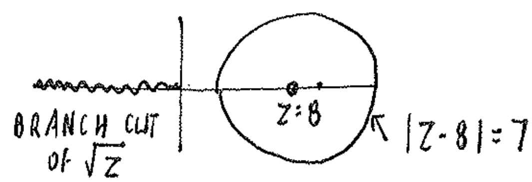



NOTICE THAT <u>ALL</u> SINGULAR POINTS ARE INSIDE C. THUS, WE CAN DEFORM C TO CR, DEFINED BY $|z| = R > 2$, TO GET $I = \int_C \frac{1}{z^4-1}\, dz = \int_{C_R} \frac{1}{z^4-1}\, dz$

NOW LET $R \to \infty$ AND ESTIMATE INTEGRAL AS $\left| \int_{C_R} \frac{1}{z^4-1}\, dz \right| \leq \frac{2\pi R}{R^4 - 1} \to 0$ AS $R \to \infty$.

THUS $I = 0$.

(iv) $I = \int_C \frac{dz}{i\sqrt{z} + 3i + 1}$ WITH $C: |z - 8| = 7$.




NOTICE PRINCIPAL BRANCH OF $\sqrt{z}$ SATISFIES

$\mathrm{RE}(\sqrt{z}) = |z|^{1/2} \cos(\varphi/2) \geq 0$ SINCE $-\pi < \varphi \leq \pi$ WITH $\varphi = \mathrm{ARG}\, z$.

THE ONLY POINT OF NON-ANALYTICITY WOULD BE IF $i\sqrt{z} + 3i + 1 = 0 \rightarrow \sqrt{z} + 3 - i = 0$.

BUT THIS GIVES $\mathrm{RE}(\sqrt{z}) + 3 = 0 \rightarrow \mathrm{RE}(\sqrt{z}) = -3$ WHICH IS IMPOSSIBLE WITH PRINCIPAL BRANCH OF $\sqrt{z}$. THUS INTEGRAND IS ANALYTIC IN $|z - 8| = 7 \Rightarrow I = 0$.

<u>NOTE</u> WE <u>CANNOT</u> CALCULATE AS $(\sqrt{z})^2 = (-3+i)^2 = 8 - 6i \rightarrow z = 8 - 6i$ INSIDE DISK BUT ON WRONG BRANCH!

(v) $I = \int_C \log\left(1 - 1/z^2\right) dz \qquad C: |z-3i| = 1.$

NOW $\log\left(1 - 1/z^2\right)$ IS <u>NOT</u> ANALYTIC WHEN

$$\mathrm{Im}\left(1 - 1/z^2\right) = -\mathrm{Im}\left(1/z^2\right) = 0 \quad \text{AND} \quad \mathrm{Re}\left(1 - 1/z^2\right) = 1 - \mathrm{Re}\left(1/z^2\right) \le 0.$$

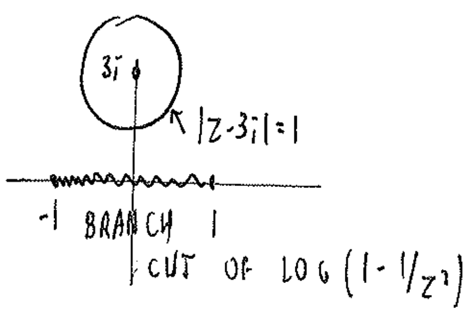
LET $z = e^{i\varphi}R$

$$\mathrm{Im}\left(z^{-2}\right) = 0 \longrightarrow \mathrm{Im}\left(e^{-2i\varphi}R^{-2}\right) = 0 \longrightarrow \sin(2\varphi) = 0 \quad \text{OR} \quad \varphi = 0, \pi/2, \pi, 3\pi/2, 2\pi.$$

NOW $\mathrm{Re}\left(1 - 1/z^2\right) = 1 - \frac{1}{R^2}\cos(2\varphi) = \begin{cases} 1 - \frac{1}{R^2} & \text{WHEN } \varphi = 0, \pi, 2\pi \\ 1 + \frac{1}{R^2} & \text{WHEN } \varphi = \pi/2, 3\pi/2. \end{cases}$

THUS $\mathrm{Re}\left(1 - 1/z^2\right) \le 0$ WHEN $\varphi = 0, \pi$ AND $0 < R < 1$.

WE CONCLUDE THAT $\log\left(1 - 1/z^2\right)$ IS ANALYTIC OUTSIDE $|z| \ge 1$.



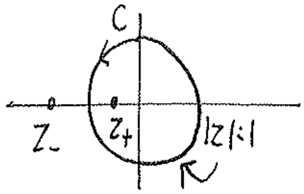

HENCE $I = 0$.

PROBLEM 2

LET $I = -i \int_C \frac{dz}{z^2+4z+1}$ $\quad C: |z|=1$ counter-clockwise.

(i) THE SINGULAR POINTS ARE AT $z^2+4z+1=0 \rightarrow z_{\pm} = \frac{-4 \pm \sqrt{16-4}}{2} \rightarrow z_{\pm} = -2 \pm \sqrt{3}$.

THUS $|z_+| < 1$ AND $|z_-| > 1$. HENCE, WE GET THE PICTURE




USING PARTIAL FRACTIONS $\frac{1}{z^2+4z+1} = \frac{A}{z-z_+} + \frac{B}{z-z_-}$ WE NEED ONLY CALCULATE $A$.

SO $1 = A(z-z_-) + B(z-z_+) \longrightarrow$ EVALUATE AT $z_+$ TO GET $A = \frac{1}{z_+-z_-} = \frac{1}{2\sqrt{3}}$.

THUS $I = -i \int_C \frac{A}{z-z_+} - i \int_C \frac{B}{z-z_-}$

$I = -i\,A(2\pi i) + 0 = 2\pi A \longrightarrow I = \frac{2\pi}{z_+-z_-} = \frac{\pi}{\sqrt{3}}$.

(ii) NOW LET $z = e^{i\varphi}$. THEN $dz = i e^{i\varphi}\, d\varphi$.

HENCE, $I = -i \int_0^{2\pi} \frac{i e^{i\varphi}\, d\varphi}{e^{2i\varphi} + 4e^{i\varphi} + 1} = \int_0^{2\pi} \frac{d\varphi}{e^{i\varphi} + e^{-i\varphi} + 4} = \int_0^{2\pi} \frac{d\varphi}{4 + 2\cos\varphi}$

SINCE $\frac{e^{i\varphi}+e^{-i\varphi}}{2} = \cos\varphi$. THUS FROM (i) WHERE $I = \pi/\sqrt{3}$, WE GET

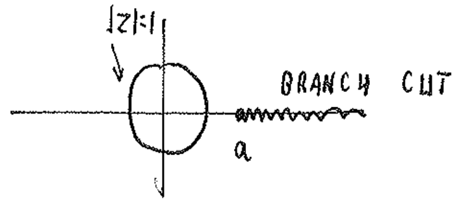
$$\int_0^{2\pi} \frac{d\varphi}{4+2\cos\varphi} = \frac{\pi}{\sqrt{3}} \quad \text{SO} \quad \int_0^{2\pi} \frac{d\varphi}{2+\cos\varphi} = \frac{2\pi}{\sqrt{3}}.$$

PROBLEM 3 LET $I = \int_C \frac{\mathrm{LOG}(a-z)}{z}\,dz$ WITH $a > 1$ REAL AND $C: |z|=1$

COUNTERCLOCKWISE. HERE $\mathrm{LOG}(\zeta)$ IS PRINCIPAL BRANCH OF $\log \zeta$.

(i) $\mathrm{LOG}(a-z)$ IS ANALYTIC EXCEPT FOR $\mathrm{IM}(z) = 0$ AND $\mathrm{RE}(z) \geq a > 1$.

THIS GIVES THE PICTURE




NOW BY CAUCHY INTEGRAL FORMULA $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$ WITH $z_0$ INSIDE $C$ AND $f$ ANALYTIC INSIDE AND ON $C$.

THUS, $$I = 2\pi i\, \mathrm{LOG}(a-z)\Big|_{z=0} = 2\pi i\, \mathrm{LOG}(a) = 2\pi i \ln a.$$

(ii) NOW LET $z = e^{i\varphi}$. $dz = i e^{i\varphi}\,d\varphi$ SO $dz/z = i\,d\varphi$.

THIS GIVES, $$I = i\int_0^{2\pi} \mathrm{LOG}(a - e^{i\varphi})\,d\varphi = 2\pi i \ln a.$$

CANCELLING THE $i$, WE GET $$\int_0^{2\pi} \mathrm{LOG}(a - e^{i\varphi})\,d\varphi = 2\pi \ln a. \quad (*)$$

NOW SINCE RHS OF $(*)$ IS REAL, WE GET
$$\int_0^{2\pi} \mathrm{RE}\left[\mathrm{LOG}(a - e^{i\varphi})\right] d\varphi = \int_0^{2\pi} \ln|a - e^{i\varphi}|\,d\varphi = 2\pi \ln a.$$

HENCE, $$\int_0^{2\pi} \ln\left[|a - \cos\varphi - i\sin\varphi|\right] d\varphi = \frac{1}{2}\int_0^{2\pi} \ln\left[(a - \cos\varphi)^2 + \sin^2\varphi\right] d\varphi = 2\pi \ln a.$$

WE SIMPLIFY, $$\int_0^{2\pi} \ln\left[a^2 - 2a\cos\varphi + \cos^2\varphi + \sin^2\varphi\right] d\varphi = 4\pi \ln a.$$

SO $$\int_0^{2\pi} \ln\left[a^2 + 1 - 2a\cos\varphi\right] d\varphi = 4\pi \ln a.$$

PROBLEM 4

SUPPOSE THAT $f(z)$ IS ANALYTIC INSIDE AND ON $C$. THEN $g(z) = f(z) - 1$ IS ANALYTIC IN THIS REGION.

(i) BY ASSUMPTION $|g(z)| < 1$ FOR $z$ ON $C$.

THUS BY THE MAX-MODULUS PRINCIPLE FOR $z$ INSIDE $C$ WE HAVE

$$|g(z)| \leq \max_{z \text{ on } C} |g(z)| < 1.$$

THUS FOR $z$ INSIDE $C$, $|g(z)| < 1$. $(*)$

BUT $g(z) = f(z) - 1$. IF $f(z_0) = 0$ FOR $z_0$ INSIDE $C$, THEN $|g(z_0)| = 1$, WHICH CONTRADICTS $(*)$. HENCE $f$ HAS NO ROOTS INSIDE $C$.

(ii) PROVE THAT $3z^3 - 2iz^2 + iz - 7 = 0$ HAS NO SOLUTIONS IN $|z| \leq 1$.

WE WRITE THIS AS FINDING ROOTS OF $f(z) = 0$ WHERE

$$f(z) = -\frac{3}{7} z^3 + \frac{2i}{7} z^2 - \frac{iz}{7} + 1.$$

WE CALCULATE $$|f(z) - 1| = \left| -\frac{3}{7} z^3 + \frac{2i}{7} z^2 - \frac{iz}{7} \right| \leq \frac{3}{7}|z|^3 + \frac{2}{7}|z|^2 + \frac{1}{7}|z|$$

BY THE TRIANGLE INEQUALITY.

THUS ON $|z| = 1$, $$|f(z) - 1| \leq \frac{3}{7} + \frac{2}{7} + \frac{1}{7} = \frac{6}{7} < 1.$$

IT FOLLOWS BY THE RESULT IN (i) THAT $f(z) = 0$ HAS NO SOLUTIONS IN $|z| \leq 1$.