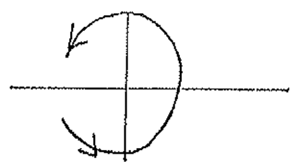
M305 SAMPLE MIDTERM PROBLEMS

SAMPLE INTEGRATION QUESTIONS

CALCULATE

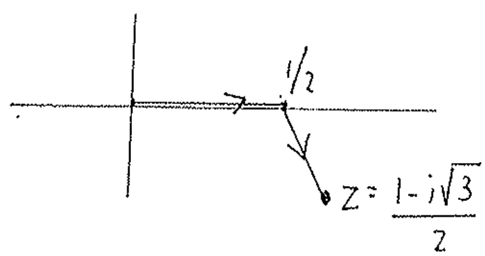
(i) $\displaystyle\int_C \sqrt{z}\, dz$ $\quad$ $C: |z|=1$ FROM $z=e^{i\pi/4}$ TO $z=e^{7\pi i/4}$
AND $\sqrt{z}$ IS BRANCH FOR WHICH
$\mathrm{IM}(\sqrt{z}) \geq 0$

(ii) $\displaystyle\int_C \mathrm{LOG}\, z\, dz$ $\quad$ $C: |z|=1$ FROM $z=e^{-3\pi i/4}$ TO $z=e^{3\pi i/4}$

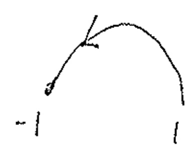


LOG = PRINCIPAL BRANCH

(iii) $\displaystyle\int_C z e^{z^2}\, dz$ where C IS PATH SHOWN CONNECTING $z=0$ TO $z=\dfrac{+1-i\sqrt{3}}{2}$



(iv) $\displaystyle\int_C (\bar{z})^2\, dz$ $\quad$ C: IS UNIT CIRCLE $|z|=1$
FROM $z=1$ TO $z=-1$



CALCULATE THE FOLLOWING where C is full circle $|z|=2$ counter-clockwise

(i) $$\int_C \frac{1}{z^2+1}\,dz$$

(ii) $$\int_C \frac{z^3}{z^5-1}\,dz \quad \text{AND} \quad \int_C \frac{z^4}{z^5-1}\,dz$$

(iii) $$\int_C \frac{\sin(\pi z)}{z^2}\,dz$$

(iv) $$\int_C \frac{e^{2z}}{(z-1)^{10}}\,dz$$

(v) $$\int_C \frac{z+1}{(z^2-4z+3)}\,dz$$

<u>EXAMPLE</u> DEFINE $f(z) = [z^2(z-1)]^{1/3}$ FIND A BRANCH OF $f(z)$ THAT IS ANALYTIC OUTSIDE THE UNIT DISK FOR WHICH $f(2) = 4^{1/3}$.

<u>EXAMPLE</u> SUPPOSE THAT WE TAKE THE BRANCH CUT FOR $\sqrt{z}$ SO THAT $\mathrm{RE}(\sqrt{z}) \geq 0$ FOR ALL z.

IS $f(z) = \mathrm{LOG}\left(\frac{1}{2} + \sqrt{z}\right)$ AN ANALYTIC FUNCTION FOR all z?

Q1 SUPPOSE THAT $f(z)$ IS AN ENTIRE FUNCTION AND SATISFIES $|f(z)| > 1$ FOR ALL Z. PROVE THAT $f(z)$ MUST BE THE CONSTANT FUNCTION.

Q2 PROVE THAT THE POLYNOMIAL

$$p(z) = z^6/6 + z^3/4 + z^2/12 + 2$$

HAS NO ZEROES INSIDE THE UNIT DISK.

Q3: FIND $\max\limits_{z \text{ in } \Omega} |e^{iz^4}|$ WHERE $\Omega: |z| \leq 2$.

Q4: SHOW THAT $z = \infty$ IS A BRANCH POINT OF $f(z) = \sqrt{1-z}$ BUT NOT A BRANCH POINT OF $g(z) = \sqrt{z(z-1)}$.

Q5 IS $z = 0$ A BRANCH POINT OF $f(z) = \sqrt{z}\,\sin(\sqrt{z})$, OF $f(z) = \cos(\sqrt{z})$?

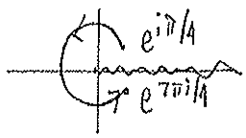
Q6 CONSIDER $f(z) = [z(z-1)(z-i)]^{1/4}$. WHAT ARE ALL B P's?

DRAW TWO SETS OF VIABLE BRANCH CUTS.

Q7 BY CHOOSING A BRANCH OF log, CONSTRUCT A BRANCH OF $f(z) = (z^3-1)^{1/3}$ THAT IS ANALYTIC IN $|z| > 1$ AND FOR WHICH $f(2) = 7^{1/3}$.

SOLUTIONS TO SOME SAMPLE MIDTERM QUESTIONS

(i) $I = \int_C \sqrt{z}\, dz$ WITH BRANCH FOR WHICH $\mathrm{IM}(\sqrt{z}) \geq 0$.



LET $z = Re^{i\varphi}$ $\quad \mathrm{IM}(\sqrt{z}) = R\sin(\varphi/2) \geq 0$

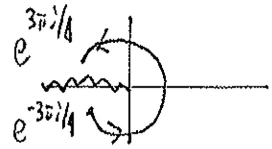
WHEN $0 \leq \varphi \leq 2\pi$

THIS GIVES BRANCH CUT ON positive real AXIS AS SHOWN.

BY ANTI-DERIVATIVE

$$I = \frac{2}{3} z^{3/2} \Big|_{z_i}^{z_f} = \frac{2}{3}\left[\left(e^{7\pi i/4}\right)^{3/2} - \left(e^{\pi i/4}\right)^{3/2} \right] = \frac{2}{3}\left[e^{21\pi i/8} - e^{3\pi i/8} \right]$$

(ii) $I = \int_C \mathrm{Log}\, z\, dz$ WITH $\mathrm{LOG}\, z$ AS PRINCIPAL BRANCH.



LET $\hat{F}(z) = z\,\mathrm{Log}\,z - z$.

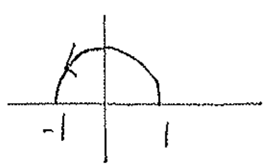
THEN $\hat{F}'(z) = \mathrm{Log}\,z$ SO THAT

$$I = \hat{F}\left(e^{3\pi i/4}\right) - \hat{F}\left(e^{-3\pi i/4}\right)$$

$$I = e^{3\pi i/4}\,\mathrm{Log}\left(e^{3\pi i/4}\right) - e^{-3\pi i/4}\,\mathrm{Log}\left(e^{-3\pi i/4}\right) = e^{-3\pi i/4} - e^{3\pi i/4}$$

(iii) $I = \int_C z e^{z^2} dz$. THE ANTI-DERIVATIVE IS $\hat{F}(z) = \frac{1}{2} e^{z^2}$. WE HAVE THAT I DEPENDS ONLY ON END POINTS $z_i = 0$, $z_f = (1 - i\sqrt{3})/2 = e^{-\pi i/3}$ $\rightarrow$ $I = \hat{F}\left(e^{-\pi i/3}\right) - \hat{F}(0)$.

(iv) $I = \int_C (\bar{z})^2\, dz$ $\quad C$ IS UNIT CIRCLE $|z| = 1$ FROM $z = 1$ TO $z = -1$



THERE IS NO ANTI-DERIVATIVE. WE MUST INTEGRATE DIRECTLY.

LET $z = e^{i\varphi}$ $\quad dz = ie^{i\varphi}\, d\varphi$

SO $$I = i\int_0^{\pi} \left(e^{-i\varphi}\right)^2 e^{i\varphi}\, d\varphi = i\int_0^{\pi} e^{-i\varphi}\, d\varphi$$

$$I = i\int_0^{\pi} (\cos\varphi - i\sin\varphi)\, d\varphi = \underbrace{\int_0^{\pi} \sin\varphi\, d\varphi}_{=2} + \underbrace{i\int_0^{\pi} \cos\varphi\, d\varphi}_{0}$$

SO $I = 2$

PROBLEM 2

LET C BE $|z| = 2$ counterclockwise

(i) $I = \int_C \frac{1}{z^2+1}\,dz$. SINGULARITIES AT $z = \pm i$ ARE INSIDE C.

THUS $\frac{1}{z^2+1}$ IS ANALYTIC OUTSIDE C.

HENCE DEFORM PATH TO $\int_C \frac{1}{z^2+1}\,dz = \lim_{R\to\infty} \int_{C_R} \frac{1}{z^2+1}\,dz$

WITH $C_R : |z| = R$ counterclockwise. SINCE $\left| \int_{C_R} \frac{dz}{z^2+1} \right| \leq \frac{1}{R^2-1} 2\pi R \to 0$

AS $R \to \infty$ WE GET $I = 0$.

(ii) BY SAME REASONING $\int_C \frac{z^3}{z^5-1}\,dz = 0$. (SEE PROBLEM (i)).

NOW $\int_C \frac{z^4}{z^5-1}\,dz = \int_{C_R} \frac{z^4}{z^5-1}\,dz = \int_{C_R} \frac{1}{z\left(1-1/z^5\right)}\,dz$ WITH $R > 2$.

FOR $R \to \infty$ WE GET $\int_{C_R} \frac{1}{z(1-1/z^5)}\,dz \sim \int_{C_R} \frac{1}{z}\left(1+1/z^5+\dots\right)dz = 2\pi i$.

THUS $I = 2\pi i$.

HERE $\lim_{R\to\infty} \int_{C_R} \frac{z^4}{z^5-1}\,dz = 2\pi i$.

(iii) RECALL $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ WHEN z_0 inside C AND f analytic inside and on C.

LET $n = 1$. THEN
$z_0 = 0$
$f(z) = \sin(\pi z)$

$$\int_C \frac{\sin(\pi z)}{z^2}\,dz = 2\pi i \frac{d}{dz} \sin(\pi z)\Big|_{z=0} = 2\pi i(\pi)\cos(\pi z)\Big|_{z=0} = 2\pi^2 i.$$

(iv) $\displaystyle\int_C \frac{e^{2z}}{(z-1)^{10}}\,dz = \frac{2\pi i}{9!}\frac{d^9}{dz^9}\left(e^{2z}\right) = \frac{2\pi i\,(2^9)\,e^2}{9!}$

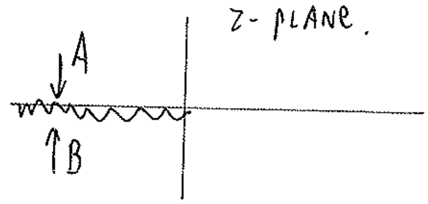
(v) $z^2 - 4z + 3 = (z-1)(z-3)$ $\quad C: |z| = 2$

THUS $\displaystyle I = \int \frac{z+1}{(z-1)(z-3)}\,dz = \underbrace{\int_C \frac{A}{z-1}\,dz}_{\text{inside } C} + \underbrace{\int_C \frac{B}{z-3}\,dz}_{=0 \text{ since outside } C} = 2\pi i A$

NOW $(z+1) = A(z-3) + B(z-1)$.

LET $z = +1 \to A = -1 \longrightarrow I = -2\pi i$.

EXAMPLE LET THE BRANCH OF $\sqrt{z}$ BE SUCH THAT $\mathrm{Re}(\sqrt{z}) \geq 0$ FOR ALL z. THEN BRANCH CUT OF $\sqrt{z}$ IS ALONG negative real AXIS AS shown



- THUS $f(z) = \mathrm{LOG}\left(\frac{1}{2} + \sqrt{z}\right)$ IS NOT ANALYTIC ALONG $\mathrm{Re}(z) < 0$, $\mathrm{IM}(z) = 0$ SINCE $\displaystyle\lim_{z\to A} f(z) \neq \lim_{z\to B} f(z)$. WHERE A IS ABOVE CUT $z = x + i\varepsilon$, B IS BELOW CUT $z = x - i\varepsilon$ $(x<0,\ \varepsilon \to 0^+)$.
- BUT LOG DOES NOT INTRODUCE ANY NEW BRANCH CUTS. ANY NEW CUT WOULD BE SUCH THAT $\mathrm{IM}\left(\frac{1}{2} + \sqrt{z}\right) = 0$ AND $\mathrm{RE}\left(\frac{1}{2} + \sqrt{z}\right) \leq 0$. BUT $\mathrm{RE}(\sqrt{z}) \geq 0$ AND SO THESE EQUATIONS CAN'T HOLD FOR ANY z.

Q1 EXAMPLE SUPPOSE $f(z)$ IS ENTIRE AND SATISFIES $|f(z)| > 1$ FOR ALL z. PROVE THAT $f(z)$ MUST BE CONSTANT FUNCTION.

PROOF THERE is NO POINT z_0 FOR WHICH $f(z_0) = 0$ (SINCE $|f(z_0)| > 1$). THUS $g(z) = 1/f(z)$ IS AN ENTIRE FUNCTION AND SATISFIES $|g(z)| < 1$ FOR ALL z. BY LIOUVILLE'S THEOREM THE ENTIRE AND BOUNDED FUNCTION $g(z)$ MUST BE A CONSTANT FUNCTION $\longrightarrow$ $f(z)$ IS THE CONSTANT FUNCTION.

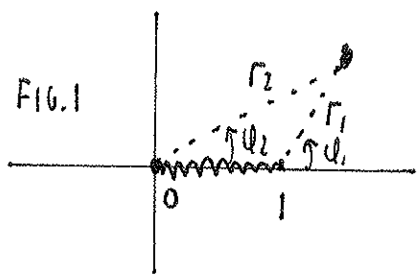
EXAMPLE DEFINE $f(z) = [z^2(z-1)]^{1/3}$.

WE WANT A BRANCH OF $f(z)$ ANALYTIC IN $|z|>1$ WITH $f(2) = 4^{1/3}$.

THE BP ARE $z = 0, 1$. NOTICE AS $|z| \to \infty$, THEN $f(z) \sim z$ AND SO THERE IS NO BRANCH POINT AT $z = \infty$.

WE WILL PUT THE BRANCH CUT BETWEEN $0 < z < 1$ WITH z REAL

FIG. 1



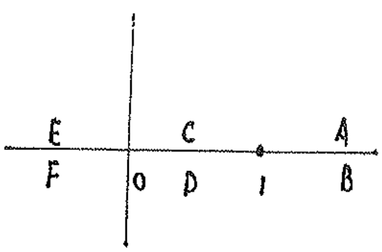
$$f(z) = z^{2/3}(z-1)^{1/3}$$

$$f(z) = r_2^{2/3} r_1^{1/3} e^{i2\varphi_2/3 + i\varphi_1/3} \qquad (*)$$

WITH $r_1 = |z-1|$, $r_2 = |z|$.

WE TRY $-\pi < \varphi_1 \le \pi$, $-\pi < \varphi_2 \le \pi$ AS THE RANGE OF ANGLES.

THEN WE TEST AS FOLLOWS:



AT A AND B: $\varphi_1 = \varphi_2 = 0 \to e^{i(\varphi_1/3 + 2\varphi_2/3)}$ CONTINUOUS

AT C: $\varphi_1 = \pi$, $\varphi_2 = 0$
AT D: $\varphi_1 = -\pi$, $\varphi_2 = 0$
$\to e^{i(\varphi_1/3 + 2\varphi_2/3)}$ IS DIFFERENT

AT E: $\varphi_1 = \pi$, $\varphi_2 = \pi$ $\quad e^{i(2\varphi_1/3 + \varphi_1/3)} = e^{i\pi}$
AT F: $\varphi_1 = -\pi$, $\varphi_1 = -\pi$ $\quad e^{i(2\varphi_1/3 + \varphi_1/3)} = e^{-i\pi}$
SAME

THUS WE HAVE BRANCH CUTS AS SHOWN IN FIG. 1

NOW AT $z = 2$, $\varphi_1 = \varphi_1 = 0$, $r_1 = 1$, $r_2 = 2$. NOW USE $(*)$

THUS $f(2) = 2^{2/3} e^{i0} = 4^{1/3}$.

SUMMARY TAKE $(*)$ WITH $-\pi < \varphi_1 \le \pi$, $-\pi < \varphi_2 \le \pi$.

Q2 PROVE THAT $p(z) = z^6/6 + z^3/4 + z^2/12 + 2$

has no zeroes inside $|z| \leq 1$.

PROOF recall that if f is analytic in $|z| \leq 1$ AND satisfies $|f(z) - 1| < 1$ FOR z ON $|z| \leq 1$, THEN BY MAX-MODULUS PRINCIPLE $f(z)$ has no zeroes in $|z| < 1$.

we write the polynomial as

$$z^6/12 + z^3/8 + z^2/24 + 1 = 0.$$

DEFINE $f(z) = z^6/12 + z^3/8 + z^2/24 + 1.$

THEN $$|f(z) - 1| = |z^6/12 + z^3/8 + z^2/24| \leq \frac{|z|^6}{12} + \frac{|z|^3}{8} + \frac{|z|^2}{24}$$

$$= \frac{1}{12} + \frac{1}{8} + \frac{1}{24} < 1$$

FOR $|z| = 1$. THUS BY CRITERION above we have no roots in $|z| \leq 1$.

Q3 FIND $\max\limits_{z \text{ IN } \Omega} |e^{iz^4}|$ WHERE $\Omega = |z| \leq 2$.

BY MAX-MODULUS PRINCIPLE

$$\max_{z \text{ IN } \Omega} |e^{iz^4}| = \max_{|z|=2} |e^{iz^4}| = \max_{0 \leq \varphi \leq 2\pi} |e^{16i e^{4i\varphi}}|$$

$$= \max_{0 \leq \varphi \leq 2\pi} e^{-16 \sin(4\varphi)} = e^{16}.$$

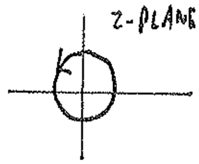
occurs when $\sin(4\varphi) = -1$.

~~(which occurs when $\varphi = 3\pi/4$)~~

NOTE $$|e^{16i[\cos(4\varphi) + i\sin(4\varphi)]}| = e^{-16\sin(4\varphi)}$$

Q5 $z=0$ IS NOT A BRANCH POINT OF $f(z)=\sqrt{z}\sin(\sqrt{z})$ (IF WE USE SAME BRANCH CUT FOR $\sqrt{z}$), NOR IS IT A BP OF $f(z)=\cos(\sqrt{z})$.

PROOF CONSIDER A PATH $z=re^{i\varphi}$, $0\le\varphi\le 2\pi$ ENCIRCLING $z=0$.



DOES $\sqrt{z}\sin(\sqrt{z})$ CHANGE VALUE AS WE RETURN TO 2π. IF YES, WE HAVE A BP, IF NO THEN $z=0$ IS NOT A BP.

AT $z=re^{i0}$ $(\varphi=0)$ $\qquad \sqrt{z}\sin(\sqrt{z})=r^{1/2}\sin(r^{1/2})$

AT $z=re^{2\pi i}$ $(\varphi=2\pi)$ $\qquad \sqrt{z}\sin(\sqrt{z})=r^{1/2}e^{i\pi}\sin(r^{1/2}e^{i\pi})$

$$=-r^{1/2}\sin(-r^{1/2})=r^{1/2}\sin(r^{1/2})$$

(SINCE $\sin(-x)=-\sin x$)

THUS $\left[\sqrt{z}\sin(\sqrt{z})\right]\Big|_C=0 \rightarrow z=0$ IS NOT A BP.

FOR $f(z)=\cos(\sqrt{z})$ WE DO IT SIMILARLY.

AT $z=re^{i0}$, $(\varphi=0)$ $\qquad \cos(\sqrt{z})=\cos(r^{1/2})$

AT $z=re^{i2\pi}$ $(\varphi=2\pi)$ $\qquad \cos(\sqrt{z})=\cos(r^{1/2}e^{i\pi})=\cos(-r^{1/2})$

$$=\cos(r^{1/2})$$

SINCE $\cos x=\cos(-x)$

THUS $\left[\cos(\sqrt{z})\right]\Big|_C=0 \rightarrow z=0$ IS NOT A BP.

Q7 BY CHOOSING A BRANCH OF $\log$, CONSTRUCT A BRANCH OF $F(z) = (z^3 - 1)^{1/3}$ THAT IS ANALYTIC IN $|z| > 1$ AND FOR WHICH $F(2) = 7^{1/3}$.

SOLUTION NOTICE AS $|z| \to \infty$, $f(z) \sim z$ SO $z = \infty$ IS NOT A BP.

THE BP ARE AT $z^3 = 1 \to z = 1, e^{2\pi i/3}, e^{4\pi i/3}$.

ALL OF THESE ARE ON UNIT DISK.

THE RANGE OF ANGLE METHOD IS A BIT TEDIOUS HERE SINCE THE BRANCH POINTS DO NOT LIE ON A LINE (ARE NOT COLINEAR), WE WILL USE THE OTHER METHOD.

WE WRITE $f(z) = [-(1-z^3)]^{1/3} = [z^3(1 - 1/z^3)]^{1/3}$.

THUS $f(z) = (z^3)^{1/3} e^{\frac{1}{3}\log(1 - 1/z^3)}$

LET'S TRY PRINCIPAL BRANCH, AND $(z^3)^{1/3} = z$.

THUS $f(z) = z e^{\frac{1}{3} \mathrm{LOG}(1 - 1/z^3)}$

THIS WILL WORK IF $F(2) = 7^{1/3}$, AND $f(z)$ IS ANALYTIC IN $|z| > 1$.

NOTE $f(2) = 2 e^{\frac{1}{3}\mathrm{LOG}(1 - 1/8)} = 2 e^{1/3 \ln(7/8)} = 2\left(\frac{7}{8}\right)^{1/3} = 7^{1/3}$ ✓

WE NEED ONLY SHOW $\mathrm{LOG}(1 - 1/z^3)$ IS ANALYTIC IN $|z| > 1$.

PROOF $\mathrm{LOG}(1 - 1/z^3)$ IS NOT ANALYTIC WHEN

$$\mathrm{RE}(1 - 1/z^3) \leq 0 \quad \text{AND} \quad \mathrm{IM}(1 - 1/z^3) = 0.$$

PUT $z = re^{i\varphi}$. THEN $1 - \frac{1}{z^3} = 1 - \frac{1}{r^3} e^{-3i\varphi}$.

$$\mathrm{IM}\left(1-\frac{1}{r^3}e^{-3i\varphi}\right)=0 \;\longrightarrow\; \sin(3\varphi)=0 \qquad 3\varphi=0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi$$

$$\mathrm{RE}\left(1-{}^{1}/_{r^3}\,e^{-3i\varphi}\right)=1-\frac{1}{r^3}\cos(3\varphi)\leq 0 \quad \text{WHERE?}$$

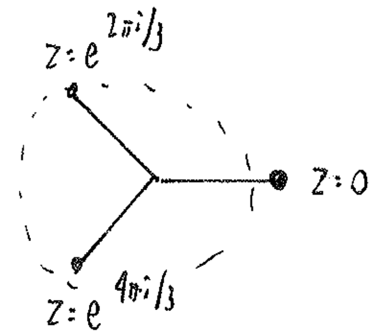
- LET $3\varphi=0,\ 2\pi,\ 4\pi \longrightarrow \cos(3\varphi)=1 \rightarrow 1-{}^{1}/_{r^3}\leq 0 \rightarrow r\leq 1.$
- LET $3\varphi=\pi,\ 3\pi,\ 5\pi \longrightarrow \cos(3\varphi)=-1 \rightarrow 1+{}^{1}/_{r^3}\leq 0 \rightarrow$ IMPOSSIBLE.

THUS $\mathrm{LOG}\left(1-{}^{1}/_{z^3}\right)$ IS <u>NOT</u> ANALYTIC ALONG $\varphi=0,\ {}^{2\pi}/_{3},\ {}^{4\pi}/_{3}$ WITH $r\leq 1.$

IT IS ANALYTIC THEN OUTSIDE UNIT DISK.

THUS $F(z)=z\,e^{\frac{1}{3}\mathrm{LOG}\left(1-{}^{1}/_{z^3}\right)}$ IS CORRECT.

THE BRANCH CUTS ARE AS SHOWN



NOTICE ALL BRANCH CUTS GO THROUGH $z=0$.

(BUT OF COURSE $z=0$ IS NOT A BRANCH POINT)