Non-radial solutions to a bi-harmonic equation with negative exponent

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Abstract

We prove the existence of non-radial entire solution to

$$\Delta^2 u + u^{-q} = 0 \quad \text{in } \mathbb{R}^3, \quad u > 0,$$

for $q > 1$. This answers an open question raised by P. J. McKenna and W. Reichel (E. J. D. E. **37** (2003) 1-13).

1 Introduction

We consider the following bi-harmonic equation with negative exponent

$$\Delta^2 u + u^{-q} = 0 \quad \text{in } \mathbb{R}^3, \quad u > 0, \tag{1}$$

where $q > 0$.

For $q = 7$, problem (1) can be seen as a fourth order analog of the Yamabe equation (see [1, 5, 21]), namely

$$\Delta^2 u = \frac{n-4}{2} u^{\frac{n+4}{n-4}} \quad \text{in } \mathbb{R}^n, \quad u > 0. \tag{2}$$

In the recent past, radial solutions to equation (1) have been studied by many authors, especially the existence and asymptotic behavior:

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Theorem A *([5, 6, 8, 11, 16, 20])*

i) There is no entire solution to (1) *for* $0 < q \leq 1$.

ii) If u *has exact linear growth at infinity, that is,*

$$\lim_{|x|\to+\infty} \frac{u(x)}{|x|} = C > 0,$$

then $q > 3$. *Moreover, for* $q = 7$, u *is given by* $u(x) = \sqrt{\sqrt{1/15 + |x|^2}}$, *and is unique up to dilation and translations.*

iii) For $q > 3$ *there exists radial solution with exact linear growth.*

iv) For $q > 1$ *there exists radial solution with exact quadratic growth, that is,*

$$\lim_{|x|\to+\infty} \frac{u(x)}{|x|^2} = C > 0.$$

v) For $1 < q < 3$ *there exists a radial solution* u *such that* $r^{-\frac{4}{q+1}}u(r) \to C(q) > 0$ *as* $r \to \infty$ *(the constant* $C(q)$ *is explicitly known).*

vi) For $q = 3$ *there exists a radial solution* u *such that* $r^{-1}(\log r)^{-\frac{1}{4}}u(r) \to 2^{\frac{1}{4}}$ *as* $r \to \infty$.

It has been shown by Choi-Xu [5] that if u is a solution to (1) with $q > 4$, and u has exact linear growth at infinity then u satisfies the integral equation

$$u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x-y|}{u^q(y)} dy + \gamma, \tag{3}$$

for some $\gamma \in \mathbb{R}$, and $\gamma = 0$ if and only if $q = 7$. In fact, every positive solution u to

$$(-\Delta)^n u + u^{-(4n-1)} = 0 \quad \text{in } \mathbb{R}^{2n-1}, \quad n \geq 2,$$

with exact linear growth at infinity satisfies

$$u(x) = c_n \int_{\mathbb{R}^{2n-1}} \frac{|x-y|}{u^{4n-1}(y)} dy,$$

where c_n is a dimensional constant, see [7], [17]. For the classification of solutions to the above integral equation we refer the reader to [12], [20].

In [16] McKenna-Reichel proved the existence of non-radial solution to

$$\Delta^2 w + w^{-q} = 0 \quad \text{in } \mathbb{R}^n, \quad w > 0 \tag{4}$$

for $n \geq 4$. This was a simple consequence of their existence results to (4) in lower dimension. More precisely, if u is a radial solution to (4) with $n \geq 3$ then $w(x) := u(x')$ is a non-radial solution to $\Delta^2 w + w^{-q} = 0$ in $\mathbb{R}^{n+1}$, where $x = (x', x'') \in \mathbb{R}^n \times \mathbb{R}$. Then they asked whether in $\mathbb{R}^3$ non-radial positive entire solution exist. (See [Open Questions (1), [16]].)

We answer this question affirmatively. (See Theorem 1.2 below.) In fact we prove the following theorems.

Theorem 1.1 *Let u be a solution to (1) for some $q > 1$. Assume that*

$$\beta := \frac{1}{8\pi} \int_{\mathbb{R}^3} u^{-q} dx < +\infty. \tag{5}$$

Then, up to a rotation and translation, we have

$$u(x) = (\beta + o(1))|x| + \sum_{i \in \mathcal{I}_1} a_i x_i^2 + \sum_{i \in \mathcal{I}_2} b_i x_i + c, \quad o(1) \xrightarrow{|x|\to\infty} 0, \tag{6}$$

where

$$\mathcal{I}_1, \mathcal{I}_2 \subseteq \{1,2,3\}, \quad \mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset, \quad a_i > 0 \text{ for } i \in \mathcal{I}_1, \quad |b_i| < \beta \text{ for } i \in \mathcal{I}_2, \quad c > 0.$$

Theorem 1.2 *Let $q > 1$. Then for every $0 < \kappa_1 < \kappa_2$ there exists a non-radial solution u to (1) such that*

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|^2} = \kappa_1 \quad \text{and} \ \limsup_{|x|\to\infty} \frac{u(x)}{|x|^2} = \kappa_2. \tag{7}$$

Theorem 1.3 *Let $q > 7$. Then for every $\kappa > 0$ there exists a non-radial solution u to (1) such that*

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \in (0,\infty) \quad \text{and} \ \limsup_{|x|\to\infty} \frac{u(x)}{|x|^2} = \kappa.$$

The non-radial solutions constructed in Theorem 1.2 also satisfies the following integral condition

$$\int_{\mathbb{R}^3} u^{-q} dx < +\infty, \tag{8}$$

for $q > \frac{3}{2}$. Note that McKenna-Reichel's non-radial example has infinite L^1 bound: $\int_{\mathbb{R}^{n+1}} w^{-q} dx = +\infty$.

The existence of infinitely many entire non-radial solutions with different growth rates for the conformally invariant equation $\Delta^2 u + u^{-7} = 0$ in $\mathbb{R}^3$ is in striking contrast to other conformally invariant equations $-\Delta u = u^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$, $n \geq 3$ and $(-\Delta)^m u = u^{\frac{n+2m}{n-2m}}$ in $\mathbb{R}^n$, $n > 2m$. In both cases all solutions are radially symmetric with respect to some point in $\mathbb{R}^n$, see [2], [4], [13] and [18].

Our motivation in the proof of Theorems 1.2-1.3 come from a similar phenomena exhibited in the following equation

$$(-\Delta)^{\frac{n}{2}} u = e^{nu} \ \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^{nu} dx < +\infty. \tag{9}$$

It has been proved that for $n \geq 4$ problem (9) admits non-radial entire solutions with polynomial growth at infinity, see [3], [9], [10], [14], [15], [19] and the references therein. It is surprising to see that conformally invariant equations with negative powers share similar phenomena.

In the remaining part of the paper we prove Theorems 1.1-1.3 respectively. We also give a new proof of *iii*)-*iv*) of Theorem A, see sub-section 2.1.

2 Proof of the theorems

We begin by proving Theorem 1.1.

Proof of Theorem 1.1 Let u be a solution to (1)-(5). We set

$$v(x) := \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x-y|-|y|}{u^q(y)}dy, \quad w := u - v. \tag{10}$$

Fixing $\varepsilon > 0$ and $R = R(\varepsilon) > 0$ so that

$$\int_{B_R^c} \frac{dx}{u^q(x)} < 8\pi\varepsilon,$$

one gets

$$v(x) \geq \frac{1}{8\pi}\int_{B_R} \frac{|x|-2|y|}{u^q(y)}dy - \frac{1}{8\pi}\int_{B_R^c} \frac{|x|}{u^q(y)}dy \geq (\beta - 2\varepsilon)|x| - C(R).$$

Using that $||x-y|-|y|| \leq |x|$, form (10), we obtain

$$|v(x)| \leq \beta|x| \quad \text{in } \mathbb{R}^3.$$

Combining these estimates we deduce that

$$\lim_{|x|\to\infty} \frac{v(x)}{|x|} = \beta.$$

It follows that w satisfies

$$\Delta^2 w = 0 \quad \text{in } \mathbb{R}^3, \quad w(x) \geq -\beta|x|,$$

and hence, w is a polynomial of degree at most 2, see for instance [14, Theorem 5]. Indeed, up to a rotation and translation, we can write

$$w(x) = \sum_{i\in\mathcal{I}_1} a_i x_i^2 + \sum_{i\in\mathcal{I}_2} b_i x_i + c_0,$$

where $\mathcal{I}_1, \mathcal{I}_2$ are two disjoint (possibly empty) subsets of $\{1,2,3\}$, $a_i \neq 0$ for $i \in \mathcal{I}_1$, $b_i \neq 0$ for $i \in \mathcal{I}_2$ and $c_0 \in \mathbb{R}$. Therefore, up to a rotation and translation, we have

$$u(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x-y|-|y|}{u^q(y)}dy + \sum_{i\in\mathcal{I}_1} a_i x_i^2 + \sum_{i\in\mathcal{I}_2} b_i x_i + c.$$

Now $u > 0$ and $|v(x)| \leq \beta|x|$ lead to $a_i > 0$ for $i \in \mathcal{I}_1$, $|b_i| \leq \beta$ for $i \in \mathcal{I}_2$ and $c = u(0) > 0$.

In order to prove that $|b_i| < \beta$ we assume by contradiction that $|b_{i_0}| = \beta$ for some $i_0 \in \mathcal{I}_2$. Up to relabelling we may assume that $i_0 = 1$. Then

$$u(x) \leq C + |b_1 x_1| + b_1 x_1 \quad \text{on } \mathcal{C} := \{x = (x_1, \bar{x}) \in \mathbb{R}\times\mathbb{R}^2 : |\bar{x}| \leq 1\},$$

a contradiction to (5).

We conclude the proof. □

Now we move on to the existence results. We look for solutions to (1) of the form $u = v + P$ where P is a polynomial of degree 2. Notice that $u = v + P$ satisfies (1) if and only if v satisfies

$$\Delta^2 v = -(v + P)^{-q}, \quad v + P > 0. \tag{11}$$

In particular, if $P \geq 0$, and v satisfies the integral equation

$$v(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \frac{1}{(P(y) + v(y))^q} dy, \tag{12}$$

then v satisfies (11). Thus, we only need to find solutions to (12) (or a variant of it), and we shall do that by a fixed point argument. Let us first define the spaces on which we shall work:

$$X := \left\{ v \in C^0(\mathbb{R}^3) : \|v\|_X < \infty \right\}, \quad \|v\|_X := \sup_{x \in \mathbb{R}^3} \frac{|v(x)|}{1 + |x|},$$

$$X_{ev} := \left\{ v \in X : v(x) = v(-x) \, \forall x \in \mathbb{R}^3 \right\}, \quad \|v\|_{X_{ev}} := \|v\|_X,$$

$$X_{rad} := \{ v \in X : v \text{ is radially symmetric} \}, \quad \|v\|_{X_{rad}} := \|v\|_X.$$

The following proposition is crucial in proving Theorem 1.2.

Proposition 2.1 *Let P be a positive function on $\mathbb{R}^3$ such that $P(-x) = P(x)$ and for some $q > 0$*

$$\int_{\mathbb{R}^3} \frac{|x|}{(P(x))^q} dx < \infty.$$

Then there exists a function $v \in X_{ev}$ satisfying $\min_{\mathbb{R}^3} v = v(0) = 0$,

$$v(x) := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{(P(y) + v(y))^q} dy, \tag{13}$$

and

$$\lim_{|x| \to \infty} \frac{v(x)}{|x|} = \alpha_{P,v} := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{dy}{(P(y) + v(y))^q}.$$

Moreover, if P is radially symmetric then there exists a solution to (13) in X_{rad}.

Proof. Let us define an operator $T : X_{ev} \to X_{ev}$, $v \mapsto \bar{v}$, (In case P is radial we restrict the operator T on X_{rad}. Notice that $T(X_{rad}) \subset X_{rad}$.) where

$$\bar{v}(x) := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{(P(y) + |v(y)|)^q} dy. \tag{14}$$

We proceed by steps.

Step 1 T is compact.

Using that $||x-y|-|y|| \le |x|$ we bound

$$\|\bar{v}\|_X \le \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{1}{(P(y))^q} dy \le C \quad \text{for every } v \in X. \tag{15}$$

Differentiating under the integral sign one gets

$$|\nabla \bar{v}(x)| \le \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{1}{(P(y))^q} dy \le C \quad \text{for every } v \in X,\ x \in \mathbb{R}^3.$$

We let (v_k) be a sequence in X_{ev}. Then $\bar{v}_k := T(v_k)$ is bounded in $C^1_{loc}(\mathbb{R}^3)$. Moreover, up to a subsequence, for some $c_i \ge 0$ with $i = 0, 1$, we have

$$\frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|y|^i}{(P(y)+|v_k(y)|)^q} dy \xrightarrow{k\to\infty} c_i.$$

We rewrite (14) (with $v = v_k$ and $\bar{v} = \bar{v}_k$) as

$$\bar{v}_k(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x|-|y|}{(P(y)+|v_k(y)|)^q} dy + \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x-y|-|x|}{(P(y)+|v_k(y)|)^q} dy =: I_{1,k}(x) + I_{2,k}(x).$$

It follows that

$$I_{1,k}(x) \to c_0|x| - c_1 \quad \text{in } X.$$

Using that $||x-y|-|x|| \le |y|$ we bound

$$|I_{2,k}(x)| \le \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|y|}{(P(y))^q} dy \le C.$$

This implies that

$$\lim_{R\to\infty} \sup_k \sup_{x\in\mathbb{R}^3\setminus B_R} \frac{I_{2,k}(x)}{1+|x|} = 0.$$

Since

$$\sup_k \sup_{x\in\mathbb{R}^3} |\nabla I_{2,k}(x)| < \infty,$$

up to a subsequence,

$$I_{2,k} \to I \quad \text{in } X_{ev},$$

for some $I \in X_{ev}$. This proves Step 1 as T is continuous.

Step 2 T has a fixed point in X_{ev}.

It follows form (15) that there exists $M > 0$ such that $T(X_{ev}) \subset \mathcal{B}_M \subset X_{ev}$. In particular, $T(\bar{\mathcal{B}}_M) \subset \mathcal{B}_M$. Hence, by Schauder fixed point theorem there exists a fixed point of T in $\mathcal{B}_M$.

Step 3 $\lim_{|x|\to\infty} \frac{\bar{v}(x)}{|x|} = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{dy}{(P(y)+|v(y)|)^q} =: \alpha(P, v)$.

Step 3 follows from

$$|\bar{v}(x) - \alpha(P,v)|x|| \le \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{||x-y|-|y|-|x||}{(P(y)+|v(y)|)^q} dy \le \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|y|}{(P(y))^q} dy \le C.$$

Step 4 If v is a fixed point of T then $v \geq 0$.

Differentiating under the integral sign, from (14) one can show that the hessian $D^2\bar{v}$ is strictly positive definite, and hence $\bar{v}$ is strictly convex. Moreover, using that $(P + |v|)$ is an even function, one obtains $\nabla\bar{v}(0) = 0$. This leads to

$$\min_{x \in \mathbb{R}^3} \bar{v}(x) = \bar{v}(0) = 0.$$

We conclude the proposition. □

In the same spirit one can prove the following proposition.

Proposition 2.2 *Let P be a positive even function on $\mathbb{R}^3$ such that for some $q > 0$*

$$\int_{\mathbb{R}^3} \frac{|x|}{(P(x))^q} dx < \infty.$$

Then there exists a positive function $v \in X_{ev}$ satisfying

$$v(x) := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x-y|}{(P(y)+v(y))^q} dy, \quad \min_{\mathbb{R}^3} v = v(0). \tag{16}$$

Proof of Theorem 1.2 Let $q > 1$ and $0 < \kappa_1 < \kappa_2$ be fixed. For every $\varepsilon > 0$ let $v_\varepsilon \in X_{ev}$ be a solution of (13), that is,

$$v_\varepsilon(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x-y| - |y|}{(P_\varepsilon(y) + v_\varepsilon(y))^q} dy, \tag{17}$$

where

$$P_\varepsilon(x) := 1 + \kappa_1 x_1^2 + \kappa_2(x_2^2 + x_3^2) + \varepsilon|x|^4, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We claim that for every multi-index $\beta \in \mathbb{N}^3$ with $|\beta| = 2$

$$|D^\beta v_\varepsilon(x)| \leq C \quad \text{on } B_2 \quad \text{and } |D^\beta v_\varepsilon(x)| \leq C f_q(x) \quad \text{on } B_2^c, \tag{18}$$

where

$$f_q(x) := \begin{cases} |x|^{-1} & \text{if } q > 3/2 \\ |x|^{-1} \log|x| & \text{if } q = 3/2 \\ |x|^{2-2q} & \text{if } q < 3/2. \end{cases}$$

For $|\beta| = 2$, differentiating under the integral sign, from (17), we obtain

$$\begin{aligned} |D^\beta v_\varepsilon((x))| &\leq C \int_{\mathbb{R}^3} \frac{1}{|x-y|} \frac{dy}{(P_\varepsilon(y) + v_\varepsilon(y))^q} \\ &\leq C \int_{\mathbb{R}^3} \frac{1}{|x-y|} \frac{dy}{(1 + \kappa_1|y|^2)^q} \\ &= C \sum_{i=1}^{3} I_i(x), \end{aligned}$$

where

$$I_i(x) := \int_{A_i} \frac{1}{|x-y|} \frac{dy}{(1+\kappa_1|y|^2)^q}, \quad A_1 := B_{\frac{|x|}{2}},\, A_2 := B_{2|x|} \setminus A_1,\, A_3 := \mathbb{R}^3 \setminus B_{2|x|}.$$

Since $q > 1$ we have $|D^\beta v_\varepsilon| \le C$ on B_2. For $|x| \ge 2$ we bound

$$I_1(x) \le \frac{2}{|x|} \int_{A_1} \frac{dy}{(1+\kappa_1|y|^2)^q} \le C f_q(x),$$

$$I_2(x) \le \frac{C}{|x|^{2q}} \int_{A_2} \frac{dy}{|x-y|} \le \frac{C}{|x|^{2q}} \int_{|y|\le 3|x|} \frac{dy}{|y|} \le C|x|^{2-2q},$$

$$I_3(x) \le 2 \int_{A_3} \frac{dy}{|y|(1+\kappa_1|y|^2)^q} \le C|x|^{2-2q}.$$

This proves (18). Since $v_\varepsilon(0) = |\nabla v_\varepsilon(0)| = 0$, by (18), we have

$$v_\varepsilon(x) \le C \begin{cases} (1+|x|)\log(2+|x|) & \text{if } q > 3/2 \\ (1+|x|)(\log(2+|x|))^2 & \text{if } q = 3/2 \\ (1+|x|)^{4-2q} & \text{if } q < 3/2. \end{cases} \tag{19}$$

Therefore, for some $\varepsilon_k \downarrow 0$ we must have $v_{\varepsilon_k} \to v$ in $C^3_{loc}(\mathbb{R}^3)$ for some v in $\mathbb{R}^3$, where v satisfies

$$\Delta^2 v = -\frac{1}{(v+P_0)^q} \quad \text{in } \mathbb{R}^3, \quad v \ge 0 \quad \text{in } \mathbb{R}^3, \quad P_0(x) := 1 + \kappa_1 x_1^2 + \kappa_2(x_2^2 + x_3^2).$$

Hence, $u = v + P_0$ is a solution to (1). Moreover, as v satisfies (19), we have

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|^2} = \liminf_{|x|\to\infty} \frac{P_0(x)}{|x|^2} = \kappa_1, \quad \limsup_{|x|\to\infty} \frac{u(x)}{|x|^2} = \limsup_{|x|\to\infty} \frac{P_0(x)}{|x|^2} = \kappa_2.$$

This completes the proof.

□

Proof of Theorem 1.3 Let $q > 7$ be fixed. Then for every $\varepsilon > 0$ there exists a positive solution v_ε to (16) with

$$P(x) = P_\varepsilon(x) := 1 + \varepsilon x_1^2 + \kappa(x_2^2 + x_3^2).$$

Setting $u_\varepsilon := v_\varepsilon + P_\varepsilon$ one gets

$$u_\varepsilon(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x-y|}{u_\varepsilon^q(y)} dy + P_\varepsilon(x), \quad \min_{\mathbb{R}^3} u_\varepsilon = u_\varepsilon(0). \tag{20}$$

Since $c_q := \frac{1}{2} - \frac{3}{q-1} > 0$ for $q > 7$, from (23), one obtains

$$\begin{aligned} 0 &= c_q \int_{\mathbb{R}^3} \frac{1}{u_\varepsilon^{q-1}(x)} dx + \frac{1}{2} \int_{\mathbb{R}^3} \frac{2x \cdot \nabla P_\varepsilon(x) - P_\varepsilon(x)}{u^q(x)} dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \frac{3P_\varepsilon(x) + 2c_q u_\varepsilon(x) - 4}{u_\varepsilon^q(x)} dx, \end{aligned}$$

which implies that $2c_q u_\varepsilon(0) < 4$, that is, $u_\varepsilon(0) \le C$. Therefore, by (20)

$$\frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|y|}{u_\varepsilon^q(y)} dy = u_\varepsilon(0) - 1 \le C. \tag{21}$$

Hence, differentiating under the integral sign, from (20)

$$|\nabla(u_\varepsilon(x) - P_\varepsilon(x))| \le \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{dy}{u_\varepsilon^q(y)} \le C.$$

Thus, $(u_\varepsilon)_{0<\varepsilon\le 1}$ is bounded in $C^1_{loc}(\mathbb{R}^3)$. This yields

$$u_\varepsilon(x) \ge \frac{1}{8\pi}\int_{B_1} \frac{|x-y|}{u_\varepsilon^q(y)} dy \ge \delta|x| \quad \text{for } |x| \ge 2,$$

for some $\delta > 0$. Using this, and recalling that $q > 4$, we deduce

$$\lim_{R\to\infty} \sup_{0<\varepsilon\le 1} \int_{|y|\ge R} \frac{|y|}{u_\varepsilon^q(y)} dy = 0.$$

Therefore, for some $\varepsilon_k \downarrow 0$, we have $u_{\varepsilon_k} \to u$, where u satisfies

$$u(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x-y|}{u^q(y)} dy + 1 + \kappa(x_2^2 + x_3^2).$$

We conclude the proof. □

2.1 A new proof of *iii)*-*iv)* of Theorem A

Proof of *iii)* Let $q > 3$ be fixed. Then by Proposition 2.1, for every $\varepsilon > 0$, there exists a radial function u_ε satisfying

$$u_\varepsilon(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x-y| - |y|}{u_\varepsilon^q(y)} dy + 1 + \varepsilon|x|^2, \quad \min_{\mathbb{R}^3} u_\varepsilon = u_\varepsilon(0) = 1.$$

Since u_ε is radially symmetric, one has (see Eq. (3.3) in [5])

$$u_\varepsilon(r) \ge \delta(1 + r^4)^{\frac{1}{q+1}},$$

for some $\delta > 0$. Therefore, as $q > 3$

$$\int_{\mathbb{R}^3} \frac{dx}{u_\varepsilon^q(x)} \le C\int_{\mathbb{R}^3} \frac{dx}{(1+|x|^4)^{\frac{q}{q+1}}} \le C,$$

which gives

$$|\nabla u_\varepsilon(x)| \le \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{1}{u_\varepsilon^q(y)} dy + 2\varepsilon|x| \le C + 2\varepsilon|x|.$$

As $u_\varepsilon(0) = 1$, one would get

$$u_\varepsilon(x) \le 1 + C|x| + C\varepsilon|x|^2.$$

Thus, the family $(u_\varepsilon)_{0<\varepsilon\le 1}$ is bounded in $C^1_{loc}(\mathbb{R}^3)$. Hence, for some $\varepsilon_k \downarrow 0$ we have $u_{\varepsilon_k} \to u$ where u satisfies

$$u(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x-y|-|y|}{u^q(y)}dy + 1, \quad \min_{\mathbb{R}^3} u = u(0) = 1.$$

Finally, as before, we have

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|} = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{dy}{u^q(y)}.$$

This completes the proof of *iii*). □

Proof of *iv*) Let $q > 1$ be fixed. Then by Proposition 2.1, for every $\varepsilon > 0$, there exists a non-negative radial function v_ε satisfying

$$v_\varepsilon(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x-y|-|y|}{(1+|y|^2+\varepsilon|y|^4+v_\varepsilon(y))^q}dy.$$

The rest of the proof is similar to that of Theorem 1.2. □

In the spirit of [5, Lemma 4.9] we prove the following Pohozaev type identity.

Lemma 2.3 (Pohozaev identity) *Let u be a positive solution to*

$$u(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|x-y|}{u^q(y)}dy + P(x), \tag{22}$$

for some non-negative polynomial P of degree at most 2 and $q > 4$. Then

$$\left(\frac{1}{2} - \frac{3}{q-1}\right)\int_{\mathbb{R}^3} \frac{1}{u^{q-1}(x)}dx + \frac{1}{2}\int_{\mathbb{R}^3} \frac{2x\cdot\nabla P(x) - P(x)}{u^q(x)}dx = 0 \tag{23}$$

Proof. Differentiating under the integral sign, from (22)

$$x\cdot\nabla u(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{x\cdot(x-y)}{|x-y|}\frac{1}{u^q(y)}dy + x\cdot\nabla P(x).$$

Multiplying the above identity by $u^{-q}(x)$ and integrating on B_R

$$\int_{B_R} \frac{x\cdot\nabla u(x)}{u^q(x)}dx = \frac{1}{8\pi}\int_{B_R}\int_{\mathbb{R}^3} \frac{x\cdot(x-y)}{|x-y|}\frac{1}{u^q(x)u^q(y)}dydx + \int_{B_R} \frac{x\cdot\nabla P(x)}{u^q(x)}dx. \tag{24}$$

Integration by parts yields

$$\begin{aligned}\int_{B_R} \frac{x\cdot\nabla u(x)}{u^q(x)}dx &= \frac{1}{1-q}\int_{B_R} x\cdot\nabla(u^{1-q}(x))dx \\ &= -\frac{3}{1-q}\int_{B_R} u^{1-q}dx + \frac{R}{1-q}\int_{\partial B_R} u^{1-q}d\sigma.\end{aligned}$$

Since $q > 4$ and $u(x) \geq \delta|x|$ for some $\delta > 0$ and $|x|$ large

$$\lim_{R\to\infty} R \int_{\partial B_R} u^{1-q} d\sigma = 0.$$

Writing $x = \frac{1}{2}((x+y)+(x-y))$, and setting

$$F(x,y) := \frac{(x+y)\cdot(x-y)}{|x-y|} \frac{1}{u^q(x)u^q(y)}$$

we get

$$\begin{aligned}
&\frac{1}{8\pi}\int_{B_R}\int_{\mathbb{R}^3} \frac{x\cdot(x-y)}{|x-y|}\frac{1}{u^q(x)u^q(y)}dydx \\
&= \frac{1}{2}\int_{B_R}\frac{1}{u^q(x)}\left(\frac{1}{8\pi}\int_{\mathbb{R}^3}\frac{|x-y|}{u^q(y)}dy\right)dx + \frac{1}{16\pi}\int_{B_R}\int_{\mathbb{R}^3} F(x,y)dydx \\
&= \frac{1}{2}\int_{B_R}\frac{1}{u^q(x)}(u(x)-P(x))dx + \frac{1}{16\pi}\int_{B_R}\int_{\mathbb{R}^3} F(x,y)dydx.
\end{aligned}$$

Notice that $F(x,y) = -F(y,x)$. Hence,

$$\int_{B_R}\int_{B_R} F(x,y)dydx = 0,$$

and

$$\lim_{R\to\infty}\int_{B_R}\int_{\mathbb{R}^3} F(x,y)dydx = \lim_{R\to\infty}\int_{B_R}\int_{B_R^c} F(x,y)dydx = 0,$$

where the last equality follows from $|x|u^{-q}(x) \in L^1(\mathbb{R}^3)$. Combining these estimates and taking $R \to \infty$ in (24) one gets (23). □

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