ON THE CLASSIFICATION OF STABLE SOLUTION TO BIHARMONIC PROBLEMS IN LARGE DIMENSIONS

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ABSTRACT. We give a new bound on the exponent for nonexistence of stable solutions to the biharmonic problem

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n$$

where $p > 1, n \geq 20$.

1. INTRODUCTION

Of concern is the following biharmonic equation

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n \tag{1.1}$$

where $n \geq 5$ and $p > 1$. Set

$$\Lambda_u(\varphi) := \int_{\mathbb{R}^n} |\Delta\varphi|^2 dx - p\int_{\mathbb{R}^n} u^{p-1}\varphi^2 dx, \quad \forall\ \varphi \in H^2(\mathbb{R}^n). \tag{1.2}$$

The Morse index of a classical solution to (1.1), $ind(u)$ is defined as the maximal dimension of all subspaces of $H^2(\mathbb{R}^n)$ such that $\Lambda_u(\varphi) < 0$ in $H^2(\mathbb{R}^n) \setminus \{0\}$. We say u is a stable solution to (1.1) if $\Lambda_u(\varphi) \geq 0$ for any test function $\varphi \in H^2(\mathbb{R}^n)$, i.e., the Morse index is zero.

In the first part of the paper, we obtain the following classification result on stable solutions of (1.1).

Theorem 1.1. *Let $n \geq 20$ and $1 < p < 1 + \frac{8p^*}{n-4}$. Then equation (1.1) has no stable solutions.*

In the above theorem, p^* stands for the smallest real root which is greater than $\frac{n-4}{n-8}$ of the following algebraic equation

$$\begin{aligned}&512(2-n)x^6 + 4(n^3 - 60n^2 + 670n - 1344)x^5 - 2(13n^3 - 424n^2 + 3064n - 5408)x^4\\&+ 2(27n^3 - 572n^2 + 3264n - 5440)x^3 - (49n^3 - 772n^2 + 3776n - 5888)x^2\\&+ 4(5n^3 - 66n^2 + 288n - 416)x - 3(n^3 - 12n^2 + 48n - 64) = 0.\end{aligned}$$

Some remarks are in order. Let us recall that for the second order problem

$$\Delta u + u^p = 0 \quad u > 0 \text{ in } \mathbb{R}^n,\ p > 1, \tag{1.3}$$

Farina [4] gave a complete classification of all finite Morse index solutions. The main result of [4] is that no stable solution exists to (1.3) if either $n \leq 10, p > 1$ or $n \geq 11,\ p < p_{JL}$. Here p_{JL} denotes the well-known Joseph-Lundgren exponent ([9]). On the other hand, stable radial solution exists for $p \geq p_{JL}$.

For the fourth order case, the nonexistence of positive solutions to (1.1) is shown if $p < \frac{n+4}{n-4}$, and all entire solutions are classified if $p = \frac{n+4}{n-4}$. See [13] and [16]. When $p > \frac{n+4}{n-4}$, radially symmetric solutions to (1.1) are completely classified in [5], [6] and [12]. The radial solutions are shown to be stable if and only if $p \geq p'_{JL}$ and $n \geq 13$, where p'_{JL} stands for the corresponding Joseph-Lundgren exponent (see[5], [6]). In the general nonradial case, Wei and Ye [17] showed the nonexistence of stable or finite Morse index solutions when either $n \leq 8, p > 1$ or $n \geq 9, p \leq \frac{n}{n-8}$. In dimensions $n \geq 9$, a perturbation argument is used to show the nonexistence of stable solutions for $p < \frac{n}{n-8} + \epsilon_n$ for some $\epsilon_n > 0$. However, no explicit value of ϵ_n is given. The proof of [17] follows an earlier idea of Cowan-Esposito-Ghoussoub [3] in which a similar problem in a bounded domain was studied. Theorem 1.1 gives an explicit value on ϵ_n for $n \geq 20$.

In the second order case, the proof of Farina uses basically the Moser iterations: namely multiply the equation (1.3) by the power of u, like $u^q, q > 1$. Moser iteration works because of the following simple identity

$$\int_{\mathbb{R}^n} u^q(-\Delta u) = \frac{4q}{(q+1)^2} \int_{\mathbb{R}^n} |\nabla u^{\frac{q+1}{2}}|^2, \forall u \in C_0^1(\mathbb{R}^n).$$

In the fourth order case, such equality does not hold, and in fact we have

$$\int_{\mathbb{R}^n} u^q(\Delta^2 u) = \frac{4q}{(q+1)^2} \int_{\mathbb{R}^n} |\Delta u^{\frac{q+1}{2}}|^2 - q(q-1)^2 \int_{\mathbb{R}^n} u^{q-3}|\nabla u|^4, \forall u \in C_0^2(\mathbb{R}^n).$$

The additional term $\int_{\mathbb{R}^n} u^{q-3}|\nabla u|^4$ makes the Moser iteration argument difficult to use. In [17], they used instead the new test function $-\Delta u$ and showed that $\int_{\mathbb{R}^2} |\Delta u|^2$ is bounded. Thus the exponent $\frac{n}{n-8}$ is obtained. In this paper, we use the Moser iteration for the fourth order problem and give a control on the term $\int_{\mathbb{R}^n} u^{q-3}|\nabla u|^4$ (Lemma 2.3). As a result, we obtain a better exponent $\frac{n}{n-8} + \epsilon_n$ where ϵ_n is explicitly given. As far as we know, this seems to be the first result for Moser iteration for a fourth order problem.

In the second part of this paper, we show that the same idea can be used to establish the regularity of extremal solutions to

$$\begin{cases} \Delta^2 u = \lambda(u+1)^p, \ \lambda > 0 & \text{in} \quad \Omega \\ u > 0, & \text{in} \quad \Omega \\ u = \Delta u = 0, & \text{on} \quad \partial\Omega \end{cases} \tag{1.4}$$

where Ω is a smooth and bounded convex domain in $\mathbb{R}^n$.

For problem (1.4), it is known ([2]) that for $p > \frac{n+4}{n-4}$ there exists a critical value $\lambda^* > 0$ depending on $p > 1$ and Ω such that

- If $\lambda \in (0, \lambda^*)$, equation (1.4) has a minimal and classical solution which is stable;
- If $\lambda = \lambda^*$, a unique weak solution, called the extremal solution u^* exists for equation (1.4);
- No weak solution of equation (1.4) exists whenever $\lambda > \lambda^*$.

The regularity of the extremal solution of problem (1.4) at $\lambda = \lambda_*$ has been studied in [3] and in [17], where they showed that the extremal solution is bounded provided $n \leq 8$ or $p < \frac{n}{n-8} + \varepsilon_n$, $n \geq 9$ (ε_n very small). Here, we also give a explicit bound for the exponent p in large dimensions and our second result is the following.

Theorem 1.2. *The extremal solution u^* of (1.4) when $\lambda = \lambda^*$ is bounded provided that $n \geq 20$ and $1 < p < 1 + \frac{8p^*}{n-4}$, where p^* is defined as above.*

As $n \to +\infty$, the value ϵ_n is asymptotically $\frac{8\sqrt{8/3}}{(n-8)^{3/2}}$ and thus the upper bound for p has the following expansion

$$1 + \frac{8}{n-8} + \frac{8\sqrt{8/3}}{(n-8)^{3/2}} + O(\frac{1}{(n-8)^2}). \tag{1.5}$$

On the other hand, for radial solutions, the Joseph-Lundgren exponent ([9]) has the following asymptotic expansion

$$1 + \frac{8}{n-8} + \frac{16}{(n-8)^{3/2}} + O(\frac{1}{(n-8)^2}). \tag{1.6}$$

In this paper, we have only considered the fourth order problems with power-like nonlinearity. Other kinds of nonlinearity, such as exponential and negative powers, also appear in many applications. See [3]. However, our technique here yields no improvements of results of [3] in the case of exponential and negative nonlinearities.

This paper is organized as follows. We prove Theorem 1.1 and Theorem 1.2 respectively in section 2 and section 3. Some technical inequalities are given in the appendix.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 through a series of Lemmas. First of all, we have following.

Lemma 2.1. *For any $\varphi \in C_0^4(\mathbb{R}^n), \varphi \geq 0$, $\gamma > 1$ and $\varepsilon > 0$ arbitrary small number, we have*

$$\int_{\mathbb{R}^n} (\Delta(u^\gamma\varphi^\gamma))^2 \leq \int_{\mathbb{R}^n} ((\Delta u^\gamma \varphi^\gamma)^2 + \varepsilon|\nabla u|^4 \varphi^{2\gamma} u^{2\gamma-4} + Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|), \tag{2.1}$$

$$\int_{\mathbb{R}^n} (\Delta(u^\gamma\varphi^\gamma))^2 \geq \int_{\mathbb{R}^n} ((\Delta u^\gamma \varphi^\gamma)^2 - \varepsilon|\nabla u|^4 \varphi^{2\gamma} u^{2\gamma-4} - Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|), \tag{2.2}$$

$$\begin{aligned}\int_{\mathbb{R}^n} ((u^\gamma)_{ij})^2\varphi^{2\gamma} \leq \int_{\mathbb{R}^n} ((u^\gamma\varphi^\gamma)_{ij})^2 + \varepsilon \int_{\mathbb{R}^n} |\nabla u|^4 u^{2\gamma-4}\varphi^{2\gamma} \\ + C\int_{\mathbb{R}^n} u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|,\end{aligned} \tag{2.3}$$

where C is a positive number only depends on γ, ε and $\|\nabla^4(\varphi^{2\gamma})\|$ is defined by

$$\|\nabla^4(\varphi^{2\gamma})\|^2 = \varphi^{-2\gamma}|\nabla\varphi^\gamma|^4 + |\varphi^\gamma(\Delta^2\varphi^\gamma)| + |\nabla^2\varphi^\gamma|^2.$$

In the following, unless otherwise, the constant C in this section always denotes a positive number which may change term by term but only depends on γ, ε.

Proof. Since φ is compactly supported, we can use integration by parts without considering the boundary terms. First, by direct calculations, we get

$$\begin{aligned}(\Delta(u^\gamma\varphi^\gamma))^2 =& [(\Delta u^\gamma)\varphi^\gamma]^2 + 4\nabla u^\gamma\nabla\varphi^\gamma\Delta\varphi^\gamma u^\gamma + 4\nabla u^\gamma\nabla\varphi^\gamma\Delta u^\gamma\varphi^\gamma \\ &+ 4(\nabla u^\gamma\nabla\varphi^\gamma)^2 + 2\Delta u^\gamma u^\gamma\Delta\varphi^\gamma\varphi^\gamma + u^{2\gamma}(\Delta\varphi^\gamma)^2.\end{aligned} \tag{2.4}$$

We now need to deal with the third and the fifth term on the right hand side of the above equality up to the integration both sides.

For the third term, we have

$$\int_{\mathbb{R}^n}\Delta u^\gamma\nabla u^\gamma\nabla\varphi^\gamma\varphi^\gamma=-\int_{\mathbb{R}^n}(u^\gamma)_i(u^\gamma)_{ij}(\varphi^\gamma)_j\varphi^\gamma$$
$$-\int_{\mathbb{R}^n}(u^\gamma)_i(u^\gamma)_j(\varphi^\gamma)_{ij}\varphi^\gamma-\int_{\mathbb{R}^n}(u^\gamma)_i(u^\gamma)_j(\varphi^\gamma)_j(\varphi^\gamma)_i,$$

where $f_i=\frac{\partial f}{\partial x_i}$ and $f_{ij}=\frac{\partial^2 f}{\partial x_j\partial x_i}$. (Here and in the sequel, we use the Einstein summation convention: an index occurring twice in a product is to be summed from 1 up to the space dimension, e.g., $u_iv_i=\sum_{i=1}^n u_iv_i, \partial_i(u_iu_j\varphi_j)=\sum_{1\le i,j\le n}\partial_i(u_iu_j\varphi_j)$.) The first term on the right hand side of the previous equation can be estimated as

$$2\int_{\mathbb{R}^n}(u^\gamma)_i(u^\gamma)_{ij}(\varphi^\gamma)_j\varphi^\gamma=\int_{\mathbb{R}^n}\partial_j((u^\gamma)_i(u^\gamma)_i(\varphi^\gamma)_j\varphi^\gamma)-\int_{\mathbb{R}^n}((u^\gamma)_i)^2(\varphi^\gamma)_{jj}\varphi^\gamma$$
$$-\int_{\mathbb{R}^n}((u^\gamma)_i)^2(\varphi^\gamma)_j(\varphi^\gamma)_j.$$

Combining these two equalities, we get

$$\begin{aligned}2\int_{\mathbb{R}^n}\Delta u^\gamma\nabla u^\gamma\nabla\varphi^\gamma\varphi^\gamma=&-\int_{\mathbb{R}^n}\partial_j((u^\gamma)_i(u^\gamma)_i(\varphi^\gamma)_j\varphi^\gamma)\\&-\int_{\mathbb{R}^n}2(u^\gamma)_i(u^\gamma)_j(\varphi^\gamma)_{ij}\varphi^\gamma-\int_{\mathbb{R}^n}2(u^\gamma)_i(u^\gamma)_j(\varphi^\gamma)_j(\varphi^\gamma)_i\\&+\int_{\mathbb{R}^n}((u^\gamma)_i)^2(\varphi^\gamma)_{jj}\varphi^\gamma+\int_{\mathbb{R}^n}((u^\gamma)_i)^2(\varphi^\gamma)_j(\varphi^\gamma)_j.\end{aligned}$$

Rewriting the above equality we have

$$\begin{aligned}4\int_{\mathbb{R}^n}\Delta u^\gamma\nabla u^\gamma\nabla\varphi^\gamma\varphi^\gamma=&2\int_{\mathbb{R}^n}|\nabla u^\gamma|^2\Delta\varphi^\gamma\varphi^\gamma+2\int_{\mathbb{R}^n}|\nabla u^\gamma|^2|\nabla\varphi^\gamma|^2\\&-4\int_{\mathbb{R}^n}(u^\gamma)_i(u^\gamma)_j(\varphi^\gamma)_{ij}\varphi^\gamma-4\int_{\mathbb{R}^n}(<\nabla u^\gamma,\nabla\varphi^\gamma>)^2.\end{aligned}\tag{2.5}$$

For the fifth term on the right hand side of Equation (2.4) we have

$$\begin{aligned}\int_{\mathbb{R}^n}\Delta u^\gamma u^\gamma\Delta\varphi^\gamma\varphi^\gamma=&-\int_{\mathbb{R}^n}u^\gamma<\nabla u^\gamma,\nabla(\Delta\varphi^\gamma)>\varphi^\gamma\\&-\int_{\mathbb{R}^n}<\nabla u^\gamma,\nabla\varphi^\gamma>u^\gamma\Delta\varphi^\gamma-\int_{\mathbb{R}^n}|\nabla u^\gamma|^2\Delta\varphi^\gamma\varphi^\gamma.\end{aligned}\tag{2.6}$$

Combining Equations (2.4), (2.5) and (2.6), one obtains

$$\begin{aligned}&\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2-\int_{\mathbb{R}^n}(\Delta u^\gamma)^2\varphi^{2\gamma}\\&=2\int_{\mathbb{R}^n}|\nabla u^\gamma|^2|\nabla\varphi^\gamma|^2-4\int_{\mathbb{R}^n}\varphi^\gamma(\nabla^2\varphi^\gamma(\nabla u^\gamma,\nabla u^\gamma))\\&+\int_{\mathbb{R}^n}u^{2\gamma}\varphi^\gamma\Delta^2(\varphi^\gamma)-2\int_{\mathbb{R}^n}u^{2\gamma}(\Delta\varphi^\gamma)^2.\end{aligned}\tag{2.7}$$

Now by the Young equality, for any $\varepsilon>0$, there exists a constant $C=C(\gamma,\varepsilon)$ such that

$$|\nabla u^\gamma|^2|\nabla\varphi^\gamma|^2\le\frac{\varepsilon}{4}|\nabla u^\gamma|^4u^{-2\gamma}\varphi^{2\gamma}+C|\nabla\varphi^\gamma|^4u^{2\gamma}\varphi^{-2\gamma}$$

and

$$|\varphi^\gamma(\nabla^2\varphi^\gamma(\nabla u^\gamma,\nabla u^\gamma))| \le \frac{\varepsilon}{8}|\nabla u^\gamma|^4 u^{-2\gamma}\varphi^{2\gamma} + Cu^{2\gamma}|\nabla^2\varphi^\gamma|^2.$$

Thus by the equation (2.7), together with the above two estimates, one gets:

$$|\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2 - \int_{\mathbb{R}^n}(\Delta u^\gamma)^2\varphi^{2\gamma}| \le \varepsilon\int_{\mathbb{R}^n}|\nabla u^\gamma|^4 u^{-2\gamma}\varphi^{2\gamma} + 6C\int_{\mathbb{R}^n} u^{2\gamma}\|\nabla^4\varphi^\gamma\|^2.$$

The estimates (2.1) and (2.2) follow from this easily.

Next we observe that $|\nabla^2 u^\gamma|^2\varphi^{2\gamma} = [\frac{1}{2}\Delta|\nabla u^\gamma|^2 - <\nabla u^\gamma, \nabla\Delta u^\gamma>]\varphi^{2\gamma}$. Thus up to the integration by parts, with the help of equation (2.5) and the estimates we just proved, the estimate (2.3) also follows by noticing the identity $\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2 = \int_{\mathbb{R}^n}|\nabla^2(u^\gamma\varphi^\gamma)|^2$. The proof of Lemma 2.1 is thus completed. □

Let us return to the equation

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n. \tag{2.8}$$

Fix $q = 2\gamma - 1 > 0$ and $\gamma > 1$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Multiplying (2.8) by $u^q\varphi^{2\gamma}$ and integration by parts, we obtain

$$\int_{\mathbb{R}^n}\Delta u\Delta(u^q\varphi^{2\gamma}) = \int_{\mathbb{R}^n} u^{p+q}\varphi^{2\gamma}. \tag{2.9}$$

For the left hand side of (2.9), we have the following lemma.

Lemma 2.2. *For any $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \ge 0$, for any $\varepsilon > 0$ and γ with q defined above, there exists a positive constant C depends on γ, ε such that*

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{\gamma^2}{q}\Delta u\Delta(u^q\varphi^{2\gamma}) \ge & \int_{\mathbb{R}^n}(\Delta u^\gamma\varphi^\gamma)^2 - \int_{\mathbb{R}^n} Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| \\ & - \int_{\mathbb{R}^n}(\gamma^2(\gamma-1)^2+\varepsilon)u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}.\end{aligned} \tag{2.10}$$

Proof. First, by direct computations, we obtain

$$\begin{aligned}\Delta u\Delta(u^{2\gamma-1}\varphi^{2\gamma}) =& \Delta u((2\gamma-1)u^{2\gamma-2}\Delta u\varphi^{2\gamma} + 2(2\gamma-1)u^{2\gamma-2}\nabla u\nabla(\varphi^{2\gamma}) \\ &+ (2\gamma-1)(2\gamma-2)u^{2\gamma-3}|\nabla u|^2\varphi^{2\gamma} + u^{2\gamma-1}\Delta\varphi^{2\gamma}),\end{aligned}$$

$$\begin{aligned}(\Delta u^\gamma\varphi^\gamma)^2 =& \gamma^2 u^{2\gamma-2}(\Delta u)^2\varphi^{2\gamma} + \gamma^2(\gamma-1)^2u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} \\ &+ 2(\gamma-1)\gamma^2u^{2\gamma-3}|\nabla u|^2\Delta u\varphi^{2\gamma}.\end{aligned}$$

Combining the above two identities, we get

$$\begin{aligned}\frac{\gamma^2}{q}\Delta u\Delta(u^q\varphi^{2\gamma}) =& (\Delta u^\gamma\varphi^\gamma)^2 + 2\gamma^2u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma} + \frac{\gamma^2}{q}u^{2\gamma-1}\Delta u\Delta\varphi^{2\gamma} \\ &- \gamma^2(\gamma-1)^2u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}.\end{aligned} \tag{2.11}$$

For the term $u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma}$, we have

$$\begin{aligned}u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma} =& \partial_i(u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_j) - (2\gamma-2)u^{2\gamma-3}(u_i)^2u_j(\varphi^{2\gamma})_j \\ &- u^{2\gamma-2}u_iu_{ij}(\varphi^{2\gamma})_j - u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_{ij}.\end{aligned}$$

We can regroup the term $u^{2\gamma-2}u_iu_{ij}(\varphi^{2\gamma})_j$ as

$$\begin{aligned}2u^{2\gamma-2}u_iu_{ij}(\varphi^{2\gamma})_j =& \partial_j(u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_j) - (2\gamma-2)u^{2\gamma-3}u_j(u_i)^2(\varphi^{2\gamma})_j \\ &- u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_{jj}.\end{aligned}$$

Therefore we get

$$\begin{aligned}2u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma} =&2\partial_i(u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_j)-\partial_j(u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_j)\\&-(2\gamma-2)u^{2\gamma-3}(u_i)^2u_j(\varphi^{2\gamma})_j+u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_{jj}\\&-2u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_{ij}.\end{aligned}\tag{2.12}$$

For the last three terms on the right hand side of (2.12), applying Young's inequality, we get

$$\begin{aligned}|u^{2\gamma-3}(u_i)^2u_j(\varphi^{2\gamma})_j|\le&\frac{\varepsilon}{6\gamma^2(\gamma-1)}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|,\\|u^{2\gamma-2}(u_i)^2(\varphi^{2\gamma})_{jj}|\le&\frac{\varepsilon}{6\gamma^2}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|,\\|u^{2\gamma-2}u_iu_j(\varphi^{2\gamma})_{ij}|\le&\frac{\varepsilon}{6\gamma^2}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}+Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|.\end{aligned}$$

By the above three inequalities and (2.12), we have

$$\int_{\mathbb{R}^n}2\gamma^2u^{2\gamma-2}\Delta u\nabla u\nabla\varphi^{2\gamma}\ge-\frac{\varepsilon}{2}\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}-C\int_{\mathbb{R}^n}u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|.\tag{2.13}$$

Similarly we get

$$\int_{\mathbb{R}^n}\frac{\gamma^2}{q}u^{2\gamma-1}\Delta u\Delta\varphi^{2\gamma}\ge-\frac{\varepsilon}{2}\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}-C\int_{\mathbb{R}^n}u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|.\tag{2.14}$$

Inequality (2.10) follows from (2.11), (2.13) and (2.14). □

As a result of (2.1) and (2.10), we have

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{\gamma^2}{q}\Delta u\Delta(u^q\varphi^{2\gamma})\ge&\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2-\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|\\&-\int_{\mathbb{R}^n}(\gamma^2(\gamma-1)^2+\varepsilon)u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}.\end{aligned}\tag{2.15}$$

Next we estimate the most difficult term $\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}$ in (2.15). This is the key step in proving Theorem 1.1.

Lemma 2.3. *If u is the classical solution to the biharmonic equation (2.8), and φ is defined as above, then for any sufficiently small $\varepsilon>0$, we have the following inequality*

$$\begin{aligned}(\frac{1}{2}-\varepsilon)\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}\le&\frac{2}{\gamma^2}\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2+\int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|\\&-\int_{\mathbb{R}^n}\frac{4}{(4\gamma-3+p)(p+1)}u^{2\gamma+p-1}\varphi^{2\gamma}.\end{aligned}\tag{2.16}$$

Proof. It is easy to see that

$$\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma}=\frac{1}{\gamma^4}\int_{\mathbb{R}^n}u^{-2\gamma}|\nabla u^\gamma|^4\varphi^{2\gamma},\tag{2.17}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} u^{-2\gamma}|\nabla u^\gamma|^4\varphi^{2\gamma} &= \int_{\mathbb{R}^n} u^{-2\gamma}|\nabla u^\gamma|^2\nabla u^\gamma\nabla u^\gamma\varphi^{2\gamma}\\ &= \int_{\mathbb{R}^n} -\nabla u^{-\gamma}|\nabla u^\gamma|^2\nabla u^\gamma\varphi^{2\gamma}\\ &= \int_{\mathbb{R}^n} u^{-\gamma}|\nabla u^\gamma|^2\Delta u^\gamma\varphi^{2\gamma} + \int_{\mathbb{R}^n} u^{-\gamma}\nabla(|\nabla u^\gamma|^2)\nabla u^\gamma\varphi^{2\gamma}\\ &\quad + \int_{\mathbb{R}^n} u^{-\gamma}|\nabla u^\gamma|^2\nabla u^\gamma\nabla\varphi^{2\gamma}, \end{aligned} \tag{2.18}$$

where in the last step we used integration by parts. For the first term in the last part of the above equality, we have

$$\int_{\mathbb{R}^n} u^{-\gamma}|\nabla u^\gamma|^2\Delta u^\gamma\varphi^{2\gamma} = \gamma^3\int_{\mathbb{R}^n}((\gamma-1)u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} + u^{2\gamma-3}|\nabla u|^2\Delta u\varphi^{2\gamma}). \tag{2.19}$$

Substituting (2.19) into (2.18), and combining with (2.17), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} &= \int_{\mathbb{R}^n}\frac{1}{\gamma^3}u^{-\gamma}\nabla(|\nabla u^\gamma|^2)\nabla u^\gamma\varphi^{2\gamma} + \int_{\mathbb{R}^n} u^{2\gamma-3}(|\nabla u|^2)\Delta u\varphi^{2\gamma}\\ &\quad + \int_{\mathbb{R}^n}\frac{1}{\gamma^3}u^{-\gamma}(|\nabla u^\gamma|^2)\nabla u^\gamma\nabla\varphi^{2\gamma}. \end{aligned} \tag{2.20}$$

The first term on the right hand side of (2.20) can be estimated as

$$\begin{aligned}u^{-\gamma}\nabla(|\nabla u^\gamma|^2)\nabla u^\gamma &= 2u^{-\gamma}((u^\gamma)_{ij}(u^\gamma)_i(u^\gamma)_j)\\ &\le 2\gamma(u^\gamma)_{ij}(u^\gamma)_{ij} + \frac{u^{-2\gamma}}{2\gamma}(u^\gamma)_i(u^\gamma)_j(u^\gamma)_i(u^\gamma)_j\\ &= 2\gamma|\nabla^2u^\gamma|^2 + \frac{u^{-2\gamma}}{2\gamma}|\nabla u^\gamma|^4. \end{aligned}$$

As a consequence, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{1}{\gamma^3}u^{-\gamma}\nabla(|\nabla u^\gamma|^2)\nabla u^\gamma\varphi^{2\gamma} &\le \int_{\mathbb{R}^n}\frac{2}{\gamma^2}|\nabla^2u^\gamma|^2\varphi^{2\gamma} + \int_{\mathbb{R}^n}\frac{1}{2\gamma^4}u^{-2\gamma}|\nabla u^\gamma|^4\varphi^{2\gamma}\\ &\le \int_{\mathbb{R}^n}\frac{2}{\gamma^2}|\nabla^2(u^\gamma\varphi^\gamma)|^2 + \int_{\mathbb{R}^n} Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|\\ &\quad + \int_{\mathbb{R}^n}\frac{1+4\gamma^2\varepsilon}{2\gamma^4}u^{-2\gamma}|\nabla u^\gamma|^4\varphi^{2\gamma}\\ &= \int_{\mathbb{R}^n}\frac{2}{\gamma^2}(\Delta(u^\gamma\varphi^\gamma))^2 + \int_{\mathbb{R}^n} Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|\\ &\quad + \int_{\mathbb{R}^n}\frac{1+4\gamma^2\varepsilon}{2\gamma^4}u^{-2\gamma}|\nabla u^\gamma|^4\varphi^{2\gamma}, \end{aligned} \tag{2.21}$$

where we used (2.3) in the last step.

For the second term on the right hand side of (2.20), applying the estimate (2.3) from [17], i.e., $(\Delta u)^2 \ge \frac{2}{p+1}u^{p+1}$, and the fact that $\Delta u < 0$ from Theorem 3.1 in

[16] or Theorem 2.1 in [18], we have

$$\begin{aligned}\int_{\mathbb{R}^n} u^{2\gamma-3}(|\nabla u|^2)\Delta u\varphi^{2\gamma} &\leq -\int_{\mathbb{R}^n}\sqrt{\frac{2}{p+1}}u^{2\gamma-3+\frac{p+1}{2}}(|\nabla u|^2)\varphi^{2\gamma}\\ &= +\int_{\mathbb{R}^n}\frac{\sqrt{\frac{2}{p+1}}}{2\gamma-2+\frac{p+1}{2}}u^{2\gamma-2+\frac{p+1}{2}}\Delta u\varphi^{2\gamma}\\ &\quad+\int_{\mathbb{R}^n}\frac{\sqrt{\frac{2}{p+1}}}{2\gamma-2+\frac{p+1}{2}}u^{2\gamma-2+\frac{p+1}{2}}\nabla u\nabla\varphi^{2\gamma}. \end{aligned}\tag{2.22}$$

Using the inequality $-\Delta u \geq \sqrt{\frac{2}{p+1}}u^{\frac{p+1}{2}}$, we get

$$\int_{\mathbb{R}^n}\frac{\sqrt{\frac{2}{p+1}}}{2\gamma-2+\frac{2}{p+1}}u^{2\gamma-2+\frac{p+1}{2}}\Delta u\varphi^{2\gamma} \leq -\int_{\mathbb{R}^n}\frac{\frac{2}{p+1}}{2\gamma-2+\frac{p+1}{2}}u^{2\gamma+p-1}\varphi^{2\gamma}. \tag{2.23}$$

On the other hand, for the second term on the right hand side of (2.22), we have

$$\begin{aligned}\int_{\mathbb{R}^n} u^{2\gamma-2+\frac{p+1}{2}}\nabla u\nabla\varphi^{2\gamma} &= -\int_{\mathbb{R}^n}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma}\\ &= -\int_{\{x|\Delta\varphi^{2\gamma}>0\}}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma}\\ &\quad-\int_{\{x|\Delta\varphi^{2\gamma}\leq 0\}}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma},\end{aligned}\tag{2.24}$$

where the first equality follows from integration by parts and $L = 2\gamma - 1 + \frac{p+1}{2}$. As for the first term on the last part of (2.24), using the inequality $\Delta u \leq -\sqrt{\frac{2}{p+1}}u^{\frac{p+1}{2}} < 0$, we have

$$\frac{\sqrt{\frac{p+1}{2}}}{L}\int_{\{x|\Delta\varphi^{2\gamma}>0\}}u^{2\gamma-1}\Delta u\Delta\varphi^{2\gamma} \leq -\int_{\{x|\Delta\varphi^{2\gamma}>0\}}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma}. \tag{2.25}$$

Similar to the proof of Lemma 2.1, it is easy to get

$$\Big|\int_{\{x|\Delta\varphi^{2\gamma}>0\}}\frac{\sqrt{\frac{p+1}{2}}}{L}u^{2\gamma-1}\Delta u\Delta\varphi^{2\gamma}\Big| \leq \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} + \int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|. \tag{2.26}$$

By (2.25) and (2.26), we have

$$\Big|\int_{\{x|\Delta\varphi^{2\gamma}>0\}}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma}\Big| \leq \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} + \int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|. \tag{2.27}$$

Similarly, we also obtain

$$\Big|\int_{\{x|\Delta\varphi^{2\gamma}\leq 0\}}\frac{1}{L}u^{2\gamma-1+\frac{p+1}{2}}\Delta\varphi^{2\gamma}\Big| \leq \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} + \int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|. \tag{2.28}$$

By (2.24), (2.27) and (2.28), we have

$$\Big|\int_{\mathbb{R}^n}u^{2\gamma-2+\frac{p+1}{2}}\nabla u\nabla\varphi^{2\gamma}\Big| \leq \varepsilon\int_{\mathbb{R}^n}u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} + \int_{\mathbb{R}^n}Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|. \tag{2.29}$$

Combining (2.22), (2.23) and (2.29), we get the following inequality

$$\int_{\mathbb{R}^n} u^{2\gamma-3}|\nabla u|^2\Delta u\varphi^{2\gamma} \leq \varepsilon \int_{\mathbb{R}^n} u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} + \int_{\mathbb{R}^n} Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| - \int_{\mathbb{R}^n} \frac{4}{(4\gamma-3+p)(p+1)}u^{2\gamma+p-1}\varphi^{2\gamma}. \tag{2.30}$$

Finally, we apply Young's inequality to the third term on the right hand side of (2.20), and get

$$\int_{\mathbb{R}^n} \frac{1}{\gamma^3}u^{-\gamma}(|\nabla u^\gamma|^2)\nabla u^\gamma\nabla\varphi^{2\gamma} = \int_{\mathbb{R}^n} u^{2\gamma-3}|\nabla u|^2\nabla u\nabla(\varphi^{2\gamma}) \leq \varepsilon \int_{\mathbb{R}^n} u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} + \int_{\mathbb{R}^n} Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|. \tag{2.31}$$

By (2.20), (2.21), (2.30) and (2.31), we finally obtain

$$(\frac{1}{2}-\varepsilon)\int_{\mathbb{R}^n} u^{2\gamma-4}|\nabla u|^4\varphi^{2\gamma} \leq \frac{2}{\gamma^2}\int_{\mathbb{R}^n}(\Delta(u^\gamma\varphi^\gamma))^2 + \int_{\mathbb{R}^n} Cu^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| - \int_{\mathbb{R}^n} \frac{4}{(4\gamma-3+p)(p+1)}u^{2\gamma+p-1}\varphi^{2\gamma}.$$

□

By (2.9), (2.15) and (2.16), since the number ε is arbitrary small in those three places, we have for $\delta > 0$ sufficiently small, the following inequality holds

$$\int_{\mathbb{R}^n}(1-4(\gamma-1)^2-\delta)(\Delta(u^\gamma\varphi^\gamma))^2 - \int_{\mathbb{R}^n}(\frac{\gamma^2}{2\gamma-1} - \frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+p)(p+1)})u^{p+2\gamma-1}\varphi^{2\gamma} \leq \int_{\mathbb{R}^n} C_\delta u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|, \tag{2.32}$$

where C_δ is a positive constant depends on δ only. Here, we need $1-4(\gamma-1)^2 > 0$, since we have assumed that $\gamma > 1$ in Lemma 2.1. So γ is required be in $(1, \frac{3}{2})$. If we can choose δ small enough to make $1-4(\gamma-1)^2-\delta$ positive, by the stability property of function u, we obtain

$$\int_{\mathbb{R}^n}(E-p\delta)u^{p+q}\varphi^{2\gamma} \leq \int_{\mathbb{R}^n} C_\delta u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\|, \tag{2.33}$$

where E is defined to be

$$E = p(1-4(\gamma-1)^2) - \frac{\gamma^2}{q} + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+p)(p+1)}. \tag{2.34}$$

Now we take $\varphi = \eta^m$ with m sufficiently large, and choose η a cut-off function satisfying $0 \leq \eta \leq 1$, $\eta = 1$ for $|x| < R$ and $\eta = 0$ for $|x| > 2R$. By Young's inequality again, we have

$$\int_{\mathbb{R}^n} u^{2\gamma}\|\nabla^4(\varphi^{2\gamma})\| \leq C_\delta R^{-4}\int_{\mathbb{R}^n} u^{2\gamma}\eta^{2\gamma m-4} \leq C_{\delta,\epsilon}R^{-\frac{4}{1-\theta}}\int_{\mathbb{R}^n} u^2\eta^{2\gamma m-\frac{4}{1-\theta}} + \epsilon C_\delta\int_{\mathbb{R}^n} u^{2\gamma+p-1}\eta^{2\gamma m}, \tag{2.35}$$

where $C_{\delta,\epsilon}$ is a positive constant depends on δ, ϵ, θ is a number such that $2(1-\theta)+(2\gamma+p-1)\theta = 2\gamma$ so that $0<\theta<1$ for $2<2\gamma<2\gamma+p-1$. By (2.33) and (2.35), we get

$$(E-p\delta-\epsilon C_\delta)\int_{\mathbb{R}^n} u^{p+2\gamma-1}\eta^{2\gamma m} \le C_{\delta,\epsilon} R^{-\frac{4}{1-\theta}} \int_{\mathbb{R}^n} u^2 \eta^{2\gamma m-\frac{4}{1-\theta}}. \tag{2.36}$$

Since θ is strictly less than 1 and will be fixed for given γ, p, we can choose m sufficiently large to make $2\gamma m - \frac{4}{1-\theta} > 0$. On the other hand, if $E>0$, we can find small δ and then small ϵ, such that $E-p\delta-\epsilon C_\delta > 0$. Therefore, by the definition of function η and (2.36), we obtain

$$(E-p\delta-\epsilon C_\delta)\int_{B_R} u^{p+2\gamma-1} \le C_{\delta,\epsilon} R^{-\frac{4}{1-\theta}} \int_{B_{2R}} u^2. \tag{2.37}$$

By (2.10) of [17], we have $\int_{B_{2R}} u^2 \le CR^{n-\frac{8}{p-1}}$, as a result, the left hand side of (2.37) is less equal than $C_{\delta,\epsilon}R^{n-\frac{8}{p-1}-\frac{4}{1-\theta}}$, which tends to 0 as R tends to ∞, provided the power $n-\frac{8}{p-1}-\frac{4}{1-\theta}$ is negative, which is equivalent to $(p+2\gamma-1) > (p-1)\frac{n}{4}$ according to the definition of θ. So, if $(p+2\gamma-1) > (p-1)\frac{n}{4}$ and $E-p\delta-C_\delta\epsilon > 0$, we have $u \equiv 0$.

Thus, we have proved the nonexistence of stable solution to (2.8) if p satisfies the condition $(p+2\gamma-1) > (p-1)\frac{n}{4}$ and $E>0$ (for δ,ϵ are arbitrary small). By Lemma 4.1 in the appendix, the power p can be in the interval $(\frac{n}{n-8}, 1+\frac{8p^*}{n-4})$. Combining with Theorem 1.1 of [17], we have proved Theorem 1.1, i.e., for any $1<p<1+\frac{8p^*}{n-4}$, $n \ge 20$, equation (2.8) has no stable solution.

3. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2. We note that it is enough to consider stable solutions u_λ to (1.4) since $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$. Now we give a uniform bound for the stable solutions to (1.4) when $0<d<\lambda<\lambda^*$, where d is a fixed positive constant from $(0,\lambda^*)$.

First, we need to analyze the solution near the boundary.

3.1. Regularity of the solution on the boundary.

In this subsection, we establish the regularity of stable solution of (3.1) and its derivative near the boundary of the following equation:

$$\begin{cases} \Delta^2 u = \lambda(u+1)^p,\ \lambda>0 & \text{in} \quad \Omega \\ u>0, & \text{in} \quad \Omega \\ u=\Delta u = 0, & \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

Theorem 3.1. *Let Ω be a bounded, smooth, and convex domain. Then there exists a constant C (independent of λ, u) and small positive number ϵ, such that for stable solutions u to (3.1) we have*

$$u(x) < C, \quad \forall x \in \Omega_\epsilon := \{z\in\Omega:\ d(z,\partial\Omega)<\epsilon\}. \tag{3.2}$$

Proof. This result is well-known. See [11]. For the sake of completeness, we include a proof here. By Lemma 3.5 of [3], we see that, there exists a constant C independent of λ, u, such that

$$\int_\Omega (1+u)^p dx \le C. \tag{3.3}$$

We write Equation (3.1) as

$$\begin{cases} \Delta u + v = 0, & \text{in} \quad \Omega \\ \Delta v + \lambda(1+u)^p = 0, & \text{in} \quad \Omega \\ u = v = 0, & \text{in} \quad \partial\Omega. \end{cases}$$

If we denote $f_1(u,v) = v,\ f_2(u,v) = \lambda(u+1)^p$, we see that $\frac{\partial f_1}{\partial v} = 1 > 0$ and $\frac{\partial f_2}{\partial u} = \lambda p(u+1)^{p-1} > 0$. Therefore, the convexity of Ω, Lemma 5.1 of [14], and the moving plane method near $\partial\Omega$ (as in the appendix of [7]) imply that there exist $t_0 > 0$ and α which depends only on the domain Ω, such that $u(x - t\nu)$ and $v(x - t\nu)$ are nondecreasing for $t \in [0, t_0]$, $\quad \nu \in \mathbb{R}^n$ satisfying $|\nu| = 1$ and $(\nu, n(x)) \ge \alpha$ and $x \in \partial\Omega$. Therefore, we can find $\rho, \epsilon > 0$ such that for any $x \in \Omega_\epsilon := \{z \in \Omega :\ d(z, \partial\Omega) < \epsilon\}$ there exists a fixed-sized cone Γ_x (with x as its vertex) with

- $\text{meas}(\Gamma_x) \ge \rho$,
- $\Gamma_x \subset \{z \in \Omega :\ d(z, \partial\Omega) < 2\epsilon\}$, and
- $u(y) \ge u(x)$ for any $y \in \Gamma_x$.

Then, for any $x \in \Omega_\epsilon$, we have

$$(1 + u(x))^p \le \frac{1}{\text{meas}(\Gamma_x)} \int_{\Gamma_x} (1+u)^p \le \frac{1}{\rho} \int_\Omega (1+u)^p \le C.$$

This implies that $(1 + u(x))^p \le C$, therefore $u(x) \le C$. □

Remark: By classical elliptic regularity theory, $u(x)$ and its derivatives up to fourth order are bounded on the boundary by a constant independent of u. See [15] for more details.

3.2. **Proof of Theorem 1.2.** In the following, we will use the idea in Section 2 to prove Theorem 1.2.

First of all, multiplying (1.4) by $(u+1)^q$ and integration by parts, we have

$$\int_\Omega \lambda(u+1)^{p+q} = \int_\Omega \Delta^2 u (u+1)^q = \int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial n} + \int_\Omega \Delta(u+1)\Delta(u+1)^q. \tag{3.4}$$

Setting $v = u + 1$, by direct calculations, we get

$$\begin{aligned} \int_\Omega (\Delta v^\gamma)^2 = \int_\Omega \gamma^2 v^{2\gamma-2}(\Delta v)^2 + \int_\Omega \gamma^2(\gamma-1)^2 v^{2\gamma-4}|\nabla v|^4 \\ + 2\int_\Omega \gamma^2(\gamma-1)v^{2\gamma-3}\Delta v|\nabla v|^2, \end{aligned} \tag{3.5}$$

$$\int_\Omega \Delta v \Delta v^q = \int_\Omega q(\Delta v)^2 v^{q-1} + \int_\Omega q(q-1)|\nabla v|^2 \Delta v v^{q-2}. \tag{3.6}$$

From (3.4), (3.5) and (3.6), we obtain

$$\int_\Omega (\frac{q}{\gamma^2}(\Delta v^\gamma)^2 - q(\gamma-1)^2|\nabla v|^4 v^{2\gamma-4}) + \int_{\partial\Omega} \frac{\partial(\Delta v)}{\partial n} = \int_\Omega \lambda v^{p+q}. \tag{3.7}$$

For the second term in (3.7), we have

$$\begin{aligned}\int_\Omega |\nabla v|^4 v^{2\gamma-4} &= \frac{1}{\gamma^4}\int_\Omega v^{-2\gamma}|\nabla v^\gamma|^4 = \frac{1}{\gamma^4}\int_\Omega |\nabla v^\gamma|^2 \nabla v^\gamma(-\nabla v^{-\gamma}) \\ &= \frac{1}{\gamma^4}\int_\Omega (-\nabla(\frac{|\nabla v^\gamma|^2\nabla v^\gamma}{v^\gamma}) + \frac{\nabla(|\nabla v^\gamma|^2)\nabla v^\gamma}{v^\gamma} + \frac{|\nabla v^\gamma|^2\Delta v^\gamma}{v^\gamma}) \\ &= \frac{1}{\gamma^4}\int_\Omega v^{-\gamma}\nabla(|\nabla v^\gamma|^2)\nabla v^\gamma + |\nabla v^\gamma|^2\Delta v^\gamma - \frac{1}{\gamma}\int_{\partial\Omega} v^{2\gamma-3}|\nabla v|^2\frac{\partial v}{\partial n}.\end{aligned} \tag{3.8}$$

Simple calculation yields

$$\frac{1}{\gamma^4}\int_\Omega v^{-\gamma}|\nabla v^\gamma|^2\Delta v^\gamma = \frac{\gamma-1}{\gamma}\int_\Omega v^{2\gamma-4}|\nabla v|^4 + \frac{1}{\gamma}\int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v. \tag{3.9}$$

Substituting (3.9) into (3.8), we get

$$\int_\Omega |\nabla v|^4 v^{2\gamma-4} = \int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v + \frac{1}{\gamma^3}\int_\Omega v^{-\gamma}\nabla(|\nabla v^\gamma|^2)\nabla v^\gamma - \int_{\partial\Omega}|\nabla v|^2\frac{\partial v}{\partial n}. \tag{3.10}$$

We now estimate the second term on the right hand side of (3.10). From the proof of Lemma 2.3, together with the identity $\frac{1}{2}\Delta|\nabla v^\gamma|^2 = |\nabla^2 v^\gamma|^2 + < \nabla\Delta v^\gamma, \nabla v^\gamma >$, the following inequality holds

$$\begin{aligned}\frac{1}{\gamma^3}\int_\Omega v^{-\gamma}\nabla(|\nabla v^\gamma|^2)\nabla v^\gamma \quad &\le \quad \frac{1}{2}\int_\Omega |\nabla v|^4 v^{2\gamma-4} + \frac{2}{\gamma^2}\int_\Omega (\Delta v^\gamma)^2 \\ &\quad + \frac{1}{\gamma^2}\int_{\partial\Omega}\frac{\partial|\nabla v^\gamma|^2}{\partial n} - \frac{2}{\gamma^2}\int_{\partial\Omega}(\Delta v^\gamma)\frac{\partial v^\gamma}{\partial n}.\end{aligned} \tag{3.11}$$

By (3.10) and (3.11), thanks to the convexity of the domain Ω, we get

$$\frac{1}{2}\int_\Omega |\nabla v|^4 v^{2\gamma-4} \le \int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v + \frac{2}{\gamma^2}\int_\Omega (\Delta v^\gamma)^2 - (2\gamma-1)\int_{\partial\Omega}|\nabla v|^2\frac{\partial v}{\partial n}. \tag{3.12}$$

For the first term on the right hand side of (3.12), since $v = u+1$, we have $\Delta v = \Delta u < 0$ by maximal principle, and the inequality $\Delta v < -\sqrt{\frac{2\lambda}{p+1}}v^{\frac{p+1}{2}} < 0$ by Lemma 3.2 of [3]. Thus

$$\int_\Omega v^{2\gamma-3}|\nabla v|^2\Delta v \le \int_\Omega -\sqrt{\frac{2\lambda}{p+1}}v^{2\gamma-3+\frac{p+1}{2}}|\nabla v|^2.$$

Moreover, we have

$$\begin{aligned}\int_\Omega -\sqrt{\frac{2\lambda}{p+1}}v^{2\gamma-3+\frac{p+1}{2}}|\nabla v|^2 &= -\int_\Omega \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma-2+\frac{p+1}{2}}\nabla(v^{2\gamma-2+\frac{p+1}{2}}\nabla v) \\ &\quad + \int_\Omega \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma-2+\frac{p+1}{2}}v^{2\gamma-2+\frac{p+1}{2}}\Delta v.\end{aligned}$$

For the second term on the right hand side of the above equality, using the inequality $\Delta v < -\sqrt{\frac{2\lambda}{p+1}} v^{\frac{p+1}{2}} < 0$ again, we have

$$\int_\Omega \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma - 2 + \frac{p+1}{2}} v^{2\gamma-2+\frac{p+1}{2}} \Delta v \leq -\int_\Omega \frac{\frac{2\lambda}{p+1}}{2\gamma - 2 + \frac{p+1}{2}} v^{2\gamma+p-1}.$$

Hence, we obtain

$$\int_\Omega v^{2\gamma-3} |\nabla v|^2 \Delta v \leq -\int_{\partial\Omega} \frac{\sqrt{\frac{2\lambda}{p+1}}}{2\gamma - 2 + \frac{p+1}{2}} \frac{\partial v}{\partial n} - \int_\Omega \frac{\frac{2\lambda}{p+1}}{2\gamma - 2 + \frac{p+1}{2}} v^{2\gamma+p-1}, \quad (3.13)$$

where we used $v|_{\partial\Omega} = u + 1|_{\partial\Omega} = 1$, for the boundary term in (3.4), (3.12) and (3.13). By the remark after Theorem 3.1, we find that there exists a constant C (the constant C appeared now and later in this section is independent of u), such that

$$\int_{\partial\Omega} (|\nabla u|^2 |\frac{\partial u}{\partial n}| + |\frac{\partial(\Delta u)}{\partial n}| + |\frac{\partial u}{\partial n}|) \leq C. \quad (3.14)$$

Combining (3.7), (3.12), (3.13) and (3.14), we get

$$(1 - 4(\gamma-1)^2) \int_\Omega (\Delta(u+1)^\gamma)^2 + (\frac{8\lambda\gamma^2(\gamma-1)^2}{(4\gamma+p-3)(p+1)} - \frac{\lambda\gamma^2}{q}) \int_\Omega (u+1)^{p+q} \leq C. \quad (3.15)$$

If $(1 - 4(\gamma-1)^2) > 0$, $p(1 - 4(\gamma-1)^2) + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma+p-3)(p+1)} - \frac{\gamma^2}{q} > 0$ and u is a stable solution to the equation (1.4), we have

$$(p(1 - 4(\gamma-1)^2) + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma+p-3)(p+1)} - \frac{\gamma^2}{2\gamma-1}) \int_\Omega (u+1)^{p+q} \leq \frac{C}{\lambda}.$$

This leads to $u + 1 \in L^{p+q}$.

If $p + q > \frac{(p-1)n}{4}$, then classical regularity theory implies that $u \in L^\infty(\Omega)$.

Therefore we have established the bound of extremal solutions of (1.4) if

$$p(1 - 4(\gamma-1)^2) + \frac{8\gamma^2(\gamma-1)^2}{(4\gamma+p-3)(p+1)} - \frac{\gamma^2}{q} > 0$$

and

$$p < \frac{8\gamma + n - 4}{n - 4}.$$

By Lemma 4.1 and Theorem 3.8 of [17], we prove the extremal solution u^*, the unique solution of equation (1.4) (where $\lambda = \lambda^*$) is bounded provided that

(1) $n \leq 8$, $p > 1$,
(2) $9 \leq n \leq 19$, there exists $\varepsilon_n > 0$ such that for any $1 < p < \frac{n}{n-8} + \varepsilon_n$,
(3) $n \geq 20$, $1 < p < 1 + \frac{8p^*}{n-4}$. ($p^*$ is defined as before.)

4. APPENDIX

In this appendix, we study the following inequalities

$$p(1-4(\gamma-1)^2)-\frac{\gamma^2}{2\gamma-1}+\frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+p)(p+1)}>0, \tag{4.1}$$

$$p<\frac{8\gamma+n-4}{n-4}. \tag{4.2}$$

In order to get a better range of the power p from (4.1) and (4.2), it is necessary for us to study the following equation (Letting $p=\frac{8\gamma+n-4}{n-4}$ in (4.1)):

$$\frac{8\gamma+n-4}{n-4}(1-4(\gamma-1)^2)-\frac{\gamma^2}{2\gamma-1}+\frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+\frac{8\gamma+n-4}{n-4})(\frac{8\gamma+n-4}{n-4}+1)}=0. \tag{4.3}$$

We can only consider the behavior of (4.3) for $\gamma\in(1,\frac{3}{2})$. Through tedious computations, we see the following equation which appeared in the introduction is the simplified form of (4.3). As a consequence, they have same roots in $(1,\frac{3}{2})$:

$$\begin{aligned}&512(2-n)\gamma^6+4(n^3-60n^2+670n-1344)\gamma^5-2(13n^3-424n^2+3064n-5408)\gamma^4\\&+2(27n^3-572n^2+3264n-5440)\gamma^3-(49n^3-772n^2+3776n-5888)\gamma^2\\&+4(5n^3-66n^2+288n-416)\gamma-3(n^3-12n^2+48n-64)=0.\end{aligned} \tag{4.4}$$

We denote the left hand side of the equation (4.3) by $h(\gamma)$. Notice that if $\gamma=\frac{n-4}{n-8}$, then $p=\frac{n}{n-8}$ and $\gamma-1=\frac{4}{n-8}$. Hence

$$h(\frac{n-4}{n-8})=\frac{8}{n-8}[n^4-18n^3-56n^2+384n-512].$$

In fact, if $n=20$, then $h(\frac{4}{3})=512>0$. On the other hand, it is also easy to see that $h(\frac{3}{2})<0$, while it is obvious that $(4\gamma-3+\frac{8\gamma+n-4}{n-4})(\frac{8\gamma+n-4}{n-4}+1)>0$ and $(2\gamma-1)>0$ when $\gamma\in(\frac{n-4}{n-8},\frac{3}{2})$. Therefore, by continuity, equation (4.3) possesses a root in $(\frac{n-4}{n-8},\frac{3}{2})$. We denote the smallest root of (4.3) which is greater than $\frac{n-4}{n-8}$ by p^*. Once we pick out a γ from the interval $(\frac{n-4}{n-8},p^*)$, $h(\gamma)$ is of course positive. By continuity, we can find a small positive number δ such that, the inequality $p(1-4(\gamma-1)^2)-\frac{\gamma^2}{2\gamma-1}+\frac{8\gamma^2(\gamma-1)^2}{(4\gamma-3+p)(p+1)}>0$ holds when $p\in(\frac{8\gamma+n-4}{n-4}-\delta,\frac{8\gamma+n-4}{n-4})$. So, we conclude that when γ runs in the whole interval $(\frac{n-4}{n-8},p^*)$, the power p can be in the whole interval $(\frac{n}{n-8},1+\frac{8p^*}{n-4})$. We summarize the result as follows:

Lemma 4.1. *When $n\geq 20$, we have p which satisfies (4.1) and (4.2) can range in $(\frac{n}{n-8},1+\frac{8p^*}{n-4})$ and this interval is not empty.*

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