The University of British Columbia

Midterm Examinations - November 2012

Mathematics 305

M. Ward

Closed book examination. No notes, texts, or calculators allowed. Time: 50 minutes

Marks

[15] **1.** Define $f(z) = (z^2+4)^{1/2}$.

(i) By using the range of angle method, construct a branch of $f(z)$ that is analytic in $|z| > 2$ and that satisfies $f(4) = \sqrt{20}$.

(ii) Show how to construct the branch in (i) by choosing a branch of the multi-valued logarithm. (You must justify your choice of the logarithm)

(iii) For this branch of $f(z)$, calculate $f(-4+2i)$.

[10] **2.** Let C be the curve $z = e^{i\theta}$ with $\pi/4 \le \theta \le 7\pi/4$ oriented counterclockwise. Calculate the following integrals:

$$\text{(i)} \quad \int_C \frac{1}{z-3}\, dz\,; \qquad\qquad \text{(ii)} \quad \int_C (\bar{z})^2\, dz$$

[15] **3.** Calculate the following integrals, providing justification for your results:

(i) $\int_C \frac{z+1}{z(z-3)(z-2)}\, dz$ where C is the curve $|z| = 1$ counterclockwise.

(ii) $\int_C \frac{ze^z}{(z+i)^2}\, dz$ where C is the the curve $|z| = 2$ counterclockwise.

(iii) $\int_C \frac{1}{z+(2+i)\sqrt{z}}\, dz$ where $\sqrt{z}$ is the principal value of the square root and C is the curve $|z-(3+4i)| = 1$ counterclockwise.

[10] **4.** Suppose that k is an integer with $k \ge 1$, and let C be the boundary of the unit circle $|z| = 3$ counterclockwise. Consider the integral I defined by

$$I = \int_C \frac{e^{1/z}}{z^k(z+2i)}\, dz.$$

(i) Prove that $|e^{1/z}| \le e^{1/R}$ for $|z| \ge R$ and any $R > 0$.

(ii) By using (i) and deforming C appropriately, prove that $I = 0$ (You must state clearly what results you are using in your derivation)

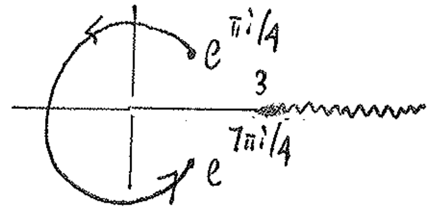
[50] **Total Marks**

The End

M305 MIDTERM 2 NOV. 2012

PROBLEM 2

(i) $I = \int_C \frac{1}{z-3}\,dz \qquad C: z = e^{i\varphi} \quad \pi/4 \le \varphi \le 7\pi/4 \quad$ C.C.



BY FTC WE HAVE ANTI-DERIVATIVE $\mathcal{F}(z) = \log(z-3)$

WHERE THE BRANCH CUT IS ON $z \ge 3$ REAL AS SHOWN. NEED $0 \le \arg(z-3) < 2\pi$

(NOTE: $\mathcal{F}(z)$ MUST BE ANALYTIC IN A REGION CONTAINING C).

THEN $$I = \mathcal{F}(e^{7\pi i/4}) - \mathcal{F}(e^{\pi i/4}) = \log(e^{7\pi i/4} - 3) - \log(e^{\pi i/4} - 3)$$

SO $$I = \ln|e^{7\pi i/4} - 3| - \ln|e^{\pi i/4} - 3| + i\left(\arg(e^{7\pi i/4} - 3) - \arg(e^{\pi i/4} - 3)\right)$$

SO $$I = i\left(\arg(e^{7\pi i/4} - 3) - \arg(e^{\pi i/4} - 3)\right)$$

(ii) $I = \int_C (\bar{z})^2\,dz \qquad C: z = e^{i\varphi} \quad \pi/4 < \varphi < 7\pi/4 \quad$ C.C.

MUST DO IT DIRECTLY: $\quad z = e^{i\varphi} \qquad dz = i e^{i\varphi}\,d\varphi$.

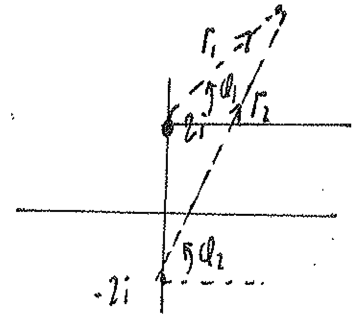
AND $\bar{z} = e^{-i\varphi}$

SO $$I = \int_{\pi/4}^{7\pi/4} i\left(e^{-2i\varphi}\right) e^{i\varphi}\,d\varphi = i\int_{\pi/4}^{7\pi/4} e^{-i\varphi}\,d\varphi = i\left(+\frac{1}{-i}e^{-i\varphi}\right)\Big|_{\pi/4}^{7\pi/4}$$

SO $$I = -e^{-i\varphi}\Big|_{\pi/4}^{7\pi/4} = -\left[e^{-i7\pi/4} - e^{-i\pi/4}\right] = -\left[e^{\pi i/4} - e^{-i\pi/4}\right]$$

SO $$I = -\left[2i\sin(\pi/4)\right] = -2i\,\sqrt{2}/2 = -i\sqrt{2}.$$

PROBLEM 1 $\quad f(z) = (z^2 + 4)^{1/2}$.



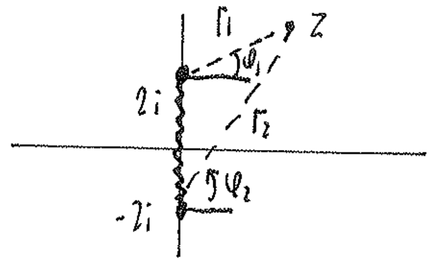
(i) BRANCH POINTS AT $z = \pm 2i$. (i.e. $z^2 + 4 = 0$).

WRITE $\quad (z^2+4)^{1/2} = (z+2i)^{1/2}(z-2i)^{1/2}$

SO $\quad f(z) = (r_1 r_2)^{1/2} e^{i(\varphi_1 + \varphi_2)/2}$

$r_1 = |z - 2i|, \quad r_2 = |z + 2i|$.

WE WANT ANALYTICITY IN $|z| \geq 2$. THUS PUT BRANCH CUT AS SHOWN



CHOOSE $\quad -\pi/2 < \varphi_1 \leq 3\pi/2$

$-\pi/2 < \varphi_2 \leq 3\pi/2$

THIS WILL GIVE BRANCH CUT AS SHOWN.

NOW WHEN $z = 4$ WE HAVE

$\varphi_1 + \varphi_2 = 0$.

SO $\quad r_1 = r_2 = \sqrt{20}$

HENCE $\quad f(4) = (20)^{1/2}$.

(ii) WE FACTOR $\quad f(z) = \pm z \left(1 + 4/z^2\right)^{1/2} = \pm z \; e^{1/2 \log(1 + 4/z^2)}$

NOW TRY $\quad f(z) = \pm z \; e^{1/2 \operatorname{LOG}(1 + 4/z^2)} \quad (*)$

THIS IS ANALYTIC IN $|z| > 2$ SINCE $\operatorname{LOG}\left(1 + 4/z^2\right)$ IS ANALYTIC IN $|z| > 2$

<u>NOTE:</u> $\operatorname{LOG}\left(1 + 4/z^2\right)$ IS NOT ANALYTIC ON $\operatorname{IM}\left(1 + 4/z^2\right) = 0$

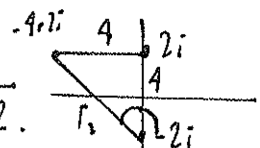
AND $\operatorname{RE}\left(1 + 4/z^2\right) \leq 0$.

THUS NOT ANALYTIC ON LINE BETWEEN $-2i \leq z \leq 2i$.

NOW CALCULATE $f(4)$: $\quad f(4) = \pm 4 \; e^{1/2 \operatorname{LOG}(1 + 4/16)} = \pm 4 \; e^{1/2 \ln(5/4)}$

SO $\quad f(4) = \pm 4 \cdot (5/4)^{1/2} = \pm\sqrt{20}$. CHOOSE + SIGN IN $(*)$

(iii) NOW FIND $f(-4 + 2i)$. THEN $\varphi_1 = \pi$.



$\varphi_2 = \pi/2 + \pi/4 = 3\pi/4$.

$r_1 = 4, \; r_2 = 4\sqrt{2}$.

SO $\quad f(-4+2i) = (16\sqrt{2})^{1/2} e^{i(\pi + 3\pi/4)/2} = (16\sqrt{2})^{1/2} e^{7\pi i/8}$.

PROBLEM 3

(i) $I = \int_C \frac{z+1}{z(z-3)(z-2)}\,dz$ C IS $|z|=1$ c.c.

METHOD 1   singularities at $z=0,2,3$. ONLY $z=0$ IS INSIDE C.

partial FRACTIONS $\frac{z+1}{z(z-3)(z-2)} = \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z-2}$.

WE NEED ONLY CALCULATE A: $z+1 = A(z-3)(z-2) + Bz(z-2) + Cz(z-3)$

SET $z=0 \rightarrow A = 1/6$.

THUS $I = \int_C \frac{A}{z}\,dz + \underbrace{\int \frac{B}{z-3}\,dz}_{=0} + \underbrace{\int \frac{C}{z-2}\,dz}_{=0}$

$I = 2\pi i A$ BY C.G.

SO $I = 2\pi i\left(\frac{1}{6}\right) = \pi i/3$.

METHOD 2 $I = \int_C \frac{f(z)}{z}\,dz$ WITH $f(z) = \frac{z+1}{(z-2)(z-3)}$

NOW $f(z)$ IS ANALYTIC inside AND ON C.

THUS BY C.I.T. THEN $I = 2\pi i\, f(0) = 2\pi i/6 = \pi i/3$.

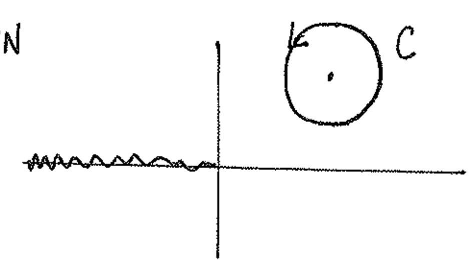
(ii) $I = \int_C \frac{ze^z}{(z+i)^2}\,dz = \int_C \frac{f(z)}{(z-z_0)^2}\,dz$ WITH $f(z) = ze^z$ $f'(z) = ze^z + e^z = e^z(z+1)$

$z_0 = -i$.

HENCE $I = 2\pi i\, f'(-i) = 2\pi i\left[ze^z + e^z\right]\Big|_{z=-i} = 2\pi i\, e^{-i}(1-i)$.

(iii) $I = \int_C \frac{1}{(z+(2+i)\sqrt{z})}\,dz$. NOTICE THAT $f(z) = \frac{1}{z+(2+i)\sqrt{z}}$ IS NOT ANALYTIC ALONG $\mathrm{Re}\,z < 0$ AND $\mathrm{Im}\,z = 0$. ALSO WE COULD HAVE A SINGULAR POINT WHEN $z + (2+i)\sqrt{z} = \sqrt{z}\left[\sqrt{z} + (2+i)\right] = 0$ THUS $\sqrt{z} = -2-i$.

HOWEVER, THIS IS IMPOSSIBLE SINCE $\mathrm{Re}(\sqrt{z}) \geq 0$ WITH THE PRINCIPAL BRANCH. HENCE $f(z)$ IS NOT ANALYTIC ONLY ALONG NEGATIVE REAL AXIS AS SHOWN



SINCE $f(z)$ IS ANALYTIC INSIDE AND ON C, THEN BY CG THEOREM $\int_C f(z)\,dz = 0$.

PROBLEM 4

$$I = \int_C \frac{e^{1/z}}{z^K(z+2i)}\,dz \qquad K = 1,2,3,\ldots$$

$C: |z| = 3$ COUNTERCLOCKWISE.

(i) PROVE THAT $|e^{1/z}| \le e^{1/R}$ FOR $|z| \ge R$ AND ANY $R > 0$.

PROOF RECALL $|e^w| \le e^{|w|}$ FOR ANY w.

(TRUE SINCE IF $w = u + iv$ $\quad |e^w| = |e^u e^{iv}| = e^u \le e^{|w|}$ SINCE $\operatorname{Re} w \le |w|$.

NOW LET $w = 1/z$. SO

$$|e^{1/z}| \le e^{|1/z|}$$

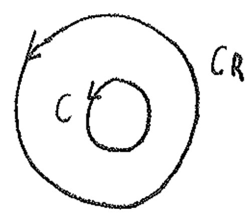
BUT IF $|z| \ge R \rightarrow \frac{1}{|z|} \le \frac{1}{R} \rightarrow e^{1/|z|} \le e^{1/R}$ SINCE e^x IS MONOTONIC.

THUS $|e^{1/z}| \le e^{1/R}$ FOR $|z| \ge R$.

(ii) NOW THE SINGULARITIES ARE AT $z = 0, -2i$ WHICH ARE BOTH IN C.

SINCE $f(z) = \frac{e^{1/z}}{z^K(z+2i)}$ IS ANALYTIC IN $|z| \ge 3$ WE DEFORM C TO C_R

WHERE $|z| = R > 3$ TO OBTAIN



$$I = \int_{C_R} \frac{e^{1/z}}{z^K(z+2i)}\,dz$$

NOW LETS ESTIMATE

$$|I| \le \max_{z \text{ on } C_R} \frac{|e^{1/z}|}{|z|^K|z+2i|}\,(2\pi R).$$

BUT $|z|^K = R^K$ ON C_R AND $|e^{1/z}| \le e^{1/R}$ ON C_R. ALSO BY REVERSE

Δ-INEQUALITY, $|z+2i| \ge \big|\,|z| - |-2i|\,\big| = |z| - 2$ FOR $|z| = R > 3$.

THUS $$|I| \le \frac{e^{1/R}}{R^K(R-2)}\,2\pi R = \frac{2\pi\, e^{1/R}}{R^{K-1}(R-2)} \longrightarrow 0 \text{ AS } R \to \infty \text{ FOR ANY } K = 1,2,3,\ldots$$

(NOTE: $e^{1/R} \to 1$ AS $R \to +\infty$).