# TYPE II FINITE TIME BLOW-UP FOR THE ENERGY CRITICAL HEAT EQUATION IN $\mathbb{R}^4$

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Abstract. We consider the Cauchy problem for the energy critical heat equation

$$\begin{cases} u_t = \Delta u + u^3 & \text{in } \mathbb{R}^4 \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^4. \end{cases} \tag{0.1}$$

We find that for given points $q_1, q_2, \ldots, q_k$ and any sufficiently small $T > 0$ there is an initial condition $u_0$ such that the solution $u(x,t)$ of (0.1) blows up at exactly those $k$ points with a type II rate, namely larger than $(T-t)^{-\frac{1}{2}}$. In fact $\|u(\cdot,t)\|_\infty \sim (T-t)^{-1}\log^2(T-t)$. The blow-up profile around each point is of bubbling type, in the form of sharply scaled Aubin-Talenti bubbles.

*Dedicated to Wei-Ming Ni on the occasion of his 70th birthday.*

## 1. Introduction

Many studies have been devoted to the analysis of blow-up phenomena in a semilinear heat equation of the form

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $p > 1$, and $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$ (or $\Omega = \mathbb{R}^n$), starting with the seminal work by Fujita [14] in the 1960's. A smooth solution of (1.1) *blows up at time* $T$ if

$$\lim_{t\to T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

We observe that for functions independent of the space variable the equation reduces to the ODE $u_t = |u|^{p-1}u$, which is solved for a suitable constant $c_p$ by the function $u(t) = c_p(T-t)^{-\frac{1}{p-1}}$ and it blows up at time $T$. It is commonly said that the blow-up of a solution $u(x,t)$ is of type I if it happens at most at the ODE rate:

$$\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < +\infty$$

while the blow-up is said of type II if

$$\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

Many results have predicted that type I is the "typical" or "generic" way in which blow-up takes place for solutions of equation (1.1). For instance it is known after a series of works, including [18–20], that type I is the only way possible if $p < p_S$ where $p_S$ is the critical Sobolev exponent,

$$p_S := \begin{cases} \frac{n+2}{n-2} & \text{if } n \geq 3, \\ +\infty & \text{if } n = 1, 2. \end{cases}$$

Stability and genericity of type I blow-up have been considered for instance in [3, 26, 28].

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Solutions with multiple type I blow-up were first built in [27] in the real line. Solutions with type II blow-up are in fact much harder to detect. The first example was discovered in [21,22], for $p > p_{JL}$ where $p_{JL}$ is the Joseph-Lundgren exponent [23],

$$p_{JL} = \begin{cases} 1 + \frac{4}{n-4-2\sqrt{n-1}} & \text{if } n \geq 11, \\ +\infty, & \text{if } n \leq 10. \end{cases}$$

See the book [30] for a survey of related results. In fact, no type II blow-up is present for radial solutions if $p_S < p < p_{JL}$ in the case of a ball or in entire space under additional assumptions, see [24,25,29]. For radial positive solutions this is not possible if $p = \frac{n+2}{n-2}$ [13]. Examples of nonradial positive blow-up solutions for $p > p_{JL}$ have been found in [1,2].

In the critical case $p = p_S$, namely for the equation

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

very few examples of type II blow-up are known. The special feature of the critical exponent $p = p_S$ is the presence of finite energy steady states of (1.2). All positive entire solutions of the stationary equation

$$\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in } \mathbb{R}^n$$

are given by the family of *Aubin-Talenti bubbles*

$$U_{\lambda,\xi}(x) = \lambda^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\lambda}\right) \tag{1.3}$$

where $U(y)$ is the *standard bubble*

$$U(y) = \varpi_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad \varpi_n = (n(n-2))^{\frac{n-2}{4}}.$$

These solutions have finite Dirichlet integral, in fact independent of the parameters:

$$\int_{\mathbb{R}^n} |\nabla U_{\lambda,\xi}(x)|^2 dx \ = \ S_n \quad \text{for all} \quad \lambda, \xi.$$

When $p = p_S$ it is typically expected that type II blow-up for a solution $u(x,T)$ of (1.2) takes the form of *bubbling*. That means that sufficiently close to one or more points $q \in \Omega$ one has

$$u(x,t) \approx U_{\lambda(t),\xi(t)}(x), \quad 0 < \lambda(t) \to 0, \quad \xi(t) \to q \quad \text{as} \quad t \to T. \tag{1.4}$$

In [13], by means of matching asymptotic expansions, Filippas, Herrero and Velázquez [13] formally derived exact profiles of radially symmetric type II blow-up solutions when $\Omega = \mathbb{R}^n$. In their analysis, bubbling blow-up seems only possible in dimensions $n = 3, 4, 5, 6$. The first rigorous example of type II blow-up in (1.2) when $n = 4$ and $\Omega = \mathbb{R}^n$ is due to Schweyer [32], who finds a radial sign-changing solution with the profile (1.4) with scaling parameter $\lambda(t) \sim \frac{T-t}{\log^2(T-t)}$. This rate is one of those formally predicted in [13].

The method in [32] seems only applicable in the radial case and an odd power in the nonlinearity. In [10] we have found the first example of type II bubbling blow-up in dimension $n = 5$, with bubbling rates $\lambda(t) \sim (T-t)^2$. Again this is one of the rates predicted in [13]. The construction in [10] does not depend on any symmetries, in fact in such a way that simultaneous blow-up takes place at any prescribed set of points $q_1, \ldots, q_k \in \Omega$.

In this paper we construct type II blow-up solutions of (1.2) for dimension $n = 4$ without any symmetries. In what follows we let $\Omega$ be a smooth bounded domain in $\mathbb{R}^4$ or $\Omega = \mathbb{R}^4$ and thus consider the equation

$$\begin{cases} u_t = \Delta u + u^3 \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{1.5}$$

Let us fix arbitrary points $q_1, q_2, \dots q_k \in \Omega$. We consider a smooth function $Z_0^* \in L^\infty(\Omega)$ with the property that

$$Z_0^*(q_j) > 0 \quad \text{for all} \quad j = 1, \dots, k.$$

**Theorem 1.** *For each $T > 0$ sufficiently small there exists an initial condition $u_0$ such that the solution of Problem (1.5) blows up at time $T$ exactly at the $k$ points $q_1, \dots, q_k$. It looks at main order like*

$$u(x,t) = \sum_{j=1}^{k} U_{\lambda_j(t),\xi_j(t)}(x) - Z_0^*(x) + \theta(x,t)$$

*where*

$$\lambda_j(t) \to 0, \quad \xi_j(t) \to q_j \quad \textit{as} \quad t \to T,$$

*and $\|\theta\|_{L^\infty} \le T^a$ for some $a > 0$. More precisely,*

$$\lambda_j(t) \sim \frac{T-t}{|\log(T-t)|^2} \quad \textit{as} \quad t \to T.$$

We observe that in particular, the solution predicted by the above result has type II blow-up since

$$b(t) := \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^4)} \sim \frac{\log^2(T-t)}{T-t}$$

and the type I blow-up rate corresponds to

$$(T-t)^{-\frac{1}{p_S - 1}} \;=\; (T-t)^{-\frac{1}{2}} \ll b(t) \quad \text{as} \quad t \to T.$$

The result in [10] is the exact analog of Theorem 1 in dimension 5. We follow the same general approach (the *inner–outer gluing method*). However substantial methods and differences arise, due to the fact that the equation that determines $\lambda(t)$ involves a delicate nonlocal integral operator. This nonlocal effect is related to the slower decay of the linear generator of dilations of the Aubin-Talenti bubbles in lower dimensions. In dimension 5 instead $\lambda(t)$ is found in a much more direct way by just solving an ODE, which is no longer the case in higher dimensions where this type of blow-up is not expected. A very similar difficulty was already faced in the work [6] on blow-up in the harmonic map flow. The similarity between these problems in the presence of symmetries had already been noticed in [31,32].

We should point out that blow-up by bubbling (at main order time dependent, energy invariant, asymptotically singular scalings of steady states) is a phenomenon that arises in various problems of parabolic and dispersive nature. It has been in particular widely studied for the energy critical wave equation

$$u_{tt} = \Delta u + |u|^{\frac{4}{n-2}} u.$$

Among other works, we refer the reader for instance to [11,15–17]. The method of this paper substantially differs from those in most of the above mentioned references for the parabolic case. It is close in spirit to the analysis in the works [4–6,8,9], where the inner–outer gluing method is employed. That approach consists of reducing the original problem to solving a basically uncoupled system, which depends in subtle ways on the parameter choices. A result related to that in this work in the $L^2$-critical nonlinear Schrödinger equation has been found in [12].

The rest of this paper will be devoted to the proof of Theorem 1.

## 2. Approximate solutions and error estimates

In this section, we shall choose a proper approximate solution to (1.5) and compute its error. For notational simplicity, we shall only carry out the construction in the case $k = 1$ and mention the minor changes for the general case when needed. We define the error operator

$$\mathcal{S}(u) := -u_t + \Delta u + u^3.$$

Then finding a solution to (1.5) is equivalent to finding $u$ such that

$$\mathcal{S}(u) = 0.$$

Recall that the Aubin-Talenti bubble

$$U(y) = \frac{\alpha_0}{1+|y|^2} \tag{2.1}$$

solves the Yamabe problem

$$\Delta_y U + U^3 = 0 \ \text{ in } \ \mathbb{R}^4,$$

where $\alpha_0 = 2\sqrt{2}$. It is well-known that the linearized operator around the bubble

$$L_0(\phi) := \Delta\phi + 3U^2\phi \tag{2.2}$$

is non-degenerate in the sense that all bounded solutions to $L_0(\phi) = 0$ are the linear combination of

$$Z_i(y) := \partial_{y_i} U(y), \ i = 1, 2, 3, 4, \ Z_5(y) := U(y) + \nabla U(y) \cdot y. \tag{2.3}$$

Our first approximation is chosen as

$$U_{\lambda(t),\xi(t)} = \lambda^{-1}(t) U\left(\frac{x - \xi(t)}{\lambda(t)}\right),$$

where $\lambda(t)$ and $\xi(t)$ are scaling and translation parameter functions to be adjusted later. Direct computations yield

$$S(U_{\lambda(t),\xi(t)}) = -\partial_t U_{\lambda(t),\xi(t)} = \lambda^{-2}(t)\dot\lambda(t)\left(-\frac{\alpha_0}{1+|y|^2} + \frac{2\alpha_0}{(1+|y|^2)^2}\right) + \lambda^{-2}(t)\nabla_y U(y)\cdot\dot\xi(t), \tag{2.4}$$

where $y = \frac{x-\xi(t)}{\lambda(t)}$. Observe that the slow decaying error in (2.4) is

$$\mathcal{E}_0 = -\frac{\alpha_0\dot\lambda(t)}{\lambda^2(t)+\rho^2} \approx -\frac{\alpha_0\dot\lambda(t)}{\rho^2},$$

where $\rho := |x - \xi(t)|$. In order to improve the approximation, we consider

$$\partial_t u_1 = \Delta u_1 + \mathcal{E}_0 \ \text{ in } \ \mathbb{R}^4 \times (0, T). \tag{2.5}$$

By similar computations as in [6], a solution to (2.5) is given explicitly by

$$u_1 = -\alpha_0 \int_{-T}^{t} \dot\lambda(s) k(\rho, t-s) ds,$$

where

$$k(\rho, t) := \frac{1 - e^{-\frac{\rho^2}{4t}}}{\rho^2}. \tag{2.6}$$

We regularize the above $u_1$ and choose a correction $\Psi_0$ to be

$$\Psi_0(x, t) = -\alpha_0 \int_{-T}^{t} \dot\lambda(s) k(\zeta(\rho, t), t-s) ds, \tag{2.7}$$

where

$$\zeta(\rho, t) = \sqrt{\rho^2 + \lambda^2(t)}.$$

Then we compute the new error produced by $\Psi_0$

$$\begin{aligned}
&\partial_t\Psi_0 - \Delta\Psi_0 - \mathcal{E}_0 \\
=& \alpha_0\left[\frac{\lambda y\cdot\dot\xi - \lambda(t)\dot\lambda(t)}{\zeta}\right]\int_{-T}^{t}\dot\lambda(s)k_\zeta(\zeta,t-s)ds \\
&+ \alpha_0\int_{-T}^{t}\dot\lambda(s)\left[-k_t(\zeta,t-s)+\frac{\rho^2}{\zeta^2}k_{\zeta\zeta}(\zeta,t-s)+\frac{3}{\zeta}k_\zeta(\zeta,t-s)+\frac{\lambda^2(t)}{\zeta^3}k_\zeta(\zeta,t-s)\right]ds.
\end{aligned}$$

Observing from (2.6) that $k(\zeta,t)$ satisfies $-k_t+k_{\zeta\zeta}+\frac{3}{\zeta}k_\zeta=0$, we get

$$\begin{aligned}
\partial_t\Psi_0 - \Delta\Psi_0 - \mathcal{E}_0 =& \alpha_0\left[\frac{y\cdot\dot\xi - \dot\lambda(t)}{(1+|y|^2)^{1/2}}\right]\int_{-T}^{t}\dot\lambda(s)k_\zeta(\zeta,t-s)ds \\
&+\frac{\alpha_0}{\lambda(t)(1+|y|^2)^{3/2}}\int_{-T}^{t}\dot\lambda(s)\left[-\zeta k_{\zeta\zeta}(\zeta,t-s)+k_\zeta(\zeta,t-s)\right]ds \\
:=& \mathcal{R}[\lambda].
\end{aligned} \tag{2.8}$$

It is thus reasonable to choose the corrected approximation as

$$u^* = U_{\lambda(t),\xi(t)} + \Psi_0$$

and its error is

$$\begin{aligned}
\mathcal{S}(u^*) &= \mathcal{S}(U_{\lambda(t),\xi(t)}) - \mathcal{E}_0 + (U_{\lambda(t),\xi(t)}+\Psi_0)^3 - U^3_{\lambda(t),\xi(t)} \\
&= \mathcal{K}[\lambda,\xi] + (U_{\lambda(t),\xi(t)}+\Psi_0)^3 - U^3_{\lambda(t),\xi(t)},
\end{aligned}$$

where $\mathcal{K}[\lambda,\xi]$ is defined as

$$\mathcal{K}[\lambda,\xi] := \frac{2\alpha_0\lambda^{-2}(t)\dot\lambda(t)}{(1+|y|^2)^2} + \lambda^{-2}(t)\nabla U(y)\cdot\dot\xi(t) - \mathcal{R}[\lambda] \tag{2.9}$$

with $\mathcal{R}[\lambda]$ given in (2.8).

## 3. The inner–outer gluing scheme

We look for solution of the following form

$$u = u^* + \mathtt{w},$$

where $\mathtt{w}$ is a small perturbation consisting of inner and outer parts

$$\mathtt{w} = \varphi_{\text{in}} + \varphi_{\text{out}},\ \varphi_{\text{in}} = \lambda^{-1}(t)\eta_R\phi(y,t),\ \varphi_{\text{out}} = \psi(x,t) + Z^*(x,t).$$

Here the cut-off function is defined by

$$\eta_R = \eta_{R(t)}(x,t) = \eta\left(\frac{|x-\xi(t)|}{\lambda(t)R(t)}\right)$$

where the smooth cut-off function $\eta(s)=1$ for $s<1$ and $\eta(s)=0$ for $s>2$, and $Z^*$ satisfies

$$\begin{cases} Z^*_t = \Delta_x Z^*, & \text{in } \Omega\times(0,T), \\ Z^*(\cdot,t) = 0, & \text{on } \partial\Omega\times(0,T), \\ Z^*(\cdot,0) = Z^*_0, & \text{in } \Omega. \end{cases}$$

Denote

$$B_{2R} = \{x\in\Omega : |x-\xi(t)| \le 2\lambda R\},\ \mathcal{D}_{2R} = B_{2R}\times(0,T),$$

and $\Psi^* = \psi + Z^*$. Then $u$ is a solution to the original problem (1.5) if

- $\phi$ solves the **inner problem**
$$\lambda^2\phi_t = \Delta_y\phi + 3U^2(y)\phi + \mathcal{H}(\phi,\psi,\lambda,\xi) \ \text{ in } \mathcal{D}_{2R} \tag{3.1}$$
where
$$\begin{aligned}\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t) := {}& 3\lambda U^2(y)[\Psi_0+\psi+Z^*](\lambda y+\xi,t)\\ &+\lambda\left[\dot\lambda(\nabla_y\phi\cdot y+\phi)+\nabla_y\phi\cdot\dot\xi\right]\\ &+\lambda^3\mathcal{N}(\mathtt{w})+\lambda^3\mathcal{K}[\lambda,\xi]\end{aligned} \tag{3.2}$$
with $\mathcal{K}[\lambda,\xi]$ defined in (2.9), and
$$\mathcal{N}(\mathtt{w}) := (U_{\lambda,\xi}+\Psi_0+\mathtt{w})^3 - U_{\lambda,\xi}^3 - 3U_{\lambda,\xi}^2(\Psi_0+\mathtt{w}). \tag{3.3}$$
- $\psi$ solves the **outer problem**
$$\psi_t = \Delta\psi + \mathcal{G}(\phi,\psi,\lambda,\xi) \ \text{ in } \Omega\times(0,T) \tag{3.4}$$
with
$$\begin{aligned}\mathcal{G}(\phi,\psi,\lambda,\xi) := {}& 3\lambda^{-2}(1-\eta_R)U^2(y)(\Psi_0+\psi+Z^*)\\ &+\lambda^{-3}\left[(\Delta_y\eta_R)\phi+2\nabla_y\eta_R\cdot\nabla_y\phi-\lambda^2\phi\partial_t\eta_R\right]\\ &+(1-\eta_R)\mathcal{K}[\lambda,\xi]+(1-\eta_R)\mathcal{N}(\mathtt{w}).\end{aligned} \tag{3.5}$$

The rest of this paper is organized as follows. In Section 4, we derive the leading orders for the parameter functions $\lambda(t)$ and $\xi(t)$. In Section 5, we establish the linear theories for the inner and outer problems. Finally, the inner–outer gluing system is formulated in Section 6, and we shall solve $(\phi,\psi,\lambda,\xi)$ by the fixed point argument.

**Notation**. Throughout the paper, we shall use the symbol " $\lesssim$ " to denote " $\le C$" for a positive constant $C$ independent of $t$ and $T$. Here $C$ might be different from line to line.

## 4. The choices of $\lambda_*$ and $\xi_*$

In this section, we shall choose the leading orders $\lambda_*(t)$, $\xi_*(t)$ of the parameter functions $\lambda(t)$ and $\xi(t)$. In Section 5.2, a linear theory for the inner problem will be developed, where approximately the following orthogonality conditions
$$\int_{\mathbb{R}^4}\mathcal{H}(\phi,\psi,\lambda,\xi)Z_j(y)dy = 0 \ \text{ for all } j=1,\cdots,5,\ t\in(0,T) \tag{4.1}$$
are required to guarantee the existence of inner solution $\phi$ with proper space-time decay. Here $Z_j$ are the kernel functions (c.f. (2.3)) of the linearized operator $L_0$ defined in (2.2). Basically, the scaling and translation parameters $\lambda(t)$ and $\xi(t)$ at main order will be derived from the orthogonality conditions (4.1).

Recall that
$$\begin{aligned}\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t) := {}& 3\lambda U^2(y)[\Psi_0+\psi+Z^*](\lambda y+\xi,t)+\lambda\left[\dot\lambda(\nabla_y\phi\cdot y+\phi)+\nabla_y\phi\cdot\dot\xi\right]\\ &+\lambda^3\mathcal{N}(\mathtt{w})+\lambda^3\mathcal{K}[\lambda,\xi].\end{aligned}$$
We single out the leading term $\mathcal{H}_*$ in $\mathcal{H}$ to derive $\lambda_*$ and $\xi_*$ and define
$$\begin{aligned}\mathcal{H}_*[\lambda,\xi,\Psi^*] :={}& 3\lambda U^2(y)[\Psi_0+\Psi^*](\lambda y+\xi,t)+\lambda^3\mathcal{K}[\lambda,\xi]\\ ={}& 3\lambda U^2(y)[\Psi_0+\Psi^*](\lambda y+\xi,t)+\frac{2\alpha_0\lambda(t)\dot\lambda(t)}{(1+|y|^2)^2}+\lambda(t)\nabla U(y)\cdot\dot\xi(t)\\ &-\frac{\alpha_0\lambda^2(t)}{(1+|y|^2)^{3/2}}\int_{-T}^t\dot\lambda(s)\left[-\zeta k_{\zeta\zeta}(\zeta,t-s)+k_\zeta(\zeta,t-s)\right]ds\\ &-\alpha_0\lambda^3(t)\left[\frac{y\cdot\dot\xi-\dot\lambda(t)}{(1+|y|^2)^{1/2}}\right]\int_{-T}^t\dot\lambda(s)k_\zeta(\zeta,t-s)ds,\end{aligned}$$

where $\Psi^* = \psi + Z^*$. The contribution of the remaining terms in $\mathcal{H} - \mathcal{H}_*$ in the orthogonality conditions turns out to be negligible compared to the leading term $\mathcal{H}_*$. This shall be dealt with in Section 6 when we finally solve the inner–outer gluing system.

For $\ell = 1, \cdots, 4$,

$$\int_{\mathbb{R}^4} \mathcal{H}_*[\lambda, \xi, \Psi^*] Z_\ell(y) dy = 0$$

imply that

$$\dot{\xi}_\ell = o(1),$$

where $o(1) \to 0$ as $t \nearrow T$. So the choice of $\xi(t)$ at main order is

$$\xi(t) = q,$$

where $q$ is a prescribed point in $\Omega$.

In order to get the reduced equation for $\lambda(t)$ from

$$\int_{\mathbb{R}^4} \mathcal{H}_*[\lambda, \xi, \Psi^*] Z_5(y) dy = 0,$$

we first evaluate

$$\begin{aligned}\int_{\mathbb{R}^4} \mathcal{R}[\lambda] Z_5(y) dy = &\frac{\alpha_0}{\lambda(t)} \int_{\mathbb{R}^4} \frac{Z_5(y)}{(1+|y|^2)^{3/2}} \left( \int_{-T}^{t} \dot{\lambda}(s) [k_\zeta(\zeta, t-s) - \zeta k_{\zeta\zeta}(\zeta, t-s)] ds \right) dy \\ &- \alpha_0 \dot{\lambda}(t) \int_{\mathbb{R}^4} \frac{Z_5(y)}{(1+|y|^2)^{1/2}} \left( \int_{-T}^{t} \dot{\lambda}(s) k_\zeta(\zeta, t-s) ds \right) dy.\end{aligned}$$

Let

$$\Upsilon = \frac{\zeta^2}{t-s} = \frac{\lambda^2(t)(1+|y|^2)}{t-s}, \ \tau = \frac{\lambda^2(t)}{t-s} \tag{4.2}$$

and $K(\Upsilon) = \frac{1-e^{-\frac{\Upsilon}{4}}}{\Upsilon}$. Then, recalling from (2.6), we have

$$k_\zeta(\zeta, t-s) - \zeta k_{\zeta\zeta}(\zeta, t-s) = -4 \left( \frac{\Upsilon}{t-s} \right)^{3/2} K_{\Upsilon\Upsilon}(\Upsilon)$$

and also

$$k_\zeta(\zeta, t-s) = -\frac{2}{\zeta^3} + \frac{e^{-\frac{\zeta^2}{4(t-s)}}}{2\zeta(t-s)} + \frac{2e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^3} = \frac{2\sqrt{\Upsilon}}{(t-s)^{3/2}} K_\Upsilon(\Upsilon).$$

Therefore, we obtain

$$\begin{aligned}\int_{\mathbb{R}^4} \mathcal{R}[\lambda] Z_5(y) dy = &- \frac{4\alpha_0}{\lambda^2(t)} \int_{\mathbb{R}^4} \frac{Z_5(y)}{(1+|y|^2)^2} \left( \int_{-T}^{t} \frac{\dot{\lambda}(s)}{t-s} \Upsilon^2 K_{\Upsilon\Upsilon}(\Upsilon) ds \right) dy \\ &- \frac{2\alpha_0 \dot{\lambda}(t)}{\lambda(t)} \int_{\mathbb{R}^4} \frac{Z_5(y)}{1+|y|^2} \left( \int_{-T}^{t} \frac{\dot{\lambda}(s)}{t-s} \Upsilon K_\Upsilon(\Upsilon) ds \right) dy.\end{aligned} \tag{4.3}$$

Expand $Z^*(\lambda y + \xi, t)$ and $\psi(\lambda y + \xi, t)$ at the point $q$

$$Z^*(\lambda y + \xi, t) = Z_0^*(q) + o(1), \ \psi(\lambda y + \xi, t) = \psi(q, 0) + o(1).$$

On the other hand, from (2.7) and (4.2), we get

$$\int_{\mathbb{R}^4} 3\lambda(t) U^2(y) Z_5(y) \Psi_0(\rho, t) dy = -3\alpha_0 \lambda(t) \int_{\mathbb{R}^4} U^2(y) Z_5(y) \left( \int_{-T}^{t} \frac{\dot{\lambda}(s)}{t-s} K(\Upsilon) ds \right) dy. \tag{4.4}$$

Then, the orthogonality condition

$$\int_{\mathbb{R}^4} \mathcal{H}_*[\lambda, \xi, \Psi^*] Z_5(y) dy = 0$$

yields

$$\int_{\mathbb{R}^4}\left(3U^2(y)[\Psi_0(\rho,t)+\psi(q,0)+Z_0^*(q)]+\lambda^2(t)\mathcal{K}[\lambda,\xi]\right)Z_5(y)dy+o(1)=0. \tag{4.5}$$

By (4.5), (2.9), (4.3), (4.4) and direct computations, we obtain

$$\begin{aligned}&4\alpha_0\int_{\mathbb{R}^4}\frac{Z_5(y)}{(1+|y|^2)^2}\left(\int_{-T}^t\frac{\dot\lambda(s)}{t-s}\Upsilon^2K_{\Upsilon\Upsilon}ds\right)dy-3\alpha_0\int_{\mathbb{R}^4}U^2(y)Z_5(y)\left(\int_{-T}^t\frac{\dot\lambda(s)}{t-s}K(\Upsilon)ds\right)dy\\&+2\alpha_0\dot\lambda(t)\lambda(t)\int_{\mathbb{R}^4}\frac{Z_5(y)}{1+|y|^2}\left(\int_{-T}^t\frac{\dot\lambda(s)}{t-s}\Upsilon K_\Upsilon(\Upsilon)ds\right)dy+2\alpha_0\dot\lambda(t)\int_{\mathbb{R}^4}\frac{Z_5(y)}{(1+|y|^2)^2}dy\\&+3[Z_0^*(q)+\psi(q,0)]\int_{\mathbb{R}^4}U^2(y)Z_5(y)dy+o(1)=0.\end{aligned} \tag{4.6}$$

The scaling parameter $\lambda(t)$ should be decreasing to 0 as $t\nearrow T$ so that a blow-up solution can be constructed. So we impose

$$\dot\lambda(t)=o(1)\ \text{ as } t\nearrow T.$$

Then (4.6) becomes

$$\begin{aligned}&4\alpha_0\int_{\mathbb{R}^4}\frac{Z_5(y)}{(1+|y|^2)^2}\left(\int_{-T}^t\frac{\dot\lambda(s)}{t-s}\Upsilon^2K_{\Upsilon\Upsilon}ds\right)dy-3\alpha_0\int_{\mathbb{R}^4}U^2(y)Z_5(y)\left(\int_{-T}^t\frac{\dot\lambda(s)}{t-s}K(\Upsilon)ds\right)dy\\&=-3[Z_0^*(q)+\psi(q,0)]\int_{\mathbb{R}^4}U^2(y)Z_5(y)dy+o(1).\end{aligned} \tag{4.7}$$

We define

$$\begin{aligned}&4\alpha_0\int_{\mathbb{R}^4}\frac{Z_5(y)}{(1+|y|^2)^2}\left(\int_{-T}^t\frac{\dot\lambda(s)}{t-s}\Upsilon^2K_{\Upsilon\Upsilon}ds\right)dy-3\alpha_0\int_{\mathbb{R}^4}U^2(y)Z_5(y)\left(\int_{-T}^t\frac{\dot\lambda(s)}{t-s}K(\Upsilon)ds\right)dy\\&:=\int_{-T}^t\frac{\dot\lambda(s)}{t-s}\Gamma\left(\frac{\lambda^2(t)}{t-s}\right)ds\end{aligned}$$

with

$$\Gamma(\tau):=\alpha_0|\mathbb{S}^3|\int_0^\infty\left(\frac{4Z_5(y)|y|^3}{(1+|y|^2)^2}\Upsilon^2K_{\Upsilon\Upsilon}(\Upsilon)-3U^2(y)Z_5(y)|y|^3K(\Upsilon)\right)\Big|_{\Upsilon=\tau(1+|y|^2)}d|y|,$$

where $|\mathbb{S}^3|$ is the area of the unit sphere $\mathbb{S}^3$. By the definition of $U(y)$ and $Z_5(y)$ as in (2.1) and (2.3) respectively, we compute

$$\Gamma(\tau)=\begin{cases}c_*+O(\tau), & \text{for } \tau<1,\\ O\left(\frac{1}{\tau}\right), & \text{for } \tau>1,\end{cases}$$

where $c_*>0$ is a constant. Therefore, (4.7) reduces to

$$c_*\int_{-T}^{t-\lambda^2(t)}\frac{\dot\lambda(s)}{t-s}ds=-3c_0[Z_0^*(q)+\psi(q,0)]+o(1), \tag{4.8}$$

where

$$c_0:=\int_{\mathbb{R}^4}U^2(y)Z_5(y)dy<0.$$

Since $\lambda(t)$ decreases to 0 as $t\nearrow T$, we impose

$$a_*:=Z_0^*(q)+\psi(q,0)<0.$$

Now we claim that a good choice of $\lambda(t)$ at main order is

$$\dot\lambda(t)=-\frac{c}{|\log(T-t)|^2}, \tag{4.9}$$

where $c > 0$ is a constant to be determined later. Indeed, we get by substituting

$$\begin{aligned}\int_{-T}^{t-\lambda^2(t)} \frac{\dot\lambda(s)}{t-s} ds &= \int_{-T}^{t-(T-t)} \frac{\dot\lambda(s)}{t-s} ds + \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t)}{t-s} ds - \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t)-\dot\lambda(s)}{t-s} ds \\ &= \int_{-T}^{t-(T-t)} \frac{\dot\lambda(s)}{t-s} ds + \dot\lambda(t)(\log(T-t) - 2\log\lambda(t)) - \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t)-\dot\lambda(s)}{t-s} ds \\ &\approx \int_{-T}^{t} \frac{\dot\lambda(s)}{T-s} ds - \dot\lambda(t)\log(T-t) := \beta(t).\end{aligned}$$

By (4.9), we then get

$$\log(T-t)\frac{d\beta}{dt}(t) = \frac{d}{dt}\left(-\log^2(T-t)\dot\lambda(t)\right) = 0,$$

which means $\beta(t)$ is a constant. Thus, equation (4.8) can be approximately solved for

$$\dot\lambda(t) = -\frac{c}{|\log(T-t)|^2}$$

with the constant $c$ chosen as

$$-c\int_{-T}^{T} \frac{ds}{(T-s)|\log(T-s)|^2} = \kappa_*,$$

where $\kappa_* := -\frac{3c_0 a_*}{c_*}$. At main order, we obtain

$$\dot\lambda(t) = \kappa_*\dot\lambda_*(t)$$

with

$$\dot\lambda_*(t) = -\frac{|\log T|}{|\log(T-t)|^2}.$$

By imposing $\lambda_*(T) = 0$, we obtain

$$\lambda_*(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2}(1+o(1)) \quad \text{as} \quad t \nearrow T.$$

## 5. Linear theories for inner and outer problems

5.1. **Linear theory for the outer problem.** In order to solve the outer problem (3.4), a linear theory for the associated linear problem is needed. We consider

$$\begin{cases} \psi_t = \Delta\psi + f, & \text{in } \Omega\times(0,T), \\ \psi = 0, & \text{on } \partial\Omega\times(0,T), \\ \psi(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{5.1}$$

where the non-homogeneous term $f$ in (5.1) is assumed to be bounded with respect to the weights appearing in the outer problem (3.4). Define the weights

$$\begin{cases} \varrho_1 := \lambda_*^{\nu-3}(t)R^{-2-\alpha}(t)\chi_{\{|x-\xi(t)|\le 2\lambda_* R\}} \\ \varrho_2 := \frac{\lambda_*^{\nu_2}}{|x-\xi(t)|^2}\chi_{\{|x-\xi(t)|\ge \lambda_* R\}} \\ \varrho_3 := 1 \end{cases} \tag{5.2}$$

where we choose $R(t) = \lambda_*^{-\beta}(t)$ for $\beta \in (0, 1/2)$ throughout the paper. We define the norms

$$\|f\|_{**} := \sup_{(x,t)\in\Omega\times(0,T)} \left(\sum_{i=1}^{3}\varrho_i(x,t)\right)^{-1} |f(x,t)| \tag{5.3}$$

$$
\begin{aligned}
\|\psi\|_* :=& \frac{\lambda_*^{1-\nu}(0)R^{\alpha}(0)}{|\log T|}\|\psi\|_{L^\infty(\Omega\times(0,T))} + \frac{\lambda_*^{2-\nu}(0)R^{1+\alpha}(0)}{|\log T|}\|\nabla\psi\|_{L^\infty(\Omega\times(0,T))}\\
&+ \sup_{(x,t)\in\Omega\times(0,T)}\left[\frac{\lambda_*^{1-\nu}(t)R^{\alpha}(t)}{|\log(T-t)|}\,|\psi(x,t)-\psi(x,T)|\right]\\
&+ \sup_{(x,t)\in\Omega\times(0,T)}\left[\frac{\lambda_*^{2-\nu}(t)R^{1+\alpha}(t)|}{|\log(T-t)|}\,|\nabla\psi(x,t)-\nabla\psi(x,T)|\right]\\
&+ \sup_{\Omega\times I_T}\frac{\lambda_*^{2\gamma+1-\nu}(t_2)R^{2\gamma+\alpha}(t_2)}{(t_2-t_1)^{\gamma}}|\psi(x,t_2)-\psi(x,t_1)|,
\end{aligned}
\tag{5.4}
$$

where $\nu,\alpha,\gamma\in(0,1)$, and the last supremum is taken over

$$
\Omega\times I_T = \left\{(x,t_1,t_2):\ x\in\Omega,\ 0\le t_1\le t_2\le T,\ t_2-t_1\le\frac{1}{10}(T-t_2)\right\}.
$$

For problem (5.1), we have the following estimates.

**Proposition 5.1.** *Let $\psi$ be the solution to problem (5.1) with $\|f\|_{**}<+\infty$. Then it holds that*

$$
\|\psi\|_*\lesssim\|f\|_{**}.
$$

Proposition 5.1 is established by the following three lemmas with different right hand sides.

**Lemma 5.1.** *Let $\psi$ solve problem (5.1) with right hand side*

$$
|f(x,t)|\lesssim\lambda_*^{\nu-3}(t)R^{-2-\alpha}(t)\chi_{\{|x-\xi(t)|\le 2\lambda_* R\}}
$$

*with $\alpha,\nu\in(0,1)$. Then it holds that*

$$
\begin{gathered}
|\psi(x,t)|\lesssim\lambda_*^{\nu-1}(0)R^{-\alpha}(0)|\log T|,\\
|\psi(x,t)-\psi(x,T)|\lesssim\lambda_*^{\nu-1}(t)R^{-\alpha}(t)|\log(T-t)|,\\
|\nabla\psi(x,t)|\lesssim\lambda_*^{\nu-2}(0)R^{-1-\alpha}(0)|\log T|,\\
|\nabla\psi(x,t)-\nabla\psi(x,T)|\lesssim\lambda_*^{\nu-2}(t)R^{-1-\alpha}(t)|\log(T-t)|,
\end{gathered}
$$

*and*

$$
|\psi(x,t_2)-\psi(x,t_1)|\lesssim\lambda_*^{\nu+\mu-3}(t_2)R^{\mu-2-\alpha}(t_2)(t_2-t_1)^{1-\mu/2},
$$

*where $0\le t_1\le t_2\le T$ with $t_2-t_1\le\frac{1}{10}(T-t_2)$ and $\mu\in(0,1)$.*

**Lemma 5.2.** *Let $\psi$ solve problem (5.1) with right hand side*

$$
|f(x,t)|\lesssim\frac{\lambda_*^{\nu_2}}{|x-\xi(t)|^2}\chi_{\{|x-\xi(t)|\ge\lambda_* R\}},
$$

*where $\nu_2\in(0,1)$. Then it holds that*

$$
\begin{gathered}
|\psi(x,t)|\lesssim T^{\nu_2}|\log T|^{-\nu_2},\\
|\psi(x,t)-\psi(x,T)|\lesssim(T-t)^{\nu_2}|\log T|^{\nu_2}|\log(T-t)|^{-2\nu_2},\\
|\nabla\psi(x,t)|\lesssim\frac{T^{\nu_2-1}|\log T|^{1-\nu_2}}{R(T)},\\
|\nabla\psi(x,t)-\nabla\psi(x,T)|\lesssim\frac{\lambda_*^{\nu_2-1}(t)|\log(T-t)|}{R(t)},
\end{gathered}
$$

*and*

$$
|\psi(x,t_2)-\psi(x,t_1)|\lesssim\frac{\lambda_*^{\nu_2}(t)|\log(T-t)|}{(\lambda_*(t)R(t))^{2\gamma}}(t_2-t_1)^{\gamma},
$$

*where $0\le t_1\le t_2\le T$ with $t_2-t_1\le\frac{1}{10}(T-t_2)$ and $\gamma\in(0,1)$.*

**Lemma 5.3.** *Let $\psi$ solve problem (5.1) with right hand side*
$$|f(x,t)| \lesssim 1.$$
*Then it holds that*
$$|\psi(x,t)| \lesssim t,$$
$$|\psi(x,t)-\psi(x,T)| \lesssim (T-t)|\log(T-t)|,$$
$$|\nabla\psi(x,t)| \lesssim T^{1/2},$$
$$|\nabla\psi(x,t)-\nabla\psi(x,T)| \lesssim (T-t)^{1/2},$$
*and*
$$|\psi(x,t_2)-\psi(x,t_1)| \lesssim (t_2-t_1)|\log(t_2-t_1)|,$$
*where $0 \le t_1 \le t_2 \le T$ with $t_2 - t_1 \le \frac{1}{10}(T-t_2)$.*

Proposition 5.1 is a direct consequence of Lemma 5.1, Lemma 5.2 and Lemma 5.3, and the proofs of Lemma 5.1, Lemma 5.2 and Lemma 5.3 are achieved by using Duhamel's formula similarly as in [6]. Here we omit the details.

5.2. **Linear theory for the inner problem.** To solve the inner problem (3.1), we develop a linear theory for the associated linear problem of the inner problem under certain topology. We consider the associated linear problem
$$\lambda^2\phi_t = \Delta_y\phi + 3U^2(y)\phi + h(y,t) \ \text{ in } \ \mathcal{D}_{2R}. \tag{5.5}$$
Recall that the linearized operator $L_0 = \Delta + 3U^2$ has only one positive eigenvalue $\mu_0$ such that
$$L_0(Z_0) = \mu_0 Z_0, \ Z_0 \in L^\infty(\mathbb{R}^4),$$
where the corresponding eigenfunction $Z_0$ is radially symmetric with the asymptotic behavior
$$Z_0(y) \sim |y|^{-3/2}e^{-\sqrt{\mu_0}|y|} \ \text{ as } \ |y| \to +\infty.$$
Multiplying equation (5.5) by $Z_0$ and integrating over $\mathbb{R}^4$, we get
$$\lambda^2(t)\dot{p}(t) - \mu_0 p(t) = q(t),$$
where
$$p(t) = \int_{\mathbb{R}^4}\phi(y,t)Z_0(y)dy \ \text{ and } \ q(t) = \int_{\mathbb{R}^4} h(y,t)Z_0(y)dy.$$
Then we get
$$p(t) = e^{\int_0^t \mu_0\lambda^{-2}(r)dr}\left(p(0) + \int_0^t q(\eta)\lambda^{-2}(\eta)e^{-\int_0^\eta \mu_0\lambda^{-2}(r)dr}d\eta\right).$$
In order to get a decaying solution, a initial condition
$$p(0) = -\int_0^T q(\eta)\lambda^{-2}(\eta)e^{-\int_0^\eta \mu_0\lambda^{-2}(r)dr}d\eta$$
is needed. The above formal argument suggests that a linear constraint should be imposed on the initial value $\phi(y,0)$. Therefore, we consider the associated linear Cauchy problem of the inner problem (3.1)
$$\begin{cases} \lambda^2\phi_t = \Delta_y\phi + 3U^2(y)\phi + h(y,t), & \text{in } \mathcal{D}_{2R}, \\ \phi(y,0) = e_0 Z_0(y), & \text{in } B_{2R(0)}, \end{cases} \tag{5.6}$$
where $R = R(t) = \lambda_*^{-\beta}(t)$ for $\beta \in (0,1/2)$. On the other hand, the parabolic operator $-\lambda^2\partial_t + L_0$ is certainly not invertible since all the time independent elements in the 5 dimensional kernel of $L_0$ (see (2.3)) also belong to the kernel of $-\lambda^2\partial_t + L_0$. In order to construct solution to (5.6) with suitable space-time decay, we expect some orthogonality conditions to hold. We shall construct a solution $(\phi, e_0)$ to problem (5.6) under the orthogonality conditions
$$\int_{B_{2R}} h(y,t)Z_\ell(y)dy = 0 \ \text{ for } \ \ell = 1,\cdots,5, \ t \in (0,T). \tag{5.7}$$

Define

$$\|h\|_{\nu,2+a} := \sup_{(y,t)\in\mathcal{D}_{2R}} \lambda_*^{-\nu}(t)(1+|y|^{2+a})\left[|h(y,t)|+(1+|y|)|\nabla h(y,t)|\right]. \tag{5.8}$$

The construction of such solution is achieved by decomposing the equation into different spherical harmonic modes. Consider an orthonormal basis $\{\Theta_i\}_{i=0}^{\infty}$ made up of spherical harmonics in $L^2(\mathbb{S}^3)$, i.e.

$$\Delta_{\mathbb{S}^3}\Theta_i + \lambda_i\Theta_i = 0 \ \text{ in } \ \mathbb{S}^3$$

with $0=\lambda_0<\lambda_1=\cdots=\lambda_4=3<\lambda_5\le\cdots$. More precisely, $\Theta_0(y)=a_0,\ \Theta_i(y)=a_1 y_i,\ i=1,\cdots,4$ for two constants $a_0$, $a_1$ and

$$\lambda_i = i(2+i) \ \text{ with multiplicity } \ \frac{(3+i)!}{6i!} \ \text{ for } \ i\ge 0.$$

For $h\in L^2(\mathcal{D}_{2R})$, we decompose

$$h(y,t)=\sum_{j=0}^{\infty} h_j(r,t)\Theta_j(y/r),\ r=|y|,\ h_j(r,t)=\int_{\mathbb{S}^3} h(r\theta,t)\Theta_j(\theta)d\theta$$

and write $h=h^0+h^1+h^\perp$ with

$$h^0=h_0(r,t),\ h^1=\sum_{j=1}^{4} h_j(r,t)\Theta_j,\ h^\perp=\sum_{j=5}^{\infty} h_j(r,t)\Theta_j.$$

Also, we decompose $\phi=\phi^0+\phi^1+\phi^\perp$ in a similar form. Then looking for a solution to problem (5.6) is equivalent to finding the pairs $(\phi^0,h^0)$, $(\phi^1,h^1)$, $(\phi^\perp,h^\perp)$ in each mode.

The key linear result for the inner problem is stated as follows.

**Proposition 5.2.** *Let the constants $a,\nu,\nu_1\in(0,1)$, $a_1\in(1,2)$. For $T>0$ sufficiently small and any $h(y,t)$ satisfying $\|h\|_{\nu,2+a}<+\infty$, $\|h^1\|_{\nu_1,2+a_1}<+\infty$ and the orthogonality conditions (5.7), there exists a pair $(\phi,e_0)$ solving (5.6), and $(\phi,e_0)=(\phi[h],e_0[h])$ defines a linear operator of $h(y,t)$ that satisfies the estimates*

$$\begin{aligned}|\phi(y,t)|+(1+|y|)|\nabla\phi(y,t)| \lesssim{}& \frac{\lambda_*^{\nu}(t)R^{2-\frac{a}{2}}\sqrt{\log R}}{1+|y|^2}\|h^0\|_{\nu,2+a}\min\{1,R^{2-\frac{a}{2}}\sqrt{\log R}|y|^{-2}\}\\ &+\frac{\lambda_*^{\nu_1}(t)}{1+|y|^{a_1}}\|h^1\|_{\nu_1,2+a_1}+\frac{\lambda_*^{\nu}(t)}{1+|y|^{a}}\|h^\perp\|_{\nu,2+a}\end{aligned}$$

*and*

$$|e_0[h]|\lesssim\|h\|_{\nu,2+a}.$$

The proof of Proposition 5.2 can be carried out in a similar manner as in [4] and [8]. Note that the restriction $a_1\in(1,2)$ is required to guarantee the integrability in the blow-up argument at translation mode. We leave the proof to the interested reader.

**Remark 5.1.** *If we define the norm*

$$\|\phi^0\|_{*,\nu,a} := \sup_{(y,t)\in\mathcal{D}_{2R}} \lambda_*^{-\nu}(t)R^{\frac{a}{2}-2}(\log R)^{-\frac{1}{2}}(1+|y|^2)\left[|\phi^0(y,t)|+(1+|y|)|\nabla\phi^0(y,t)|\right], \tag{5.9}$$

*then Proposition 5.2 implies that*

$$\|\phi^0\|_{*,\nu,a}\lesssim\|h^0\|_{\nu,2+a}.$$

*We shall use the norm (5.9) when we solve the inner–outer gluing system.*

## 6. Solving the inner–outer gluing system

In this section, we shall solve the inner–outer gluing system by the linear theories developed in Section 5, and the Schauder fixed point theorem. Our aim is to find a solution $(\phi, \psi, \lambda, \xi)$ to the inner–outer gluing system in Section 3 such that the desired blow-up solution is constructed. We shall solve the inner–outer gluing system in the function space $\mathcal{X}$ defined in (6.52). We first make some assumptions about the parameter functions. Write

$$\lambda_*(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2}$$

and assume that for some numbers $c_1, c_2 > 0$,

$$c_1|\dot\lambda_*(t)| \le |\dot\lambda(t)| \le c_2|\dot\lambda_*(t)| \quad \text{for all} \quad t \in (0,T).$$

Throughout the paper, we take $R(t) = \lambda_*^{-\beta}(t)$ for $\beta \in (0, 1/2)$.

In Section 6.1 and Section 6.2, for given $\|\phi^0\|_{*,\nu,a}$, $\|\phi^1\|_{\nu_1,a_1}$, $\|\phi^\perp\|_{\nu,a}$, $\|\psi\|_*$, $\|Z^*\|_\infty$, $\|\lambda\|_F$, $\|\xi\|_G$ bounded, we shall first estimate right hand sides $\mathcal{G}(\phi,\psi,\lambda,\xi)$ and $\mathcal{H}(\phi,\psi,\lambda,\xi)$ in the inner and outer problems. Here the above norms are defined in (5.9), (5.8), (5.4), (6.50) and (6.51).

### 6.1. The outer problem: estimates of $\mathcal{G}$.

Consider the outer problem

$$\psi_t = \Delta\psi + \mathcal{G}(\phi,\psi,\lambda,\xi) \quad \text{in } \Omega \times (0,T)$$

where

$$\begin{aligned}\mathcal{G}(\phi,\psi,\lambda,\xi) := {} & 3\lambda^{-2}(1-\eta_R)U^2(y)(\Psi_0 + \psi + Z^*) \\ & + \lambda^{-3}\left[(\Delta_y\eta_R)\phi + 2\nabla_y\eta_R\cdot\nabla_y\phi - \lambda^2\phi\partial_t\eta_R\right] \\ & + (1-\eta_R)\mathcal{K}[\lambda,\xi] + (1-\eta_R)\mathcal{N}(\mathtt{w})\end{aligned}$$

with $\mathcal{K}[\lambda,\xi]$ and $\mathcal{N}(\mathtt{w})$ defined in (2.9) and (3.3) respectively.

In order to apply the linear theory Proposition 5.1, we estimate all the terms in $\mathcal{G}(\phi,\psi,\lambda,\xi)$ in the $\|\cdot\|_{**}$-norm, defined in (5.3). Define

$$\mathcal{G}(\phi,\psi,\lambda,\xi) = g_1 + g_2 + g_3$$

with

$$\begin{aligned} g_1 &:= 3\lambda^{-2}(1-\eta_R)U^2(y)(\Psi_0+\psi+Z^*) \\ g_2 &:= \lambda^{-3}\left[(\Delta_y\eta_R)\phi + 2\nabla_y\eta_R\cdot\nabla_y\phi - \lambda^2\phi\partial_t\eta_R\right] \\ g_3 &:= (1-\eta_R)\mathcal{K}[\lambda,\xi] + (1-\eta_R)\mathcal{N}(\mathtt{w}).\end{aligned}$$

To estimate $g_1$, we first estimate the non-local correction $\Psi_0$ in (2.7).

**Estimates of $\Psi_0$**

Decompose

$$\Psi_0 = -\alpha_0\int_{-T}^{t}\dot\lambda(s)\frac{1-e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2}ds = -\alpha_0\left(\int_{-T}^{t-\frac{\zeta^2}{4}} + \int_{t-\frac{\zeta^2}{4}}^{t}\right)\dot\lambda(s)\frac{1-e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2}ds. \tag{6.1}$$

For the first integral above, we have

- For $T-t > \frac{\zeta^2}{4}$, we further decompose

$$\int_{-T}^{t-\frac{\zeta^2}{4}}\dot\lambda(s)\frac{1-e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2}ds = \left(\int_{-T}^{t-(T-t)} + \int_{t-(T-t)}^{t-\frac{\zeta^2}{4}}\right)\dot\lambda(s)\frac{1-e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2}ds.$$

Since $T-s<2(t-s)$ and $\frac{\zeta^2}{4(t-s)}<1$, we have

$$\begin{aligned}
&\int_{-T}^{t-(T-t)} \dot\lambda(s)\frac{1-e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2}ds \lesssim \int_{-T}^{t-(T-t)} \frac{|\dot\lambda(s)|}{T-s}ds\\
&\lesssim |\log T|\int_{-T}^{t-(T-t)} \frac{1}{(T-s)|\log(T-s)|^2}ds\\
&\lesssim |\log T|\left|\frac{1}{|\log 2(T-t)|}-\frac{1}{\log 2T}\right|\lesssim 1.
\end{aligned}\tag{6.2}$$

Similarly, for the second integral

$$\begin{aligned}
&\int_{t-(T-t)}^{t-\frac{\zeta^2}{4}} \dot\lambda(s)\frac{1-e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2}ds \lesssim \int_{t-(T-t)}^{t-\frac{\zeta^2}{4}} \frac{|\log T|}{(t-s)|\log(T-s)|^2}ds\\
&\lesssim \frac{|\log T|}{|\log(T-t)|^2}\left|\log\left(\frac{\zeta^2}{4}\right)-\log(T-t)\right|\lesssim |\dot\lambda|\left[\log(\rho^2+\lambda^2)+1\right].
\end{aligned}\tag{6.3}$$

- For $T-t<\frac{\zeta^2}{4}$, since $s<t-\frac{\zeta^2}{4}<t-(T-t)$, we evaluate

$$\int_{-T}^{t-\frac{\zeta^2}{4}} \frac{|\dot\lambda(s)|}{t-s}ds \lesssim \int_{-T}^{t-(T-t)} \frac{|\dot\lambda(s)|}{T-s}ds\lesssim 1.\tag{6.4}$$

Next we evaluate

$$\int_{t-\frac{\zeta^2}{4}}^{t} \dot\lambda(s)\frac{1-e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2}ds \lesssim \frac{1}{\zeta^2}\int_{t-\frac{\zeta^2}{4}}^{t}|\dot\lambda(s)|ds\lesssim 1.\tag{6.5}$$

Combining (6.1)–(6.5), we conclude that

$$|\Psi_0|\lesssim |\dot\lambda|\left[\log(\rho^2+\lambda^2)+1\right].\tag{6.6}$$

**Estimate of $g_1$.**

Since $\psi\in X_\psi$ defined in (6.47), we have

$$\begin{aligned}
g_1 &= 3\lambda^{-2}(1-\eta_R)U^2(y)(\Psi_0+\psi+Z^*)\\
&\lesssim \frac{R^{-2}(t)\lambda_*^{\nu-1}(0)R^{-\alpha}(0)|\log T|}{|x-\xi(t)|^2}\chi_{\{|x-\xi(t)|\ge\lambda_* R\}}\|\psi\|_*\\
&\quad+\frac{R^{-2}(t)}{|x-\xi(t)|^2}\chi_{\{|x-\xi(t)|\ge\lambda_* R\}}\|Z^*\|_\infty+\frac{R^{-2}(t)|\log(T-t)|}{|x-\xi(t)|^2}\chi_{\{|x-\xi(t)|\ge\lambda_* R\}}\|\dot\lambda\|_\infty.
\end{aligned}$$

So by the choice of the weight $\varrho_2$ as in (5.2), we get

$$\|g_1\|_{**}\lesssim T^{\epsilon_0}(\|\psi\|_*+\|Z^*\|_\infty+\|\dot\lambda\|_\infty+1)\tag{6.7}$$

provided

$$\begin{cases}\nu-1+\beta(2+\alpha)-\nu_2>0,\\ 2\beta-\nu_2>0.\end{cases}\tag{6.8}$$

Here $\epsilon_0$ is a small positive number.

**Estimate of $g_2$.**

Due to the cut-off, $g_2$ is supported in $\{(x,t)\in\Omega\times(0,T):\ \lambda_*R\le|x-\xi(t)|\le 2\lambda_*R\}$, and we compute

$$\begin{aligned}
g_2 &= \lambda^{-3}\left[(\Delta_y\eta_R)\phi+2\nabla_y\eta_R\cdot\nabla_y\phi-\lambda^2(\partial_t\eta_R)\phi\right]\\
&\lesssim \lambda_*^{\nu-3}R^{-2-a}\chi_{\{|x-\xi(t)|\sim\lambda_* R\}}\left(\|\phi^0\|_{*,\nu,a}+\|\phi^1\|_{\nu_1,a_1}+\|\phi^\perp\|_{\nu,a}\right)\\
&\lesssim R^{\alpha-a}\varrho_1\left(\|\phi^0\|_{*,\nu,a}+\|\phi^1\|_{\nu_1,a_1}+\|\phi^\perp\|_{\nu,a}\right).
\end{aligned}$$

So it follows that

$$\|g_2\|_{**} \lesssim T^{\epsilon_0} \left(\|\phi^0\|_{*,\nu,a} + \|\phi^1\|_{\nu_1,a_1} + \|\phi^\perp\|_{\nu,a}\right) \tag{6.9}$$

provided

$$0 < \alpha < a < 1. \tag{6.10}$$

Here $\epsilon_0$ is a small positive number.

**Estimate of $g_3$.**

We now estimate $g_3 = (1-\eta_R)\mathcal{K}[\lambda,\xi] + (1-\eta_R)\mathcal{N}(\mathtt{w})$. Recall from (2.9) that

$$\mathcal{K}[\lambda,\xi] = \frac{2\alpha_0\lambda^{-2}(t)\dot\lambda(t)}{(1+|y|^2)^2} + \lambda^{-2}(t)\nabla U(y)\cdot\dot\xi(t) - \mathcal{R}[\lambda].$$

We first estimate $\mathcal{R}[\lambda]$ defined in (2.8). By direct computations similar to the estimate of $\Psi_0$, we obtain

$$\mathcal{R}[\lambda] \lesssim |\dot\lambda| \left(\frac{1}{(T-t)(1+|y|^2)} + \frac{\lambda^2}{(T-t)^2} + \frac{1}{\lambda^2(1+|y|^2)^2} + \frac{\lambda(y\cdot\dot\xi+\dot\lambda)}{T-t} + \frac{y\cdot\dot\xi+\dot\lambda}{\lambda(1+|y|^2)}\right). \tag{6.11}$$

We next evaluate the first term $(1-\eta_R)\mathcal{K}[\lambda,\xi]$ in $g_3$ by using (6.11). Thanks to the cut-off $(1-\eta_R)$, we get

$$\begin{aligned}(1-\eta_R)\mathcal{K}[\lambda,\xi] &= (1-\eta_R)\left[\frac{2\alpha_0\lambda^{-2}(t)\dot\lambda(t)}{(1+|y|^2)^2} + \lambda^{-2}(t)\nabla U(y)\cdot\dot\xi(t) - \mathcal{R}[\lambda]\right]\\ &\lesssim R^{-2}\lambda_*^{-\nu_2}\|\dot\lambda\|_\infty\varrho_2 + \lambda_*^{\upsilon-\nu_2}R^{-1}\varrho_2\|\xi\|_G + \lambda_*^{1-\nu_2}\varrho_2 + T^{\epsilon_0}\|\dot\lambda\|_\infty\varrho_3,\end{aligned}$$

where the $\|\cdot\|_G$-norm is defined in (6.51). We then see that if

$$\begin{cases}\nu_2 - 2\beta < 0,\\ \beta + \upsilon - \nu_2 > 0,\\ \nu_2 - 1 < 0,\end{cases} \tag{6.12}$$

then

$$\|(1-\eta_R)\mathcal{K}[\lambda,\xi]\|_{**} \lesssim T^{\epsilon_0}\left(\|\dot\lambda\|_\infty + \|\xi\|_G + 1\right) \tag{6.13}$$

for some $\epsilon_0 > 0$.

For the nonlinear terms, we have

$$\begin{aligned}(1-\eta_R)\mathcal{N}(\mathtt{w}) &= (1-\eta_R)\left[(U_{\lambda,\xi}+\Psi_0+\mathtt{w})^3 - U_{\lambda,\xi}^3 - 3U_{\lambda,\xi}^2(\Psi_0+\mathtt{w})\right]\\ &\lesssim (1-\eta_R)U_{\lambda,\xi}(\Psi_0+\mathtt{w})^2\\ &\lesssim \frac{\lambda^{-1}(1-\eta_R)}{1+|y|^2}\left[(\lambda^{-1}\eta_R\phi)^2 + \Psi_0^2 + \psi^2 + (Z^*)^2\right]\\ &\lesssim \left(\lambda_*^\nu R^{\alpha-a}\log R\|\phi^0\|_{*,\nu,a}^2 + \lambda_*^{2\nu_1-\nu}R^{\alpha-2a_1}\|\phi^1\|_{\nu_1,a_1}^2 + \lambda_*^\nu R^{\alpha-2a}\|\phi^\perp\|_{\nu,a}^2\right)\varrho_1\\ &\quad + \lambda_*^{1-\nu_2}(t)\lambda_*^{2\nu-2}(0)R^{-2\alpha}(0)|\log T|^2\varrho_2\|\psi\|_*^2 + \lambda^{1-\nu_2}\varrho_2\|Z^*\|_\infty^2\\ &\quad + |\log(T-t)|^2|\dot\lambda_*|^2\lambda_*^{1-\nu_2}\varrho_2.\end{aligned}$$

Therefore, we obtain that for $\epsilon_0 > 0$

$$\|(1-\eta_R)\mathcal{N}(\mathtt{w})\|_{**} \lesssim T^{\epsilon_0}\left(\|\phi^0\|_{*,\nu,a}^2 + \|\phi^1\|_{\nu_1,a_1}^2 + \|\phi^\perp\|_{\nu,a}^2 + \|\psi\|_*^2 + \|Z^*\|_\infty^2 + \|\dot\lambda\|_\infty + 1\right) \tag{6.14}$$

provided

$$\begin{cases}2\nu_1 - \nu + \beta(2a_1-\alpha) > 0,\\ \nu_2 < 1,\\ 2\nu - \nu_2 - 1 + 2\alpha\beta > 0.\end{cases} \tag{6.15}$$

Collecting (6.7), (6.8), (6.9), (6.10), (6.13), (6.12), (6.14) and (6.15), we conclude that for a fixed number $\epsilon_0 > 0$

$$\|\mathcal{G}\|_{**} \lesssim T^{\epsilon_0}\left(\|\psi\|_* + \|Z^*\|_\infty + \|\phi^0\|_{*,\nu,a} + \|\phi^1\|_{\nu_1,a_1} + \|\phi^\perp\|_{\nu,a} + \|\lambda\|_\infty + \|\xi\|_G + 1\right) \tag{6.16}$$

if the parameters $\beta, a, a_1, \alpha, \nu, \nu_1, \nu_2$ are chosen in the following range

$$\begin{cases} \nu - 1 + \beta(2+\alpha) - \nu_2 > 0, \\ 2\beta - \nu_2 > 0, \\ 0 < \alpha < a < 1, \\ \beta + \upsilon - \nu_2 > 0, \\ 2\nu_1 - \nu + \beta(2a_1 - \alpha) > 0, \\ \nu_2 < 1, \\ 2\nu - \nu_2 - 1 + 2\alpha\beta > 0. \end{cases} \tag{6.17}$$

6.2. **The inner problem: estimate of $\mathcal{H}$.** Consider the inner problem

$$\lambda^2\phi_t = \Delta_y\phi + 3U^2(y)\phi + \mathcal{H}(\phi,\psi,\lambda,\xi) \ \text{ in } \mathcal{D}_{2R}$$

where

$$\begin{aligned} \mathcal{H}(\phi,\psi,\lambda,\xi)(y,t) := {} & 3\lambda U^2(y)[\Psi_0 + \psi + Z^*](\lambda y + \xi, t) \\ & + \lambda\left[\dot\lambda(\nabla_y\phi\cdot y + \phi) + \nabla_y\phi\cdot\dot\xi\right] \\ & + \lambda^3\mathcal{N}(\mathtt{w}) + \lambda^3\mathcal{K}[\lambda,\xi] \end{aligned}$$

with $\mathcal{N}(\mathtt{w})$ and $\mathcal{K}[\lambda,\xi]$ defined in (3.3) and (2.9).

From the linear theory in Section 5.2, we know that for $\mathcal{H} = \mathcal{H}^0 + \mathcal{H}^1 + \mathcal{H}^\perp$ satisfying

$$\|\mathcal{H}^0\|_{\nu,2+a},\ \|\mathcal{H}^1\|_{\nu_1,2+a_1},\ \|\mathcal{H}^\perp\|_{\nu,2+a} < +\infty,$$

there exists a solution $(\phi^0, \phi^1, \phi^\perp, c^0, c^\ell)$ $(\ell = 1, \cdots, 4)$ solving the projected inner problems

$$\begin{cases} \lambda^2\phi^0_t = \Delta_y\phi^0 + 3U^2(y)\phi^0 + \mathcal{H}^0(\phi,\psi,\lambda,\xi) + c^0 Z_5 & \text{in } \mathcal{D}_{2R}, \\ \phi^0(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{6.18}$$

$$\begin{cases} \lambda^2\phi^1_t = \Delta_y\phi^1 + 3U^2(y)\phi^1 + \mathcal{H}^1(\phi,\psi,\lambda,\xi) + \sum\limits_{\ell=1}^4 c^\ell Z_\ell & \text{in } \mathcal{D}_{2R}, \\ \phi^1(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{6.19}$$

$$\begin{cases} \lambda^2\phi^\perp_t = \Delta_y\phi^\perp + 3U^2(y)\phi^\perp + \mathcal{H}^\perp(\phi,\psi,\lambda,\xi) & \text{in } \mathcal{D}_{2R}, \\ \phi^\perp(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{6.20}$$

and the inner solution $\phi[\mathcal{H}] = \phi^0[\mathcal{H}^0] + \phi^1[\mathcal{H}^1] + \phi^\perp[\mathcal{H}^\perp]$ with proper space-time decay can be obtained for the inner–outer gluing to be carried out. We first choose all the constants such that

$$\|\mathcal{H}^0\|_{\nu,2+a},\ \|\mathcal{H}^1\|_{\nu_1,2+a_1},\ \|\mathcal{H}^\perp\|_{\nu,2+a} < +\infty.$$

We have the following estimates.

- By (6.6), we have

$$\begin{aligned} & \left|3\lambda U^2(y)[\Psi_0(\lambda y + \xi, t) + \psi(\lambda y + \xi, t) + Z^*(\lambda y + \xi, t)]\right| \\ \lesssim {} & \frac{\lambda_*(t)}{1+|y|^4}\left[|\dot\lambda_*|\left(\log\lambda_* + \log(1+|y|)\right) + \lambda_*^{\nu-1}(0)R^{-\alpha}(0)|\log T|\|\psi\|_* + \|Z^*\|_\infty\right]. \end{aligned} \tag{6.21}$$

- Directly computing, one has

$$\begin{aligned}&\left|\lambda\left[\dot\lambda(\nabla_y\phi\cdot y+\phi)+\nabla_y\phi\cdot\dot\xi\right]\right|\\ \lesssim{}&\lambda_*|\dot\lambda_*|\left(\frac{\lambda_*^\nu R^{2-\frac{a}{2}}\sqrt{\log R}}{1+|y|^2}\|\phi^0\|_{*,\nu,a}+\frac{\lambda_*^{\nu_1}}{1+|y|^{a_1}}\|\phi^1\|_{\nu_1,a_1}+\frac{\lambda_*^\nu}{1+|y|^a}\|\phi^\perp\|_{\nu,a}\right)\\ &+\lambda_*|\dot\xi|\left(\frac{\lambda_*^\nu R^{2-\frac{a}{2}}\sqrt{\log R}}{1+|y|^3}\|\phi^0\|_{*,\nu,a}+\frac{\lambda_*^{\nu_1}}{1+|y|^{1+a_1}}\|\phi^1\|_{\nu_1,a_1}+\frac{\lambda_*^\nu}{1+|y|^{1+a}}\|\phi^\perp\|_{\nu,a}\right).\end{aligned}\tag{6.22}$$

- Using (6.6) and (6.11), we evaluate

$$\begin{aligned}&\left|\lambda^3\mathcal N(\mathtt w)+\lambda^3\mathcal K[\lambda,\xi]\right|\\ \lesssim{}&\left|\frac{\lambda^2}{1+|y|^2}\left[(\lambda^{-1}\eta_R\phi)^2+\Psi_0^2+\psi^2+(Z^*)^2\right]+\lambda^3\mathcal K[\lambda,\xi]\right|\\ \lesssim{}&\frac{\lambda_*^{2\nu}R^{4-a}\log R}{1+|y|^6}\|\phi^0\|_{*,\nu,a}^2+\frac{\lambda_*^{2\nu_1}}{1+|y|^{2+2a_1}}\|\phi^1\|_{\nu_1,a_1}^2+\frac{\lambda_*^{2\nu}}{1+|y|^{2+2a}}\|\phi^\perp\|_{\nu,a}^2\\ &+\frac{\lambda_*^2}{1+|y|^2}|\dot\lambda_*|^2|\log(T-t)|^2+\frac{\lambda_*^2}{1+|y|^2}\|Z^*\|_\infty^2\\ &+\frac{\lambda_*^2(t)\lambda_*^{2\nu-2}(0)R^{-2\alpha}(0)|\log T|^2}{1+|y|^2}\|\psi\|_*^2+\frac{\lambda_*\dot\lambda_*}{(1+|y|^2)^2}+\frac{\lambda_*|\dot\xi|}{1+|y|^3}\\ &+|\dot\lambda_*|\left(\frac{\lambda_*^5}{(T-t)^2}+\frac{\lambda_*}{(1+|y|^2)^2}+\frac{\lambda_*^4(y\cdot\dot\xi+\dot\lambda)}{T-t}+\frac{\lambda_*^2\left(y\cdot\dot\xi+\dot\lambda\right)}{1+|y|^2}\right).\end{aligned}\tag{6.23}$$

From (6.21)–(6.23), we obtain

$$\begin{aligned}\|\mathcal H\|_{\nu,2+a}\lesssim{}&\lambda_*^{1-\nu}|\dot\lambda_*||\log(T-t)|+\lambda_*^{1-\nu}(t)\lambda_*^{\nu-1}(0)R^{-\alpha}(0)|\log T|\|\psi\|_*+\lambda_*^{1-\nu}\|Z^*\|_\infty\\ &+\lambda_*|\dot\lambda_*|R^{2+\frac{a}{2}}\sqrt{\log R}\|\phi^0\|_{*,\nu,a}+\lambda_*^{1+\nu_1-\nu}|\dot\lambda_*|R^{2+a-a_1}\|\phi^1\|_{\nu_1,a_1}+\lambda_*|\dot\lambda_*|R^2\|\phi^\perp\|_{\nu,a}\\ &+\lambda_*^\nu R^{4-a}\log R\|\phi^0\|_{*,\nu,a}^2+\lambda_*^{2\nu_1-\nu}\|\phi^1\|_{\nu_1,a_1}^2+\lambda_*^\nu\|\phi^\perp\|_{\nu,a}^2+\lambda_*^{2-\nu}|\dot\lambda_*|^2|\log(T-t)|^2R^a\\ &+\lambda_*^{2-\nu}R^a\|Z^*\|_\infty^2+\lambda_*^{2-\nu}(t)R^a(t)\lambda_*^{2\nu-2}(0)R^{-2\alpha}(0)|\log T|^2\|\psi\|_*^2\\ &+\lambda_*^{1-\nu}|\dot\lambda_*|+\lambda_*^{1-\nu}|\dot\xi|+|\dot\lambda_*|\Bigg(\frac{\lambda_*^{5-\nu}}{(T-t)^2}R^{2+a}+\lambda_*^{1-\nu}+\frac{\lambda_*^{4-\nu}R^{2+a}(R|\dot\xi|+\dot\lambda)}{T-t}\\ &+\lambda_*^{2-\nu}R^a(R|\dot\xi|+\dot\lambda)\Bigg)\end{aligned}$$

from which we conclude that for some fixed $\epsilon_0>0$

$$\|\mathcal H\|_{\nu,2+a}\lesssim T^{\epsilon_0}\left(\|\phi^0\|_{*,\nu,a}+\|\phi^1\|_{\nu_1,a_1}+\|\phi^\perp\|_{\nu,a}+\|\psi\|_*+\|Z^*\|_\infty+\|\lambda\|_\infty+\|\xi\|_G+1\right)\tag{6.24}$$

provided

$$\begin{cases}0<\nu<1,\\ 1-\beta(2+\frac{a}{2})>0,\\ 1+\nu_1-\nu-\beta(2+a-a_1)>0,\\ 1-2\beta>0,\\ \nu-\beta(4-a)>0,\\ 2\nu_1-\nu>0,\\ 2-\nu-a\beta>0,\\ \nu-\beta(a-2\alpha)>0,\\ 2-\nu-\beta(1+a)>0.\end{cases}\tag{6.25}$$

Similar computations give that for some fixed $\epsilon_0 > 0$

$$\|\mathcal{H}^1\|_{\nu_1,2+a_1} \lesssim T^{\epsilon_0} \left( \|\phi^1\|_{\nu_1,a_1} + \|\psi\|_* + \|Z^*\|_\infty + \|\lambda\|_\infty + \|\xi\|_G + 1 \right) \tag{6.26}$$

provided

$$\begin{cases} 0 < \nu_1 < 1, \\ \nu - \nu_1 + \alpha\beta > 0, \\ 2 - \nu_1 - a_1\beta > 0, \\ 2\nu - \nu_1 + 2\alpha\beta - a_1\beta > 0, \\ 1 - \nu_1 - \beta(a_1 - 1) > 0. \end{cases} \tag{6.27}$$

6.3. **The parameter problems.** From (6.18)–(6.20), it remains to adjust the parameter functions $\lambda(t)$, $\xi(t)$ such that

$$c^0[\lambda, \xi, \Psi^*] = 0, \ c^\ell[\lambda, \xi, \Psi^*] = 0, \ \ell = 1, \cdots, 4, \ \forall \ t \in (0, T),$$

where

$$c^0[\lambda, \xi, \Psi^*] = -\frac{\int_{B_{2R}} \mathcal{H} Z_5 dy}{\int_{B_{2R}} |Z_5|^2 dy}, \tag{6.28}$$

$$c^\ell[\lambda, \xi, \Psi^*] = -\frac{\int_{B_{2R}} \mathcal{H} Z_\ell dy}{\int_{B_{2R}} |Z_\ell|^2 dy} \quad \text{for} \quad \ell = 1, \cdots, 4. \tag{6.29}$$

It turns out that we can easily achieve at the translation mode (6.29), but the scaling mode (6.28) is more delicate.

6.3.1. *The reduced problem of* $\xi(t)$. We first consider the reduced equation for $\xi(t)$. Observe that (6.29) is equivalent to

$$\int_{B_{2R}} \mathcal{H}(\phi, \psi, \lambda, \xi)(y, t) Z_\ell(y) dy = 0, \quad \text{for all } t \in (0, T), \ \ell = 1, \cdots, 4.$$

Write $\Psi^* = \psi + Z^*$ and $\xi(t) = (\xi_1(t), \cdots, \xi_4(t))$. Then for $\ell = 1, \cdots, 4$,

$$\int_{B_{2R}} \mathcal{H}(\phi, \psi, \lambda, \xi)(y, t) Z_\ell(y) dy = 0$$

give that

$$\dot{\xi}_\ell = b_\ell[\lambda, \xi, \phi, \Psi^*], \tag{6.30}$$

where

$$b_\ell[\lambda, \xi, \phi, \Psi^*] = \int_{B_{2R}} \left( \mathcal{H}[\lambda, \xi, \phi, \Psi^*](y, t) - \lambda U_{y_\ell}(y) \dot{\xi}_\ell \right) Z_\ell(y) dy.$$

Furthermore, the size of $b_\ell[\lambda,\xi,\phi,\Psi^*]$ is controlled by

$$\begin{aligned}|b_\ell[\lambda,\xi,\phi,\Psi^*]| \lesssim\ & \left(\lambda_*|\dot\lambda_*||\log(T-t)|+\lambda_*\|Z^*\|_\infty\right)(1+O(R^{-3}))\\
&+\lambda_*(t)\lambda_*^{\nu-1}(0)R^{-\alpha}(0)|\log T|\|\psi\|_*(1+O(R^{-3}))\\
&+\lambda_*^{1+\nu}|\dot\lambda_*|R^{2-\frac{a}{2}}\sqrt{\log R}\|\phi^0\|_{*,\nu,a}(1+O(R^{-1}))+\lambda_*^{1+\nu_1}|\dot\lambda_*|\|\phi^1\|_{\nu_1,a_1}(1+O(R^{1-a_1}))\\
&+\lambda_*^{1+\nu}|\dot\lambda_*|\|\phi^\perp\|_{\nu,a}(1+O(R^{1-a}))+\lambda_*^{1+\nu}|\dot\xi|R^{2-\frac{a}{2}}\sqrt{\log R}\|\phi^0\|_{*,\nu,a}(1+O(R^{-2}))\\
&+\lambda_*^{1+\nu_1}|\dot\xi|\|\phi^1\|_{\nu_1,a_1}(1+O(R^{-a_1}))+\lambda_*^{1+\nu}|\dot\xi|\|\phi^\perp\|_{\nu,a}(1+O(R^{-a}))\\
&+\lambda_*^2|\dot\lambda_*|^2|\log(T-t)|^2(1+O(R^{-1}))+\lambda_*^{2\nu}R^{4-a}\log R\|\phi^0\|^2_{*,\nu,a}(1+O(R^{-5}))\\
&+\lambda_*^{2\nu_1}\|\phi^1\|^2_{\nu_1,a_1}(1+O(R^{-1-2a_1}))+\lambda_*^{2\nu}\|\phi^\perp\|^2_{\nu,a}(1+O(R^{-1-a}))\\
&+\lambda_*^2(t)\lambda_*^{2\nu-2}(0)R^{-2\alpha}(0)|\log T|^2\|\psi\|_*^2(1+O(R^{-1}))\\
&+\lambda_*^2\|Z^*\|^2_\infty(1+O(R^{-1}))+\lambda_*|\dot\lambda_*|(1+O(R^{-3}))\\
&+\lambda_*^2|\dot\lambda_*|(1+O(R^{-1}))+\lambda_*^3|\dot\lambda_*|R+\lambda_*^3|\dot\lambda_*||\dot\xi|R^2\\
&+\lambda_*^2|\dot\lambda_*||\dot\xi|R(1+O(R^{-1})).\end{aligned}\tag{6.31}$$

Next, we analyze the reduced problem (6.30), which defines operators $\Xi_\ell$ ($\ell=1,\cdots,4$) that return the solutions $\xi_\ell$ ($\ell=1,\cdots,4$) respectively. Here we write

$$\Xi=(\Xi_1,\Xi_2,\Xi_3,\Xi_4)\tag{6.32}$$

and $\xi(t)=q+\xi^1(t)$ where $q=(q_1,\cdots,q_4)$ is a prescribed point in $\Omega$. We shall solve $\xi^1(t)$ under the norm

$$\|\xi\|_G=\|\xi\|_{L^\infty(0,T)}+\sup_{t\in(0,T)}\lambda_*^{-\upsilon}(t)|\dot\xi(t)|$$

for some fixed $\upsilon\in(0,1)$. From (6.30), we have

$$|\xi_\ell(t)|\le|q_\ell|+\|b_\ell[\lambda,\xi,\phi,\Psi^*]\|_{L^\infty(0,T)}(T-t).$$

Therefore, we obtain

$$\|\Xi_\ell\|_G\le|q_\ell|+(T-t)^{-\upsilon}\|b_\ell[\lambda,\xi,\phi,\Psi^*]\|_{L^\infty(0,T)}.\tag{6.33}$$

By (6.31) and (6.33), we conclude that for some constant $C>0$

$$\begin{aligned}\|\Xi_\ell\|_G\le|q_\ell|+C(T-t)^{-\upsilon}\Bigg[&\lambda_*(t)\lambda_*^{\nu-1}(0)R^{-\alpha}(0)|\log T|\|\psi\|_*+\lambda_*\|Z^*\|_\infty\\
&+\lambda_*^{1+\nu}R^{2-\frac{a}{2}}\left(|\dot\lambda_*|+|\dot\xi|\right)\sqrt{\log R}\|\phi^0\|_{*,\nu,a}+\lambda_*^{1+\nu_1}\left(|\dot\lambda_*|+|\dot\xi|\right)\|\phi^1\|_{\nu_1,a_1}\\
&+\lambda_*^{1+\nu}\left(|\dot\lambda_*|R^{1-a}+|\dot\xi|\right)\|\phi^\perp\|_{\nu,a}+\lambda_*^{2\nu}R^{4-a}\log R\|\phi^0\|^2_{*,\nu,a}+\lambda_*^{2\nu_1}\|\phi^1\|^2_{\nu_1,a_1}+\lambda_*^{2\nu}\|\phi^\perp\|^2_{\nu,a}\\
&+\lambda_*|\dot\lambda_*||\log(T-t)|+\left(\lambda_*^{1+\nu+\upsilon}R^{\frac{10-2a}{3}}+\lambda_*^{2+\upsilon}|\dot\lambda_*|R\right)\|\xi\|_G\Bigg].\end{aligned}\tag{6.34}$$

6.3.2. *The reduced problem of $\lambda(t)$.* Since the reduced problem of $\lambda(t)$ is essentially the same as that of [6], we shall follow the strategy and logic in [6].

Direct computations show that (6.28) gives a non-local integro-differential equation

$$\int_{-T}^t\frac{\dot\lambda(s)}{t-s}\Gamma\left(\frac{\lambda^2(t)}{t-s}\right)ds+\mathbf{c_0}\dot\lambda=a[\lambda,\xi,\Psi^*](t)+\mathsf{a}_r[\lambda,\xi,\phi,\Psi^*](t),\tag{6.35}$$

where $\mathbf{c_0} = 2\alpha_0 \int_{\mathbb{R}^4} \frac{Z_5(y)}{(1+|y|^2)^2} dy$,

$$a[\lambda, \xi, \Psi^*] = -\int_{B_{2R}} 3U^2(y)\left(\Psi_0 + \Psi^*\right) Z_5(y) dy, \tag{6.36}$$

and the remainder term $\mathtt{a}_r[\lambda, \xi, \phi, \Psi^*](t)$ turns out to be smaller order and is controlled by

$$\begin{aligned}|\mathtt{a}_r[\lambda, \xi, \phi, \Psi^*](t)| \lesssim\ & \lambda_*^\nu R^{2-\frac{a}{2}}\left(|\dot\lambda_*||\log(T-t)| + |\dot\xi|\right)\sqrt{\log R}\|\phi^0\|_{*,\nu,a} + \lambda_*^\nu\left(|\dot\lambda_*|R^{2-a_1} + |\dot\xi|\right)\|\phi^1\|_{\nu_1,a_1}\\ & + \lambda_*^\nu\left(|\dot\lambda_*|R^{2-a} + |\dot\xi|R^{1-a}\right)\|\phi^\perp\|_{\nu,a} + \lambda_*^{2\nu-1}R^{4-a}\log R\|\phi^0\|^2_{*,\nu,a} + \lambda_*^{2\nu_1-1}\|\phi^1\|^2_{\nu_1,a_1}\\ & + \lambda_*^{2\nu-1}R^{2-2a}\|\phi^\perp\|^2_{\nu,a} + \lambda_*|\dot\lambda_*|^2|\log(T-t)|^3 + \lambda_*|\log(T-t)|\|Z^*\|^2_\infty\\ & + \lambda_*(t)|\log(T-t)|\lambda_*^{2\nu-2}(0)R^{-2\alpha}(0)|\log T|^2\|\psi\|^2_*.\end{aligned}$$

To solve $\lambda(t)$, we introduce the following norms

- $$\|f\|_{\Theta,l} := \sup_{t\in[0,T]} \frac{|\log(T-t)|^l}{(T-t)^\Theta}|f(t)|,$$
  where $f \in C([-T,T];\mathbb{R})$ with $f(T) = 0$, and $\Theta \in (0,1)$, $l \in \mathbb{R}$.
- $$[g]_{\gamma,m,l} := \sup_{I_T} \frac{|\log(T-t)|^l}{(T-t)^m(t-s)^\gamma}|g(t)-g(s)|,$$
  where $I_T = \left\{-T \le s \le t \le T : t-s \le \frac{1}{10}(T-t)\right\}$, $g \in C([-T,T];\mathbb{R})$ with $g(T) = 0$ and $0 < \gamma < 1$, $m > 0$, $l \in \mathbb{R}$.

Also, we define

$$\mathcal{B}_0[\lambda](t) := \int_{-T}^t \frac{\dot\lambda(s)}{t-s}\Gamma\left(\frac{\lambda^2(t)}{t-s}\right) ds + \mathbf{c_0}\dot\lambda \tag{6.37}$$

and write

$$c^0[\mathcal{H}] = \frac{\mathcal{B}_0[\lambda] - \left(a[\lambda,\xi,\Psi^*] + \mathtt{a}_r[\lambda,\xi,\phi,\Psi^*]\right)}{\int_{B_{2R}} |Z_5(y)|^2 dy}. \tag{6.38}$$

We invoke a key proposition proved in [6] concerning the solvability of $\lambda(t)$.

**Proposition 6.1.** *Let $\omega, \Theta \in (0, \frac{1}{2})$, $\gamma \in (0,1)$, $m \le \Theta - \gamma$ and $l \in \mathbb{R}$. If $a(t)$ satisfies $a(T) < 0$ with $1/C \le a(T) \le C$ for some constant $C > 1$, and*

$$T^\Theta|\log T|^{1+c-l}\|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \le C_1 \tag{6.39}$$

*for some $c > 0$, then there exist two operators $\mathcal{P}$ and $\mathcal{R}_0$ such that $\lambda = \mathcal{P}[a] : [-T,T] \to \mathbb{R}$ satisfies*

$$\mathcal{B}_0[\lambda](t) = a(t) + \mathcal{R}_0[a](t) \tag{6.40}$$

*with*

$$|\mathcal{R}_0[a](t)| \lesssim \left(T^{\frac{1}{2}+c} + T^\Theta\frac{\log|\log T|}{|\log T|}\|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}\right)\frac{(T-t)^{m+(1+\omega)\gamma}}{|\log(T-t)|^l}.$$

6.4. **Inner–outer gluing system.** By the discussions in Section 6.3.2, we transform the inner–outer problems (3.1), (3.4) into the problems of finding solutions $(\psi, \phi^0, \phi^1, \phi^\perp, \lambda, \xi)$ solving the following *inner–outer gluing system*

$$\begin{cases}\psi_t = \Delta\psi + \mathcal{G}(\phi^0 + \phi^1 + \phi^\perp, \psi + Z^*, \lambda, \xi), & \text{in } \Omega\times(0,T),\\ \psi = 0, & \text{on } \partial\Omega\times(0,T),\\ \psi(x,0) = 0, & \text{in } \Omega,\end{cases} \tag{6.41}$$

$$\begin{cases}\lambda^2\phi^0_t = \Delta_y\phi^0 + 3U^2(y)\phi^0 + \mathcal{H}^0(\phi,\psi,\lambda,\xi) + \tilde c^0[\mathcal{H}^0]Z_5 & \text{in } \mathcal{D}_{2R},\\ \phi^0(\cdot,0) = 0 & \text{in } B_{2R},\end{cases} \tag{6.42}$$

$$\begin{cases} \lambda^2 \phi^1_t = \Delta_y \phi^1 + 3U^2(y)\phi^1 + \mathcal{H}^1(\phi,\psi,\lambda,\xi) + \sum\limits_{\ell=1}^{4} c^\ell[\mathcal{H}^1] Z_\ell & \text{in } \mathcal{D}_{2R}, \\ \phi^1(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{6.43}$$

$$\begin{cases} \lambda^2 \phi^\perp_t = \Delta_y \phi^\perp + 3U^2(y)\phi^\perp + \mathcal{H}^\perp(\phi,\psi,\lambda,\xi) + c^0_*[\lambda,\xi,\Psi^*] Z_5 & \text{in } \mathcal{D}_{2R}, \\ \phi^\perp(\cdot,0) = 0 & \text{in } B_{2R}, \end{cases} \tag{6.44}$$

$$c^0[\mathcal{H}](t) - \tilde{c}^0[\lambda,\xi,\Psi^*](t) = 0 \ \text{ for all } \ t \in (0,T), \tag{6.45}$$

$$c^1[\mathcal{H}](t) = 0 \ \text{ for all } \ t \in (0,T), \tag{6.46}$$

where $\mathcal{G}$ is defined in (3.5), $\mathcal{H}^0$, $\mathcal{H}^1$, $\mathcal{H}^\perp$ are the projections of $\mathcal{H}$ (see (3.2)) on different modes. It is direct to see that if $(\psi,\phi^0,\phi^1,\phi^\perp,\lambda,\xi)$ satisfies the system (6.41)–(6.46), then

$$\Psi^* = \psi + Z^*, \ \phi = \phi^0 + \phi^1 + \phi^\perp$$

solve the inner–outer problems (3.1), (3.4) and thus the desired blow-up solution is obtained.

6.5. **The fixed point formulation.** The inner–outer gluing system (6.41)–(6.46) can be formulated as a fixed point problem for operators we will describe below.

We first define the following function spaces

$$\begin{aligned} X_{\phi^0} &:= \left\{\phi^0 \in L^\infty(\mathcal{D}_{2R}) : \ \nabla_y \phi^0 \in L^\infty(\mathcal{D}_{2R}), \ \|\phi^0\|_{*,\nu,a} < +\infty \right\}, \\ X_{\phi^1} &:= \left\{\phi^1 \in L^\infty(\mathcal{D}_{2R}) : \ \nabla_y \phi^1 \in L^\infty(\mathcal{D}_{2R}), \ \|\phi^1\|_{\nu_1,a_1} < +\infty \right\}, \\ X_{\phi^\perp} &:= \left\{\phi^\perp \in L^\infty(\mathcal{D}_{2R}) : \ \nabla_y \phi^\perp \in L^\infty(\mathcal{D}_{2R}), \ \|\phi^\perp\|_{\nu,a} < +\infty \right\}, \\ X_\psi &:= \left\{\psi \in L^\infty(\Omega \times (0,T)) : \ \|\psi\|_* < +\infty \right\}. \end{aligned} \tag{6.47}$$

In order to introduce the space for the parameter function $\lambda(t)$, we recall from (6.37) that the integral operator $\mathcal{B}_0$ takes the following approximate form

$$\mathcal{B}_0[\lambda] = \int_{-T}^{t-\lambda_*^2(t)} \frac{\dot\lambda(s)}{t-s} ds + O(\|\dot\lambda\|_\infty).$$

Proposition 6.1 defines an approximate inverse operator $\mathcal{P}$ of the integral operator $\mathcal{B}_0$ such that for $a$ satisfying (6.39), $\lambda := \mathcal{P}[a]$ satisfies

$$\mathcal{B}_0[\lambda] = a + \mathcal{R}_0[a] \ \text{ in } \ [-T,T],$$

where $\mathcal{R}_0[a]$ is a small remainder. Also, the proof as in [6] provides the decomposition

$$\mathcal{P}[a] = \lambda_{0,\kappa} + \mathcal{P}_1[a] \tag{6.48}$$

with

$$\lambda_{0,\kappa} := \kappa |\log T| \int_t^T \frac{1}{|\log(T-s)|^2} ds, \ t \le T,$$

$\kappa = \kappa[a] \in \mathbb{R}$, and the function $\lambda_1 = \mathcal{P}_1[a]$ satisfies

$$\|\lambda_1\|_{*,3-\iota} \lesssim |\log T|^{1-\iota} \log^2(|\log T|) \tag{6.49}$$

for $0 < \iota < 1$, where the $\|\cdot\|_{*,3-\iota}$-norm is defined by

$$\|f\|_{*,k} := \sup_{t\in[-T,T]} |\log(T-t)|^k |\dot f(t)|.$$

Therefore, we define

$$X_\lambda := \{\lambda_1 \in C^1([-T,T]) : \lambda_1(T) = 0, \|\lambda_1\|_{*,3-\iota} < \infty\}.$$

Here by $(\kappa,\lambda_1)$ we represent $\lambda$ in the form

$$\lambda = \lambda_{0,\kappa} + \lambda_1,$$

and from [6], one can write the norm

$$\|\lambda\|_F = |\kappa| + \|\lambda_1\|_{*,3-\iota}. \tag{6.50}$$

For the translation parameter function $\xi(t)$, we write $\xi(t) = q + \xi^1(t)$ and define the following space for $\xi^1(t)$

$$X_\xi = \left\{ \xi \in C^1((0,T);\mathbb{R}^4),\ \dot\xi(T) = 0,\ \|\xi\|_G < +\infty \right\}$$

with

$$\|\xi\|_G = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in(0,T)} \lambda_*^{-\upsilon}(t)|\dot\xi(t)| \tag{6.51}$$

for some fixed $\upsilon \in (0,1)$.

Define

$$\mathcal{X} = X_{\phi^0} \times X_{\phi^1} \times X_{\phi^\perp} \times X_\psi \times \mathbb{R} \times X_\lambda \times X_\xi. \tag{6.52}$$

We will solve the inner–outer gluing system in a closed ball $\mathcal{B}$ in which $(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi^1) \in \mathcal{X}$ satisfies

$$\begin{cases} \|\phi^0\|_{*,\nu,a} + \|\phi^1\|_{\nu_1,a_1} + \|\phi^\perp\|_{\nu,a} \le 1 \\ \|\psi\|_* \le 1 \\ |\kappa - \kappa_0| \le |\log T|^{-1/2} \\ \|\lambda_1\|_{*,3-\iota} \le C|\log T|^{1-\iota}\log^2(|\log T|) \\ \|\xi\|_G \le 1 \end{cases} \tag{6.53}$$

for some large and fixed constant $C$, where $\kappa_0 = Z_0^*(0)$. The inner–outer gluing system (6.41)–(6.46) can be formulated as the following fixed point problem. We define an operator $\mathcal{F}$ which returns the solution from $\mathcal{B}$ to $\mathcal{X}$

$$\begin{aligned} \mathcal{F} : \mathcal{B} \subset \mathcal{X} &\to \mathcal{X} \\ v &\mapsto \mathcal{F}(v) = (\mathcal{F}_{\phi^0}(v), \mathcal{F}_{\phi^1}(v), \mathcal{F}_{\phi^\perp}(v), \mathcal{F}_\psi(v), \mathcal{F}_\kappa(v), \mathcal{F}_{\lambda_1}(v), \mathcal{F}_\xi(v)) \end{aligned}$$

with

$$\begin{aligned} \mathcal{F}_{\phi^0}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_0(\mathcal{H}^0[\lambda,\xi,\Psi^*]) \\ \mathcal{F}_{\phi^1}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_1(\mathcal{H}^1[\lambda,\xi,\Psi^*]) \\ \mathcal{F}_{\phi^\perp}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_\perp\left(\mathcal{H}^\perp[\lambda,\xi,\Psi^*] + c_*^0[\lambda,\xi,\Psi^*]Z_5\right) \\ \mathcal{F}_\psi(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{T}_\psi\left(\mathcal{G}(\phi^0+\phi^1+\phi^\perp,\Psi^*,\lambda,\xi)\right) \\ \mathcal{F}_\kappa(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \kappa\left[a^0[\lambda,\xi,\Psi^*]\right] \\ \mathcal{F}_{\lambda_1}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \mathcal{P}_1\left[a^0[\lambda,\xi,\Psi^*]\right] \\ \mathcal{F}_\xi(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1) &= \Xi(\phi^0,\phi^1,\phi^\perp,\psi,\lambda,\xi) \end{aligned} \tag{6.54}$$

Here $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_\perp$ are the operators given in Proposition 5.2 which solve different modes of the inner problems (6.42)–(6.44). The operator $\mathcal{T}_\psi$ defined by Proposition 5.1 deals with the outer problem (6.41). Operators $\kappa[a]$, $\mathcal{P}_1$ and $\Xi$ handle the equations for $\lambda$ and $\xi$ which are defined in Proposition 6.1, (6.48) and (6.32), respectively.

6.6. **Choices of constants.** In this section, we shall list all the constraints of the constants $\beta$, $\alpha$, $a$, $a_1$, $\nu$, $\nu_1$, $\nu_2$ which are sufficient for the inner–outer gluing scheme to work.

First, we indicate all the parameters used in different norms.

- $R(t) = \lambda_*^{-\beta}(t)$ with $\beta \in (0, 1/2)$.
- The norm for $\phi^0$ solving mode 0 of the inner problem (6.42) is $\|\cdot\|_{*,\nu,a}$ which is defined in (5.9), where we require that $\nu, a \in (0,1)$.
- The norm for $\phi^1$ solving modes 1 to 4 of the inner problem (6.43) is $\|\cdot\|_{\nu_1,a_1}$ which is defined in (5.8), where we require that $\nu_1 \in (0,1)$ and $a_1 \in (1,2)$.
- The norm for $\phi^\perp$ solving higher modes ($j \ge 5$) of the inner problem (6.44) is $\|\cdot\|_{\nu,a}$ which is defined in (5.8), where $\nu, a \in (0,1)$.

- The norm for $\psi$ solving the outer problem (6.41) is $\|\cdot\|_*$ which is defined in (5.4), while the $\|\cdot\|_{**}$-norm for the right hand side of the outer problem (6.41) is defined in (5.3). Here we require that $\nu, \alpha, \nu_2, \gamma \in (0,1)$ .
- In Proposition 6.1, we have the parameters $\omega, \Theta, m, l, \gamma$. Here $\omega$ is the parameter used to describe the remainder $\mathcal{R}_\omega$ and $\omega \in (0, 1/2)$. To apply Proposition 6.1 in our setting, we let

$$\Theta = \nu - 1 + \alpha\beta,\ m = \nu - 2 - \gamma + \beta(2+\alpha),\ l < 1 + 2m,$$

and require that $\beta > \frac{1-\omega}{2}$ such that $m + (1+\omega)\gamma > \Theta$ is guaranteed.

In order to get the desired estimates for the outer problem (6.41), by the computations in Section 6.1, we need the following restrictions

$$\begin{cases} \nu - 1 + \beta(2+\alpha) - \nu_2 > 0, \\ 2\beta - \nu_2 > 0, \\ 0 < \alpha < a < 1, \\ \beta + \upsilon - \nu_2 > 0, \\ 2\nu_1 - \nu + \beta(2a_1 - \alpha) > 0, \\ \nu_2 < 1, \\ 2\nu - \nu_2 - 1 + 2\alpha\beta > 0. \end{cases}$$

In order to get the desired estimates for the inner problems at different modes (6.42)–(6.44), by the computations in Section 6.2, we require the restrictions

$$\begin{cases} 0 < \nu < 1, \\ 1 - \beta(2 + \frac{a}{2}) > 0, \\ 1 + \nu_1 - \nu - \beta(2 + a - a_1) > 0, \\ 1 - 2\beta > 0, \\ \nu - \beta(4 - a) > 0, \\ 2\nu_1 - \nu > 0, \\ 2 - \nu - a\beta > 0, \\ \nu - \beta(a - 2\alpha) > 0, \\ 2 - \nu - \beta(1 + a) > 0, \\ 0 < \nu_1 < 1, \\ \nu - \nu_1 + \alpha\beta > 0, \\ 2 - \nu_1 - a_1\beta > 0, \\ 2\nu - \nu_1 + 2\alpha\beta - a_1\beta > 0, \\ 1 - \nu_1 - \beta(a_1 - 1) > 0. \end{cases}$$

It turns out that suitable choices of the parameters satisfying all the restrictions in this section can be found. Here we give a specific example:

$$\beta \approx \frac{1}{4}\ (\beta > \frac{1}{4}),\ \alpha \approx a \approx a_1 \approx 1,\ \nu \approx \nu_1 \approx 1,\ \nu_2 \approx 0.$$

6.7. **Proof of Theorem 1.** Consider the operator

$$\mathcal{F} = (\mathcal{F}_{\phi^0}, \mathcal{F}_{\phi^1}, \mathcal{F}_{\phi^\perp}, \mathcal{F}_\psi, \mathcal{F}_\kappa, \mathcal{F}_{\lambda_1}, \mathcal{F}_\xi) \tag{6.55}$$

given in (6.54). To prove Theorem 1, our strategy is to show the existence of a fixed point for the operator $\mathcal{F}$ in $\mathcal{B}$ by the Schauder fixed point theorem, where the closed ball $\mathcal{B}$ is defined in (6.53). By collecting the estimates (6.16), (6.24), (6.26), (6.34), (6.49), and using Proposition 5.1, Proposition 5.2, Proposition 6.1, we conclude that for $(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\in\mathcal{B}$

$$\begin{cases} \|\mathcal{F}_{\phi^0}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\|_{*,\nu,a}\le CT^\epsilon \\ \|\mathcal{F}_{\phi^1}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\|_{\nu_1,a_1}\le CT^\epsilon \\ \|\mathcal{F}_{\phi^\perp}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\|_{\nu,a}\le CT^\epsilon \\ \|\mathcal{F}_{\psi}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\|_{*}\le CT^\epsilon \\ \left|\mathcal{F}_{\kappa}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)-\kappa_0\right|\le C|\log T|^{-1} \\ \|\mathcal{F}_{\lambda_1}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\|_{*,3-\iota}\le C|\log T|^{1-\iota}\log^2(|\log T|) \\ \|\mathcal{F}_{\xi}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\|_{G}\le CT^\epsilon \end{cases} \tag{6.56}$$

where $C>0$ is a constant independent of $T$, and $\epsilon>0$ is a small fixed number. On the other hand, compactness of the operator $\mathcal{F}$ defined in (6.55) can be proved by proper variants of (6.56). Indeed, if we vary the parameters $\beta,\alpha,a,a_1,\nu,\nu_1,\nu_2$ slightly such that all the restrictions in Section 6.6 are still satisfied, then we get (6.56) with the norms in the left hand side defined by the new parameters, while the closed ball $\mathcal{B}$ remains the same. To be more specific, for fixed $\nu',a'$ which are close to $\nu,a$, one can show that if $(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\in\mathcal{B}$, then

$$\|\mathcal{F}_{\phi^0}(\phi^0,\phi^1,\phi^\perp,\psi,\kappa,\lambda_1,\xi^1)\|_{*,\nu',a'}\le CT^{\epsilon'}.$$

Furthermore, one can show that for $\nu'>\nu$ and $\nu'-\beta(2-\frac{a'}{2})>\nu-\beta(2-\frac{a}{2})$, one has a compact embedding in the sense that if a sequence $\{\phi_n^0\}$ is bounded in the $\|\cdot\|_{*,\nu',a'}$-norm, then there exists a subsequence which converges in the $\|\cdot\|_{*,\nu,a}$-norm. Thus, the compactness follows directly from a standard diagonal argument by Arzelà–Ascoli's theorem. Arguing in a similar manner, the compactness for the rest operators can be proved. Therefore, the existence of the desired blow-up solution for $k=1$ is concluded from the Schauder fixed point theorem.

The general case of multiple-bubble blow-up is essentially identical. The ansatz is modified as follows: we let

$$u^*(x,t)=\sum_{j=1}^k U_{\lambda_j(t),\xi_j(t)}+\Psi_{0j}(x,t)$$

where $\Psi_{0j}$ is defined as in (2.7) with $\lambda,\xi$ replaced by $\lambda_j,\xi_j$. Then we look for a solution of the form

$$u(x,t)=u^*(x,t)+\sum_{j=1}^k\lambda_j^{-1}(t)\eta_{R(t)}(y_j)\phi_j(y_j,t)+Z^*(x,t)+\psi(x,t),\ \ y_j=\frac{x-\xi_j(t)}{\lambda_j(t)},$$

and are led to one outer problem and $k$ inner problems with exactly analogous estimates. A string of fixed point problems can be solved in the same manner. We omit the details. □

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