RADIAL SOLUTIONS AND PHASE SEPARATION IN A SYSTEM OF TWO COUPLED SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider the nonlinear elliptic system

$$\begin{cases} -\Delta u + u - u^3 - \beta v^2 u = 0 & \text{in } \mathbb{B}, \\ -\Delta v + v - v^3 - \beta u^2 v = 0 & \text{in } \mathbb{B}, \\ u, v > 0 \quad \text{in } \mathbb{B}, \qquad u = v = 0 & \text{on } \partial\mathbb{B}, \end{cases}$$

where $N \leq 3$ and $\mathbb{B} \subset \mathbb{R}^N$ is the unit ball. We show that, for every $\beta \leq -1$ and $k \in \mathbb{N}$, the above problem admits a radially symmetric solution (u_β, v_β) such that $u_\beta - v_\beta$ changes sign precisely k times in the radial variable. Furthermore, as $\beta \to -\infty$, after passing to a subsequence, $u_\beta \to w^+$ and $v_\beta \to w^-$ uniformly in $\mathbb{B}$, where $w = w^+ - w^-$ has precisely k nodal domains and is a radially symmetric solution of the scalar equation $\Delta w - w + w^3 = 0$ in $\mathbb{B}$, $w = 0$ on $\partial\mathbb{B}$. Within a Hartree-Fock approximation, the result provides a theoretical indication of phase separation into many nodal domains for Bose-Einstein double condensates with strong repulsion.

1. INTRODUCTION

The present paper is concerned with the study of solitary wave solutions for the coupled Gross-Pitaevskii equations

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1 \text{ for } y \in \Omega, t > 0, \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2 \text{ for } y \in \Omega, t > 0, \\ \Phi_1(y,t) = \Phi_2(y,t) = 0 \quad \text{for } y \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where μ_1, μ_2 are positive constants, Ω is a domain in $\mathbb{R}^N$, $N \leq 3$, and β is a coupling constant. System (1.1) arises in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$, see [15]. Physically, Φ_1 and Φ_2 are the corresponding condensate amplitudes, μ_1 and μ_2 are proportional to the intraspecies scattering lengths, and β is proportional to the interspecies scattering length. The sign of μ_j determines whether collisions of particles of the single state $|j\rangle$ result in a *repulsive* or *attractive* interaction, while the sign of β determines the interaction of particles of state $|1\rangle$ and state $|2\rangle$. If $\mu_j > 0$ as considered here, we are dealing with an attractive self-interaction of the single states $|j\rangle$, $j = 1, 2$. When $\beta < 0$, the interaction of state $|1\rangle$ and $|2\rangle$ is repulsive (as discussed in [37]). In contrast, when $\beta > 0$, the interaction of state $|1\rangle$ and $|2\rangle$ is attractive.

When $\Omega = \mathbb{R}^N$, system (1.1) also arises in the study of incoherent solitons in nonlinear optics. We refer to [27,28] for experimental results and to [3,9,19–21] for a comprehensive list of references.

To obtain solitary wave solutions of the system (1.1), we set $\Phi_1(x,t) = e^{i\lambda_1 t}\, u(x)$, $\Phi_2(x,t) = e^{i\lambda_2 t}\, v(x)$, and the system (1.1) is transformed to an elliptic system given by

$$(1.2) \qquad \begin{cases} -\Delta u + \lambda_1 u - \mu_1 u^3 - \beta v^2 u = 0 & \text{in } \Omega,\\ -\Delta v + \lambda_2 v - \mu_2 v^3 - \beta u^2 v = 0 & \text{in } \Omega,\\ \;u, v > 0 \quad \text{in } \Omega, \qquad u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

As shown by recent results, the structure of the solution set of (1.2) depends strongly on the value of β. For a bounded domain $\Omega \subset \mathbb{R}^N, N \leq 3$, a least energy solution of (1.2) exists within the range $\beta \in (-\infty, \beta_0]$, where $0 < \beta_0 < \sqrt{\mu_1, \mu_2}$ is a constant. This is proved in [23], where also the asymptotic behavior of this solution is studied as the domain Ω becomes large. When $\Omega = \mathbb{R}^N$, the existence of least energy and other finite energy solutions of (1.2) is proved in [2,5,25,35] for β belonging to different subintervals of $(0, \infty)$. It is important to note that when Ω is a ball or $\Omega = \mathbb{R}^N$ and $\beta > 0$, then all solutions of (1.2) are radially symmetric (up to translation if $\Omega = \mathbb{R}^N$), and both components are decreasing in the radial variable, see [38]. In contrast, different classes of nonradial solutions, distinguished by their shape and symmetries, have been constructed for $\Omega = \mathbb{R}^N$ and $\beta < 0$, $|\beta|$ small in [24] and for $\beta \leq -1$ in [43]. In the present paper we analyze another class of solutions of (1.2) which only exist for negative β, namely radial but not radially decreasing solutions when $\Omega = \mathbb{B}$ is the unit ball in $\mathbb{R}^N$. We focus on the symmetric case $\lambda_1 = \lambda_2, \mu_1 = \mu_2$, assuming without loss of generality that $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. Hence we study radial solutions of the following nonlinear elliptic system:

$$(1.3) \qquad \begin{cases} -\Delta u + u - u^3 - \beta v^2 u = 0 & \text{in } \mathbb{B},\\ -\Delta v + v - v^3 - \beta u^2 v = 0 & \text{in } \mathbb{B},\\ \;u, v > 0 \quad \text{in } \mathbb{B}, \qquad u = v = 0 & \text{on } \partial\mathbb{B}. \end{cases}$$

Our results establish a connection between radial solutions of (1.3) and *sign changing* radial solutions of the scalar problem

$$(1.4) \qquad -\Delta w + w - w^3 = 0 \quad \text{in } \mathbb{B}, \qquad w = 0 \quad \text{on } \partial\mathbb{B}.$$

Let H_r be the Hilbert space of all radially symmetric functions in $H_0^1(\mathbb{B})$ endowed with the norm $\|u\|^2 := \int_{\mathbb{B}}(|\nabla u|^2 + |u|^2)\, dx$. Radial solutions of (1.3) are critical points of the energy functional $E : H_r \times H_r \to \mathbb{R}$ given by

$$E(u,v) = \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{1}{4}\int (u^4 + v^4)\, dx - \frac{\beta}{2}\int u^2 v^2\, dx,$$

Moreover, radial solutions of (1.4) are critical points of the functional

$$E_S : H_r \to \mathbb{R}, \qquad E_S(w) = \frac{1}{2}\|w\|^2 - \frac{1}{4}\int w^4\, dx.$$

To state our main results, we recall that, for every $k \in \mathbb{N}$, (1.4) admits a radial solution with precisely k nodal domains, i.e., $k-1$ sign changes in the radial variable, see [40, 41]. In dimension $N = 1$ this solution is unique (see [39]), but for $N > 1$ this is unknown. We put

$$c_k := \inf_{w \in \mathcal{S}_k} E_S(w), \qquad (k \in \mathbb{N}), \tag{1.5}$$

where $\mathcal{S}_k \subset H_r$ is the set of radial solutions of (1.4) with precisely k nodal domains. There exists a different characterization of c_k via a variational principle introduced by Nehari [30], see Proposition 2.1 below. Our first main result is the following.

Theorem 1.1. *Let $N \leq 3$. Then for every $\beta \leq -1$ and every integer $k \geq 2$, (1.3) admits a solution $(u, v) \in H_r \times H_r$ such that $E(u, v) \leq c_k$ and $u - v$ changes sign precisely $k - 1$ times in the radial variable.*

Theorem 1.1 yields the existence of infinitely many radial solutions (u, v) of (1.3) which are distinguished by the number of intersections of u and v. For fixed k, these solutions satisfy an energy bound *independent* of the coupling parameter β. Our second main result provides a description of the limit shape of these solutions as β tends to minus infinity.

Theorem 1.2. *Let $N \leq 3$, $k \geq 2$, and let $\beta_n \leq -1$, $n \in \mathbb{N}$ be a sequence of numbers with $\beta_n \to -\infty$ as $n \to \infty$. Let also $(u_n, v_n) \in H_r \times H_r$ be solutions of (1.3) with $\beta = \beta_n$ such that $u_n - v_n$ changes sign precisely $k - 1$ times (in the radial variable) and $E(u_n, v_n) \leq c_k$.*
Then, after passing to a subsequence, $u_n \to w^+$ and $v_n \to w^-$ in H_r and $C(\overline{\mathbb{B}})$, where w is a solution of (1.4) with precisely $k - 1$ interior zeros and $E(w) = c_k$.

Here and in the following, $w^+ = \max\{w, 0\}$ and $w^- = -\min\{w, 0\}$ denote the positive and negative part of a function $w : \mathbb{B} \to \mathbb{R}$.

In the context of Bose-Einstein condensates (where $\Phi_1(x, t) = e^{it}\, u(x)$, $\Phi_2(x, t) = e^{it}\, v(x)$ stand for the amplitudes of the different hyperfine states $|1\rangle$ and $|2\rangle$), the limit shape considered in Theorem 1.2 models the spatial separation of $|1\rangle$ and $|2\rangle$ in the presence of strong repulsion. This phase separation has drawn the attention both from experimental and theoretical physicists [17, 29, 37], but rigorous mathematical results are rare. In fact, for a general bounded domain Ω and an *arbitrary* uniformly bounded solution sequence (u_β, v_β) of (1.2) corresponding to $\beta \to -\infty$, the corresponding limit profile (u, v), i.e., the weak limit in $[H_0^1(\Omega)]^2$ of a subsequence, is not well understood. It is easy to see that the nodal sets $N_u = \{x \in \Omega \ : \ u(x) > 0\}$ and $N_v = \{x \in \Omega \ : \ v(x) > 0\}$ are disjoint. Moreover, it is natural to expect that u and v are continuous and therefore N_u and N_v are open subsets of Ω, but to our knowledge this has not been proved yet. For a related system with different parameter values, Chang-Lin-Lin-Lin [8] proved that u and v solve scalar limit equations in N_u and N_v *under the crucial assumption* that N_u, N_v are open in Ω. Via numerical computations, they investigate further properties of the corresponding nodal domains, i.e., the connected components of N_u and N_v.

In the radial case, Theorems 1.1 and 1.2 exhibit a large class of solutions which converge uniformly as $\beta \to -\infty$ and give rise to continuous limit profiles with arbitrarily many nodal domains. Moreover, these limit profiles have matching derivatives of u and v at the common boundary of N_u and N_v.

It is worth pointing out that spatial segregation has been studied already for different classes of competing species systems with simpler coupling terms, see e.g. [13, 14]. Moreover, the asymptotic behaviour of *least energy solutions* to a related class of superlinear elliptic systems with strong competition is studied in [12]. In fact, although the nonlinear terms in system (1.2) do not satisfy the growth conditions assumed in [12], it seems that many of the arguments in [12] also apply to least energy solutions of (1.2).

We briefly describe the paper's organisation and the methods used in the proofs. In Section 2 we collect preliminaries on the variational framework for (1.3), and we discuss properties of a parabolic system corresponding to (1.3). A crucial property is the nonincrease of the number of intersections of u and v along trajectories of the associated parabolic semiflow. This nonincrease is an easy consequence of the zero number diminishing property for the scalar problem derived in [32]. In Section 3 we use the parabolic flow, together with a slightly modified version of the classical Krasnoselskii genus, to prove Theorem 1.1. For *scalar* elliptic equations, special solutions have already been constructed via a corresponding parabolic flow and comparison principles, see [10, 11, 33]. The approach presented here differs from these existing techniques but could also be applied to scalar equations with odd nonlinearities.

Section 4 contains the proof of Theorem 1.2. Here we combine Nehari's variational principle with comparison arguments and ordinary differential equations techniques. In particular, a Ljapunov function for radial solutions of (1.3) is used as a crucial tool to control the number of intersections of u and v while passing to the limit $\beta \to -\infty$.

We finally remark that it is open whether an existence result similar to Theorem 1.1 also holds for the *nonsymmetric* system (1.2) in $\Omega = \mathbb{B}$. Since our method uses the genus, it does not apply to (1.2). For a class of superlinear ODE-systems, solutions with a prescribed number of *zeroes* of each component were constructed in [36] without assuming oddness of the nonlinearity. It is tempting to rewrite system (1.2) in $x = u - v$ and $y = u + v$ in order to apply a similar approach as in [36] to the resulting system. However, even in the symmetric case one obtains a system of the form $-\Delta x + x = \left(\frac{1+\beta}{4}\right)x^3 + \left(\frac{3-\beta}{4}\right)y^2 x,\ -\Delta y + y = \left(\frac{1+\beta}{4}\right)y^3 + \left(\frac{3-\beta}{4}\right)x^2 y$, where, for $\beta < -1$, the nonlinear terms have precisely the opposite sign as in (1.3). Therefore this system has completely different properties than the class of systems considered in [36]. Moreover, the condition $u, v > 0$ translates into the somewhat unnatural constraint $|x| < y$.

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2. Preliminaries and the corresponding parabolic problem

Throughout the remainder of this paper we assume that $N \le 3$. In this section we consider a fixed coupling constant $\beta \le -1$ in (1.3). Multiplying the first equation in (1.3) with u, the second with v and integrating, we find that all nontrivial solutions of (1.3) are contained in the set

$$\mathcal{M} = \left\{ \begin{array}{l} (u,v) \in H_0^1(\mathbb{B}) \times H_0^1(\mathbb{B}), \\ u, v \ge 0,\ u, v \not\equiv 0 \end{array} \middle| \begin{array}{l} \|u\|^2 - \beta |uv|_2^2 = |u|_4^4, \\ \|v\|^2 - \beta |uv|_2^2 = |v|_4^4. \end{array} \right\}$$

Here and in the following, we write $|u|_p$ for the usual L^p-Norm of a function $u \in L^p(\mathbb{B})$. We note that

$$E(u,v) = \frac{1}{4}(\|u\|^2 + \|v\|^2) \qquad \text{for } (u,v) \in \mathcal{M}. \tag{2.1}$$

Similarly, all nontrivial solutions of (1.4) are contained in

$$\mathcal{M}_S := \{ w \in H_0^1(\mathbb{B}),\ w \ne 0 \ :\ \|w\|^2 = |w|_4^4 \},$$

and $E_S(w) = \frac{\|w\|^2}{4}$ for $w \in \mathcal{M}_S$.

Next, we consider the set $\Gamma_k \subset H_r$ of all functions $w \in H_r$ such that there exists radii $0 = r_0 < r_1 < ... < r_{k-1} < r_k = 1$ with $w \cdot 1_{\{r_j \le |x| \le r_{j+1}\}} \in \mathcal{M}_S$ for $j = 0, ..., k-1$. The following highly useful variational principle goes back to Nehari [30] in the one-dimensional case. Later it was generalized to radial functions in higher space dimensions, see [6, 40, 41].

Proposition 2.1. *The value c_k defined in (1.5) admits the variational characterization*

$$c_k = \inf_{w \in \Gamma_k} E_S(w). \tag{2.2}$$

Moreover, if $w \in \Gamma_k$ satisfies $E_S(w) = c_k$ and

$$(-1)^j w(x) \ge 0 \ \text{ for } r_j \le |x| \le r_{j+1},\ \ j = 0, ..., k-1 \qquad \text{or}$$
$$(-1)^j w(x) \le 0 \ \text{ for } r_j \le |x| \le r_{j+1},\ \ j = 0, ..., k-1,$$

then w is a radial solution of (1.4) with precisely $k-1$ interior zeros.

Next we fix $3 < p < \infty$, and we consider the function spaces

$$W_r = \{ u \in W_0^{1,p}(\mathbb{B}) \ :\ u \text{ radially symmetric} \},$$
$$C_r = \{ u \in C(\overline{\mathbb{B}}) \ :\ u \text{ radially symmetric, } u|_{\partial\mathbb{B}} = 0 \},$$
$$C_r^1 = \{ u \in C^1(\overline{\mathbb{B}}) \ :\ u \text{ radially symmetric, } u|_{\partial\mathbb{B}} = 0 \}.$$

We have embeddings $C_r^1 \hookrightarrow W_r$ and $W_r \hookrightarrow C_r$, since $N \le 3 < p$. Here the second arrow is the usual Sobolev embedding restricted to radial functions. We also put

$$X = W_r \times W_r, \qquad Y = C_r^1 \times C_r^1, \qquad X_+ = \{ (u,v) \in X \ :\ u, v \ge 0 \}.$$

We remark that, if the pair $(u,v) \in X_+$ is a weak solution of the coupled equations

$$-\Delta u + (1 - \beta v^2) u = u^3 \ge 0, \qquad -\Delta v + (1 - \beta u^2) v = v^3 \ge 0 \qquad \text{in } \mathbb{B}$$

and $u \neq 0$, $v \neq 0$, then (u,v) is a solution of (1.3) by the strong maximum principle. We now collect some results on the parabolic problem

$$
\begin{cases} u_t - \Delta u + u - u^3 - \beta v^2 u = 0 & \text{in } \mathbb{B}, \\ v_t - \Delta v + v - v^3 - \beta u^2 v = 0 & \text{in } \mathbb{B}, \\ u = v = 0 \quad \text{on } \partial\mathbb{B}, \quad u(0,x) = u_0(x), \qquad v(0,x) = v_0(x). \end{cases} \tag{2.3}
$$

For the Cauchy problem (2.3) in the space X, we have the following.

Proposition 2.2. *For every $(u_0, v_0) \in X$, the Cauchy problem (2.3) has a unique (mild) solution $(u(t), v(t)) = \varphi(t, u_0, v_0) \in C([0,T), X)$ with maximal existence time $T := T(u_0, v_0) > 0$ which is a classical solution for $0 < t < T$. The set $\mathcal{G} := \{(t, u_0, v_0) : 0 \le t < T(u_0, v_0)\}$ is open in $[0,\infty) \times X$, and φ is a semiflow on $\mathcal{G}$. Moreover we have:*
(i) For every $(u_0, v_0) \in X$ and every $0 < t < T(u_0, v_0)$ there is a neighborhood $U \subset X$ of (u_0, v_0) in X such that $T(u,v) > t$ for $(u,v) \in U$, and $\varphi(t,\cdot,\cdot) : (U, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a continuous map.
(ii) If $(u_0, v_0) \in X_+$, then $\varphi(t, u_0, v_0) \in X_+$ for $0 \le t < T(u_0, v_0)$.

Proof. The proposition can be derived from abstract results of Amann concerning local existence and regularity, see [1]. For this we note that the substitution operator F_* induced by the nonlinearity

$$
(u,v) \mapsto F(u,v) = (u - u^3 - \beta v^2 u, v - v^3 - \beta u^2 v). \tag{2.4}
$$

is locally Lipschitz continuous as a map $W^{\tau,p}(\mathbb{B}) \times W^{\tau,p}(\mathbb{B}) \to L^p(\mathbb{B}) \times L^p(\mathbb{B})$ whenever $\tau > \frac{N}{p}$. Hence the local existence, the semiflow properties of φ and (i) follow from [1, Theorem 2.1 and Theorem 2.4].
Property (ii) is just a consequence of the parabolic maximum principle, since u and v both satisfy equations of the form $w_t - \Delta w = f(x,t)w$ in $\mathbb{B}$ with locally bounded f, together with homogeneous Dirichlet boundary conditions. □

In the following we will frequently write $\varphi^t(u)$ instead of $\varphi(t,u)$. For a classical solution of (2.3), we have

$$
\begin{aligned}
\frac{d}{dt}E(u,v) &= \int_{\mathbb{B}} (\nabla u \nabla u_t + (u - u^3 - \beta v^2 u)u_t)\,dx + \int_{\mathbb{B}} (\nabla v \nabla v_t + (v - v^3 - \beta u^2 v)v_t)\,dx \\
&= \int_{\mathbb{B}} (-\Delta u + u - u^3 - \beta v^2 u)u_t\,dx + \int_{\mathbb{B}} (-\Delta v + v - v^3 - \beta u^2 v)v_t\,dx \\
&= -\int_{\mathbb{B}} [(u_t)^2 + (v_t)^2]\,dx, \qquad (2.5)
\end{aligned}
$$

hence E is strictly decreasing along non-constant trajectories $t \mapsto \varphi^t(u_0, v_0)$ in X. We need the following compactness property.

Proposition 2.3. *Let $(u_0, v_0) \in X$ and $T = T(u_0, v_0)$ be such that the function $t \mapsto E(\varphi^t(u_0, v_0))$ is bounded from below in $(0,T)$. Then $T = \infty$, and for every $\delta > 0$ the set $\{\varphi^t(u_0, v_0) : t \ge \delta\}$ is relatively compact in Y.*

Proof. Let $(u(t), v(t)) = \varphi^t(u_0, v_0)$, and recall that the nonlinearity F defined in (2.4) has cubic growth. Hence, in view of Amann's abstract criterion for global existence and relative compactness (see [1, Theorem 5.3 and Remark 5.4]), it suffices to show that

$$\sup_{0\le t<T} (|u(t)|_\lambda + |v(t)|_\lambda) < \infty \qquad \text{for some } \lambda \text{ satisfying } 3 < 1 + \frac{2}{N}\lambda. \tag{2.6}$$

We restrict our attention to the case $N = 3$, since the case $N \le 2$ is easier. We claim that (2.6) holds with $\lambda = \frac{10}{3}$. The following argument is similar to the method in [7], see in particular estimates (2.12) and (2.15) below. To shorten notation, we put $E_{\inf} = \inf_{0<t<T} E(u(t), v(t))$,

$$\dot{E} = \frac{d}{dt}E(u,v) = -(|u_t|_2^2 + |v_t|_2^2) \qquad \text{and} \qquad h = |u|_2^2 + |v|_2^2.$$

Then

$$\frac{dh}{dt} = 2\int_{\mathbb{B}} (uu_t + vv_t)\,dx \le 2\,(|u|_2|u_t|_2 + |v|_2|v_t|_2) \le h - \dot{E}, \tag{2.7}$$

and, by multiplying (2.3) with u resp. v and integrating,

$$\begin{aligned}\int_{\mathbb{B}} (uu_t + vv_t)\,dx &= -(\|u\|^2 + \|v\|^2) + |u|_4^4 + |v|_4^4 + 2\beta|uv|_2^2 \\ &= -4E(u,v) + \|u\|^2 + \|v\|^2.\end{aligned} \tag{2.8}$$

Consequently,

$$\begin{aligned}\|u\|^2 + \|v\|^2 &\le 4E(u_0,v_0) + \int_{\mathbb{B}} (uu_t + vv_t)\,dx \le C_1 + |u|_2|u_t|_2 + |v|_2|v_t|_2 \\ &\le C_1 + \sqrt{h}(|u_t|_2 + |v_t|_2).\end{aligned} \tag{2.9}$$

Here and in the following, $C_1, C_2, \dots$ are positive constants independent of t.
We first consider the case where $T < \infty$. From (2.7) we derive

$$\frac{d}{dt}(e^{-t}h(t)) = e^{-t}\Big(\frac{dh}{dt}(t) - h(t)\Big) \le -e^{-t}\dot{E}(t) \le -\dot{E}(t),$$

so that

$$h(t) \le e^t\Big(h(0) - \int_0^t \dot{E}(s)\,ds\Big) \le e^T[h(0) + E(u_0,v_0) - E_{\inf}] \le C_2$$

for $t \in [0,T)$. Hence (2.9) implies

$$\|u(t)\|^2 + \|v(t)\|^2 \le C_3(1 + |u_t(t)|_2 + |v_t(t)|_2)$$

and therefore

$$\|u(t)\|^4 + \|v(t)\|^4 \le C_4(1 + |u_t(t)|_2^2 + |v_t(t)|_2^2) = C_4(1 - \dot{E}(t)) \qquad \text{for } t \in [0,T). \tag{2.10}$$

Thus we obtain for $0 \le t < T$

$$\int_0^t (\|u\|^4 + \|v\|^4)\,ds \le C_4[T + E(u_0,v_0) - E_{\inf}] =: C_5, \tag{2.11}$$

which implies, for $\lambda = \frac{10}{3}$,

$$\frac{1}{\lambda}\Big(|u(t)|_\lambda^\lambda + |v(t)|_\lambda^\lambda - \big(|u(0)|_\lambda^\lambda + |v(0)|_\lambda^\lambda\big)\Big) = \int_0^t \Big(|u|^{\lambda-2}uu_t + |v|^{\lambda-2}vv_t\Big)\,ds$$

$$\le \int_0^t \Big(|u|_2^{\frac{1}{3}}|u|_6^2|u_t|_2 + |v|_2^{\frac{1}{3}}|v|_6^2|v_t|_2\Big)\,ds \le \int_0^t h^{\frac{1}{6}}\Big(|u|_6^2|u_t|_2 + |v|_6^2|v_t|_2\Big)\,ds$$

$$\le C_6 \int_0^t \Big(|u|_6^4 + |u_t|_2^2 + |v|_6^4 + |v_t|_2^2\Big)\,ds \le C_7 \int_0^t \Big(\|u\|^4 + \|v\|^4 - \dot{E}\Big)\,ds$$

$$\le C_7[C_5 + E(u_0,v_0) - E_{\inf}] =: C_8. \tag{2.12}$$

Here we used the Sobolev embedding $H_r \hookrightarrow L^6(\mathbb{B})$. This concludes the proof of (2.6) if $T < \infty$.

Next we consider the case $T = \infty$. Then there exists a sequence $(t_n)_n$ with $n \le t_n \le n+1$ and

$$-\Big(|u_t(t_n)|_2^2 + |v_t(t_n)|_2^2\Big) = \dot{E}(t_n) \to 0 \qquad \text{as } n \to \infty.$$

Combining this with (2.9), we get

$$\|u(t_n)\|^2 + \|v(t_n)\|^2 \le C_1 + \sqrt{h(t_n)}(|u_t(t_n)|_2 + |v_t(t_n)|_2) \le C_1 + o(1)\sqrt{\|u(t_n)\|^2 + \|v(t_n)\|^2}$$

which implies that

$$\|u(t_n)\| + \|v(t_n)\| \le C_9 \qquad \text{for all } n. \tag{2.13}$$

Moreover, for $t_n \le t \le t_{n+1}$, we derive from (2.7)

$$\frac{\partial}{\partial t}(e^{-(t-t_n)}h(t)) = e^{-(t-t_n)}\Big(\frac{\partial h}{\partial t}(t) - h(t)\Big) \le -e^{-(t-t_n)}\dot{E}(t) \le -\dot{E}(t),$$

so that, by (2.13),

$$h(t) \le e^{t-t_n}\Big(h(t_n) - \int_{t_n}^t \dot{E}(s)\,ds\Big) \le e^2(C_9^2 + E(u_0,v_0) - E_{\inf}) \le C_{10}.$$

Hence (2.9) implies

$$\|u(t)\|^2 + \|v(t)\|^2 \le C_{11}(1 + |u_t(t)|_2 + |v_t(t)|_2)$$

and therefore

$$\|u(t)\|^4 + \|v(t)\|^4 \le C_{12}(1 - \dot{E}(t))$$

for all $t \ge 0$. Thus we obtain, for $t_n \le t \le t_{n+1}$, as in (2.11),

$$\int_{t_n}^t (\|u\|^4 + \|v\|^4)\,ds \le C_{13}, \tag{2.14}$$

and thus, similarly as before,

$$
\begin{aligned}
\frac{1}{\lambda}(|u(t)|_\lambda^\lambda + |v(t)|_\lambda^\lambda) &= \frac{1}{\lambda}(|u(t_n)|_\lambda^\lambda + |v(t_n)|_\lambda^\lambda) + \int_{t_n}^t \Big(|u|^{\lambda-2}uu_t + |v|^{\lambda-2}vv_t\Big)\,ds \\
&\le \frac{1}{\lambda}(|u(t_n)|_\lambda^\lambda + |v(t_n)|_\lambda^\lambda) + C_{14}\int_{t_n}^t \Big(\|u\|^4 + \|v\|^4 - \dot{E}\Big)\,ds \\
&\le \frac{1}{\lambda}(|u(t_n)|_\lambda^\lambda + |v(t_n)|_\lambda^\lambda) + C_{15} \le C_{16}, \qquad (2.15)
\end{aligned}
$$

where we used (2.13) and the Sobolev embedding $H_r \hookrightarrow L^\lambda(\mathbb{B})$ in the last step. The proof of (2.6) finished, and hence the claim follows. □

The following Corollary is a consequence of (2.5) and Proposition 2.3.

Corollary 2.4. *If, for some $(u_0, v_0) \in X_+$ and $T = T(u_0, v_0)$, the function $t \mapsto E(\varphi^t(u_0, v_0))$ is bounded from below on $(0, T)$, then $T = \infty$ and the ω-limit set*

$$\omega(u_0, v_0) = \bigcap_{t>0} \operatorname{clos}_Y\Big(\{\varphi^s(u_0, v_0) \ : \ s \ge t\}\Big)$$

is a nonempty compact subset of Y consisting of radial solutions of (1.3). Here clos_Y *stands for the closure with respect to the Y-topology.*

We also need a variant of Sturm's lap number theorem similar to the one available for scalar parabolic equations, see [4, 18, 26, 31] for the one-dimensional case and [32] for the radial case in higher dimensions. Given $(u, v) \in X$, we define the number of (strict) intersections $i(u, v)$ of u and v as the maximal $k \in \mathbb{N} \cup \{0, \infty\}$ such that there exist points $x_1, \dots, x_{k+1} \in \mathbb{B}$ with $0 \le |x_1| < \cdots < |x_{k+1}| < 1$ and

$$[u(x_i) - v(x_i)][u(x_{i+1}) - v(x_{i+1})] < 0 \qquad \text{for } i = 1, \dots, k.$$

Lemma 2.5. *Let $(u_0, v_0) \in X$ and $T := T(u_0, v_0)$. Then $t \mapsto i(\varphi^t(u_0, v_0))$ is nonincreasing in $t \in [0, T)$.*

This Lemma can easily be derived from [32, Theorem 2.1]. In fact, the general result in [32] for scalar equations implies a stronger monotonicity property than the one stated in Lemma 2.5. Since we only need the weak version stated above, we give a short proof following an argument of Sattinger (cf. [34, Theorem 4]).

Proof. We write $(u(t), v(t)) = \varphi^t(u_0, v_0)$, so that (u, v) is a solution of (2.3). In view of the semiflow properties, it suffices to show the inequality $i(u(\tau), v(\tau)) \le i(u_0, v_0)$ for fixed $0 < \tau < T$. We consider the function $\tilde{w} = u - v$ which is continuous on $\mathbb{B} \times [0, \tau]$ and satisfies the equation $\tilde{w}_t - \Delta\tilde{w} + f(x, t)\tilde{w} = 0$ in $\mathbb{B} \times (0, \tau]$, where $f(\cdot, t) = 1 - [u^2(t) + v^2(t)] + (\beta - 1)u(t)v(t)$ is bounded in $\mathbb{B} \times [0, \tau]$. Fix $\lambda > 0$ such that $g(x, t) := f(x, t) + \lambda$ is positive on $\mathbb{B} \times [0, \tau]$, and consider $w(x, t) = e^{-\lambda t}\tilde{w}(x, t)$. Then w is continuous on $\mathbb{B} \times [0, \tau]$ and satisfies the equation

$$w_t - \Delta w + g(x, t)w = 0 \qquad \text{in } \mathbb{B} \times (0, \tau]. \qquad (2.16)$$

Let

$$U^+ = \{(x, t) \in \mathbb{B} \times [0, \tau] \ : \ w(x, t) > 0\}, \qquad U^- = \{(x, t) \in \mathbb{B} \times [0, \tau] \ : \ w(x, t) < 0\}.$$

We show that every connected component of U^+ intersects $S_0 := \mathbb{B} \times \{0\}$. Indeed, suppose by contradiction that there is a component U such that $U \cap S_0 = \varnothing$. Since $w \equiv 0$ on the relative boundary of U in $\mathbb{B} \times [0, \tau]$, there exists $(x_0, t_0) \in U$ with $w(x_0, t_0) = \max_U w > 0$. Hence $\Delta w(x_0, t_0) \le 0$. Moreover, since $t_0 > 0$, we have $w_t(x_0, t_0) = 0$ if $t_0 < \tau$ and $w_t(x_0, t_0) \ge 0$ if $t_0 = \tau$. This however contradicts (2.16), since $g > 0$ on $\mathbb{B} \times [0, \tau]$. Similarly, we show that every connected component of U^- intersects S_0.
Now let $k = i(u(\tau), v(\tau))$, and choose $x_1, \dots, x_{k+1}$ with $0 \le |x_1| < \dots < |x_{k+1}| < 1$ and

$$w(x_i, \tau) w(x_{i+1}, \tau) < 0 \qquad \text{for } i = 1, \dots, k.$$

We may assume that $w(x_1, \tau) > 0$ and that $k + 1 = 2j$ is even, the other cases are treated similarly. Then there are corresponding components $U_1^+, \dots, U_j^+$ of U^+ and $U_1^-, \dots, U_j^-$ of U^- such that $(x_{2i-1}, \tau) \in U_i^+$ and $(x_{2i}, \tau) \in U_i^-$ for $i = 1, \dots, j$. Since $U_i^\pm \cap S_0 \neq \varnothing$ for every i, we may pick $(y_{2i-1}, 0) \in U_i^+ \cap S_0$ and $(y_{2i}, 0) \in U_i^- \cap S_0$. From the fact that $w(\cdot, t)$ is a radial function for all $0 \le t \le \tau$, we deduce that $0 \le |y_1| < |y_2| < \dots < |y_{k+1}|$, while $w(y_i, 0) w(y_{i+1}, 0) < 0$ for $i = 1, \dots, k$. Hence $i(u_0, v_0) \ge k$, as claimed. □

By Proposition 2.2 and the principle of linearized stability, the constant solution $(u, v) \equiv (0, 0)$ is stable in X, so that the set

$$\mathcal{A}_* := \{(u, v) \in X_+ \,:\, T(u, v) = \infty \text{ and } \varphi^t(u, v) \to (0, 0) \text{ in } X \text{ as } t \to \infty\} \tag{2.17}$$

is a relatively open neighborhood of $(0, 0)$ in X_+.

Lemma 2.6. $\{(u, u) \,:\, u \in W_r, u \ge 0\} \subset \mathcal{A}_*$.

Proof. Let $u_0 \in W_r$, $u_0 \ge 0$. By uniqueness of the solution of the Cauchy problem (1.3), we have $\varphi^t(u_0, u_0) = (u(x, t), u(x, t))$, where $u(x, t)$ is the unique solution of the Cauchy problem

$$u_t - \Delta u = (1 + \beta) u^3 - u \quad \text{in } \mathbb{B}, \qquad u = 0 \quad \text{on } \partial\mathbb{B}, \qquad u(0) = u_0. \tag{2.18}$$

A comparison with the solution $y = y(t)$ of the ordinary differential equation $\dot{y} = (1 + \beta) y^3 - y$ satisfying $y(0) = |u_0|_\infty$ yields $0 \le u(x, t) \le y(t)$ for all $x \in \mathbb{B}$, $t \ge 0$, whereas $y(t) \to 0$ as $t \to \infty$ since $\beta \le -1$. This shows that $|u(\cdot, t)|_\infty$ is uniformly bounded in $t \in [0, T(u_0, u_0))$, so that $E(\varphi^t(u_0, u_0))$ remains bounded from below. Hence $T(u_0, u_0) = \infty$ by Proposition 2.3, and for $\delta > 0$ the set $\{\varphi^t(u_0, u_0) \,:\, t \ge \delta\}$ is relatively compact in Y. Since $|u(\cdot, t)|_\infty \le y(t) \to 0$ as $t \to \infty$, we conclude that $\varphi^t(u_0, u_0) \to 0$ in the Y-topology and therefore also in the X-topology. Hence $(u_0, u_0) \in \mathcal{A}_*$, as claimed. □

3. Existence of solutions with a given number of intersections

We keep using the notation of Section 2. Let $\partial\mathcal{A}_*$ denote the relative boundary of the set $\mathcal{A}_*$ (see (2.17)) in X_+ with respect to the X-topology. The continuity of the semiflow φ and Proposition 2.2(ii) imply that $\partial\mathcal{A}_*$ is positively invariant under

φ. Moreover, $E(u,v) \ge 0$ and $T(u,v) = \infty$ for every $(u,v) \in \partial\mathcal{A}_*$ by Proposition 2.3. We now define

$$Y_k := \{(u,v) \in Y : i(u,v) \le k-1\} \qquad \text{and} \qquad \mathcal{A}_k := \{(u,v) \in \partial\mathcal{A}_* : i(u,v) \le k-1\}$$

By definition, $\mathcal{A}_k$ is a closed subset of X, and by Lemma 2.5 it is a positively invariant set for the flow φ. Our aim is to find solutions of (1.3) in $\mathcal{A}_k \setminus \mathcal{A}_{k-1}$ for every $k \ge 2$.

We remark the following.

Lemma 3.1. *If $(u,v) \in \mathcal{A}_k$ is a radial solution of (1.3), then $(u,v) \in \text{int}_Y(Y_k)$, where $\text{int}_Y(Y_k)$ denotes the interior of Y_k with respect to the Y-topology.*

Proof. If (u,v) is a radial solution of (1.3), then $(u,v) \in Y$ by standard elliptic regularity. Moreover, as a function of the radial variable, $w = u - v$ is a solution of the one-dimensional boundary value problem

$$-w_{rr} - \frac{N-1}{r} w_r + f(r)w = 0, \quad r \in (0,1), \qquad w_r(0) = 0,\ w(1) = 0,$$

where $f(r) = 1 - [u^2(r) + v^2(r)] + (\beta - 1)u(r)v(r)$. Hence $w(0) \neq 0$, and $r \mapsto w(r)$ has only simple zeros in $(0,1]$. In fact, w has $l \le k-1$ zeros since $(u,v) \in \mathcal{A}_k$. But then there is a neighborhood of w in the C^1-topology containing only functions with precisely l simple zeros. Hence $(u,v) \in \text{int}_Y(Y_k)$, as claimed. □

Next we note that the set $\partial\mathcal{A}_*$ and the sets $\mathcal{A}_k$, $k \ge 1$ are symmetric with respect to the involution $(u,v) \mapsto \sigma(u,v) = (v,u)$, and the semiflow φ^t is σ-equivariant. We also note that σ has no fixed points in $\partial\mathcal{A}_*$ by Lemma 2.6. For a closed σ-symmetric subset $A \subset \partial\mathcal{A}_*$ we define the genus $\gamma(A)$ corresponding to σ as the least $k \in \mathbb{N} \cup \{0\}$ such that there is a continuous map $h : A \to \mathbb{R}^k \setminus \{0\}$ with $h(v,u) = -h(u,v)$. As usual, we define $\gamma(A) = \infty$ if no such k exists. The genus has many useful properties. In the following we only list the properties we need.

Lemma 3.2. *Let $A, B \subset \partial\mathcal{A}_*$ be closed and σ-symmetric.*

(i) *If $A \subset B$, then $\gamma(A) \le \gamma(B)$.*
(ii) *$\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.*
(iii) *If $h : A \to \partial\mathcal{A}_*$ is continuous and σ-equivariant, then $\gamma(A) \le \gamma(\overline{h(A)})$.*
(iv) *If $\gamma(A) < \infty$, then there exists a relatively open σ-symmetric neighborhood N of A in $\partial\mathcal{A}_*$ such that $\gamma(A) = \gamma(\overline{N})$.*
(v) *If S is the boundary of a bounded symmetric neighborhood of the origin in a k-dimensional normed vector space und $\psi : S \to \partial\mathcal{A}_*$ is a continuous map satisfying $\psi(-u) = \sigma(\psi(u))$, then $\gamma(\psi(S)) \ge k$.*

Note that in (v) the set $\psi(S)$ is closed since S is compact.

Proof. Properties (i) and (iii) follow immediately from the definition of γ. Moreover, (ii) and (iv) can be proved using the Tietze extension theorem similarly as in [42, p. 96]. Property (v) is proved by contradiction, assuming that there exists a continuous map $h : \psi(S) \to \mathbb{R}^{k-1} \setminus \{0\}$ with $h(v,u) = -h(u,v)$. Then $h \circ \psi : S \to \mathbb{R}^{k-1} \setminus \{0\}$ is an odd and continuous map, which contradicts the Borsuk-Ulam Theorem (see e.g. [44, Theorem D.17.]). □

Lemma 3.3. *$\gamma(\mathcal{A}_k) \le k$.*

Proof. We proceed by induction, starting with $k = 1$. By definition, $\mathcal{A}_1$ is precisely the set of vectors $(u,v) \in \partial\mathcal{A}_*$ such that $u - v$ does not change sign. By Lemma 2.6, $\{(u,u) \,:\, u \in W_r, u \ge 0\} \cap \mathcal{A}_1 = \varnothing$, which implies that $\mathcal{A}_1 = B_+ \cup B_-$ with disjoint subsets $B_\pm$ defined by

$$B_+ = \{(u,v) \in \mathcal{A}_* \,:\, u \ge v,\; u - v \not\equiv 0\}, \qquad B_- = \{(u,v) \in \mathcal{A}_* \,:\, u \le v,\; u - v \not\equiv 0\}.$$

Since the sets $\mathcal{B}_\pm$ are relatively open in $\mathcal{A}_1$, the map

$$h : \mathcal{A}_1 \to \mathbb{R} \setminus \{0\}, \qquad h(u,v) = \begin{cases} 1 & (u,v) \in B_+, \\ -1 & (u,v) \in B_- \end{cases}$$

is continuous, and it is also σ-symmetric. We conclude that $\gamma(\mathcal{A}_1) \le 1$, as claimed.
Next we consider $k > 1$ and assume that $\gamma(\mathcal{A}_{k-1}) \le k - 1$. We use the fact that $\mathcal{A}_k = \tilde{A} \cup \mathcal{A}_{k-1}$, where $\tilde{A} = \{(u,v) \in \mathcal{A}_* \,:\, i(u,v) = k-1\}$. Let $\tilde{B}_\pm$ be the set of all $(u,v) \in \tilde{A}$ such that, for some $x_1 \in \mathbb{B}$,

$$\pm(u(x_1) - v(x_1)) > 0 \qquad \text{and} \qquad \pm\,(u(x) - v(x)) \ge 0 \quad \text{for } 0 \le x \le |x_1|.$$

Then $\tilde{A} = \tilde{B}_+ \cup \tilde{\mathcal{B}}_-$. We claim that the sets $\tilde{B}_\pm$ are relatively open in $\tilde{A}$. Indeed, if $(u,v) \in \tilde{B}_+$, then there are points $x_1, \dots, x_k$ with $0 \le |x_1| < \dots < |x_k| < 1$ such that

$$u(x) - v(x) \ge 0, \quad u(x_1) - v(x_1) > 0, \quad \text{and } [u(x_i) - v(x_i)][u(x_{i+1}) - v(x_{i+1})] < 0$$

for $0 \le |x| \le |x_1|$ and $i = 1, \dots, k-1$. Hence there is a neighborhood $U \subset X_+$ of (u,v) such that $[\tilde{u}(x_i) - \tilde{v}(x_i)][\tilde{u}(x_{i+1}) - \tilde{v}(x_{i+1})] < 0$ for every $(\tilde{u},\tilde{v}) \in U$, $i = 1, \dots, k-1$. This implies that $\tilde{u}(x) - \tilde{v}(x) \ge 0$ for $0 \le |x| \le |x_1|$ and every $(\tilde{u},\tilde{v}) \in U \cap \tilde{B}_+$, since $i(\tilde{u},\tilde{v}) = k-1$. Hence $\tilde{B}_+$ is relatively open in $\tilde{A}$. A similar argument shows that $\tilde{B}_-$ is relatively open in $\tilde{A}$. Consequently, the map

$$\tilde{h} : \tilde{A} \to \mathbb{R} \setminus \{0\}, \qquad \tilde{h}(u,v) = \begin{cases} 1 & (u,v) \in \tilde{B}_+, \\ -1 & (u,v) \in \tilde{B}_- \end{cases}$$

is continuous and σ-symmetric. To conclude the proof, we let $N \subset \partial\mathcal{A}_*$ be a relatively open σ-symmetric neighborhood of $\mathcal{A}_{k-1}$ such that

$$\gamma(\overline{N}) = \gamma(\mathcal{A}_{k-1}) \le k - 1,$$

as provided by Lemma 3.2(iv). Since $\mathcal{A}_k \setminus N$ is a closed σ-symmetric subset of $\tilde{A}$ and therefore $\gamma(\mathcal{A}_k \setminus N) \le 1$ via the map $\tilde{h}$ defined above, we conclude that

$$\gamma(\mathcal{A}_k) \le \gamma(\overline{N}) + \gamma(\mathcal{A}_k \setminus N) \le k.$$

□

Proposition 3.4. *For every $k \ge 2$, there exists a solution $(u,v) \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}$ of (1.3) with $E(u,v) \le c_k$.*

Proof. It is known (see [40, 41]) that there is a radial solution $\bar{w}$ of the equation

$$\Delta w - w + w^3 = 0 \quad \text{in } \mathbb{B}, \qquad w = 0 \quad \text{on } \partial\mathbb{B} \tag{3.1}$$

with $E_S(\bar{w}) = c_k$ and such that $\bar{w}$, viewed as a function of the radial variable, has precisely $k-1$ interior zeros $0 < r_1 < \cdots < r_{k-1} < 1$. Put $r_0 = 0$ and $r_k = 1$, and consider $w_j = \bar{w} \cdot 1_{\{r_j \le |x| \le r_{j+1}\}} \in W_r$ for $j = 0, \dots, k-1$. Multiplying (3.1) by w_j and integrating over $\{r_j \le |x| \le r_{j+1}\}$, we find that $\|w_j\|^2 = |w_j|_4^4$ and therefore $E_S(w_j) = \frac{1}{4}\|w_j\|^2$. Hence we have

$$E_S(sw_j) = \frac{1}{2}(s^2 - \frac{s^4}{2})\|w_j\|^2 \le \frac{1}{4}\|w_j\|^2 = E_S(w_j) \qquad \text{for every } s \in \mathbb{R} \tag{3.2}$$

and

$$E_S(sw_j) \to -\infty \qquad \text{as } |s| \to \infty. \tag{3.3}$$

We consider the $k-$dimensional subspace $W \subset W_r$ spanned by the functions w_j, $j = 0, \dots, k-1$, and the map

$$\psi : W \to X_+, \qquad \psi(w) = (w^+, w^-),$$

where $w^+ = \max\{w, 0\}, w^- = -\min\{w, 0\}$. Clearly ψ is continuous, and $\psi(-w) = \sigma(\psi(w))$ for all $w \in W$. Using (3.2), we find that

$$E(\psi(\sum_{j=1}^{k} s_j w_j)) = \sum_{j=1}^{k} E_S(s_j w_j) \le \sum_{j=1}^{k} E_S(w_j) = E_S(\bar{w}) = c_k \tag{3.4}$$

for all $(s_1, \dots, s_k) \in \mathbb{R}^k$, while

$$\lim_{\|w\| \to \infty} E(\psi(w)) = -\infty$$

by (3.3). Hence $\mathcal{O} := \{w \in W \ : \ \psi(w) \in \mathcal{A}_*\}$ is a symmetric bounded open neighborhood of 0 in W, and $\psi(\partial\mathcal{O}) \subset \mathcal{A}_k$. Lemma 3.2(v) implies that $\gamma(\psi(\partial\mathcal{O})) \ge k$. On the other hand, defining the closed subsets

$$\mathcal{C}_{k-1}^t := \{(u, v) \in \partial\mathcal{A}_* \ : \ \varphi^t(u, v) \in \mathcal{A}_{k-1}\} \subset \partial\mathcal{A}_* \qquad \text{for } t > 0,$$

we infer $\gamma(\mathcal{C}_{k-1}^t) \le k-1$ by Lemma 3.2(iii) and Lemma 3.3 for every $t > 0$. In particular, for every positive integer n there exists $(u_n, v_n) \in \psi(\partial\mathcal{O}) \setminus \mathcal{C}_{k-1}^n$, so that $\varphi^n(u_n, v_n) \notin \mathcal{A}_{k-1}$. Since $\psi(\partial\mathcal{O})$ is compact, we may pass to a subsequence such that $(u_n, v_n) \to (\bar{u}, \bar{v})$ as $n \to \infty$. We claim that

$$\varphi^t(\bar{u}, \bar{v}) \notin \text{int}_Y(Y_{k-1}) \qquad \text{for every } t > 0. \tag{3.5}$$

Indeed, assuming by contradiction that $\varphi^{t_0}(\bar{u}, \bar{v}) \in \text{int}_Y(Y_{k-1})$ for some $t_0 > 0$, the continuity of φ^t as stated in Proposition 2.2(i) implies that

$$\varphi^{t_0}(u_n, v_n) \in \text{int}_Y(Y_{k-1}) \cap \partial\mathcal{A}_* \subset \mathcal{A}_{k-1} \qquad \text{for } n \text{ large enough,}$$

hence $\varphi^n(u_n, v_n) \in \mathcal{A}_{k-1}$ for n large by the positive invariance of $\mathcal{A}_{k-1}$. This contradicts the choice of (u_n, v_n). Hence (3.5) is true.

Now (3.5) implies that the ω-limit set $\omega(\bar{u}, \bar{v})$ does not intersect $\text{int}_Y(Y_{k-1})$. Since $\omega(\bar{u}, \bar{v})$ consists of radial solutions of (1.3), we conclude by Lemma 3.1 that $\omega(\bar{u}, \bar{v}) \subset$

$\mathcal{A}_k \setminus \mathcal{A}_{k-1}$. Moreover, $E(u,v) \le E(\bar u, \bar v) \le c_k$ for every $(u,v) \in \omega(\bar u, \bar v)$ by (3.4). So every $(u,v) \in \omega(\bar u, \bar v)$ has the asserted properties. □

Theorem 1.1 follows directly from Proposition 3.4.

4. Asymptotic behaviour as $\beta \to \infty$

This section is devoted to the proof of Theorem 1.2. For fixed $k \ge 2$, let $\beta_n \le -1$, $n \in \mathbb{N}$ be such that $\beta_n \to -\infty$ as $n \to \infty$, and let $(u_n, v_n) \in H_r \times H_r$ be solutions of (1.3) with $\beta = \beta_n$ such that $u_n - v_n$ changes sign precisely $k-1$ times in the radial variable and $E(u_n, v_n) \le c_k$. In the following, $C_0, C_1, \dots$ always stand for positive constants independent of n. By (2.1), the energy bound yields a uniform H^1-bound for the sequence $(u_n, v_n)_n$. Passing to a subsequence, we may therefore assume that

$$u_n \rightharpoonup u, \qquad v_n \rightharpoonup v \qquad \text{weakly in } H_r.$$

Since β_n is negative and u_n, v_n are bounded in $H^1(\mathbb{B})$, we deduce from standard elliptic subsolution estimates (e.g. Theorem 8.17 of [16]) that

$$|u_n|_\infty, |v_n|_\infty \le C_0. \tag{4.1}$$

We consider the radial functions

$$H_n : \mathbb{B} \to \mathbb{R}, \qquad H_n := |u_n'|^2 + |v_n'|^2 - (u_n^2 + v_n^2) + \frac{1}{2}(u_n^4 + v_n^4) + \beta_n u_n^2 v_n^2,$$

where the prime stands for the radial derivative $\frac{d}{dr}$.The following monotonocity property in $r = |x|$ is crucial:

$$\begin{aligned} H_n'(r) &= 2u_n'(r)[u_n''(r) - u_n(r) + u_n^3(r) + \beta_n v_n^2(r) u_n(r)] \\ &\quad + 2v_n'(r)[v_n''(r) - v_n(r) + v_n^3(r) + \beta_n u_n^2(r) v_n(r)] \\ &= -\frac{2(N-1)}{r}([u_n'(r)]^2 + [v_n'(r)]^2) \le 0 \qquad \text{for } r > 0. \end{aligned} \tag{4.2}$$

The second equality follows from (1.3). Since $\beta_n < 0$ and $u_n'(0) = v_n'(0) = 0$, we have

$$H_n(0) \le \frac{1}{2}(u_n^4(0) + v_n^4(0)) \le C_1 \tag{4.3}$$

and therefore

$$0 < |u_n'(1)|^2 + |v_n'(1)|^2 = H_n(1) \le H_n(0) \le C_1 \tag{4.4}$$

We thus conclude that the functions H_n are positive, nonincreasing and uniformly bounded in $[0,1]$. Integrating, we also get

$$C_1 \ge H_n(0) - H_n(1) = 2(N-1) \int_0^1 \frac{[u_n'(r)]^2 + [v_n'(r)]^2}{r}\, dr.$$

Viewing u_n, v_n as functions of $r \in [0,1]$, we deduce

$$\|u_n\|_{H^1([0,1])}, \|v_n\|_{H^1([0,1])} \le C_2 \tag{4.5}$$

for $N \ge 2$, while for $N = 1$ this is already known. We therefore conclude that

$$u_n \to u, \qquad v_n \to v \qquad \text{uniformly in } \mathbb{B}. \tag{4.6}$$

In particular, u and v are continuous. In the next three lemmas, we collect further properties of the sequence $(u_n, v_n)_n$ and its limit (u, v).

Lemma 4.1. *Let* $P(u) = \{x \in \mathbb{B} \,:\, u(x) > 0\}$, $P(v) = \{x \in \mathbb{B} \,:\, v(x) > 0\}$.
(i) For any $\tau > 0$,

$$|\beta_n|^\tau v_n \to 0 \quad \textit{uniformly on compact subsets of } P(u),$$
$$|\beta_n|^\tau u_n \to 0 \quad \textit{uniformly on compact subsets of } P(v).$$

(ii) On $P(u)$ *resp.* $P(v)$, u *resp.* v *solve the equations*

$$-\Delta u + u = u^3, \qquad -\Delta v + v = v^3,$$

respectively, in classical sense.

The following proof does not use the radial symmetry of u_n and v_n. It only relies on (4.6).

Proof. (i) We only prove the first statement. Let $K \subset P(u)$ be compact, and let $\varepsilon > 0$ be such that

$$K_\varepsilon := \{x \in \mathbb{R}^N \,: \mathrm{dist}(x, K) \le \varepsilon\} \subset \{x \in P(u) \,:\, u(x) > \varepsilon\}.$$

In K_ε, we have

$$\Delta v_n \ge (1 - v_n^2 - \frac{\beta_n \varepsilon^2}{2}) v_n \ge \Big(\frac{|\beta_n|\varepsilon^2}{2} - C_3\Big) v_n \ge \frac{|\beta_n|\varepsilon^2}{4} v_n \qquad \text{for } n \text{ sufficiently large.}$$

Now fix $x_0 \in K$. Since $B_\varepsilon(x_0) \subset K_\varepsilon$, we have

$$\begin{cases} \Delta v_n \ge M_n v_n & \text{in } B_\varepsilon(x_0), \\ \quad v_n \ge 0 & \text{in } B_\varepsilon(x_0), \\ \quad v_n \le C_0 & \text{on } \partial B_\varepsilon(x_0), \end{cases}$$

where $M_n := \frac{|\beta_n|\varepsilon^2}{4}$. Applying [13, Lemma 4.4] with $\alpha = \frac{1}{2}$, we conclude that

$$v_n(x_0) \le C_4 e^{-\frac{\varepsilon}{2}\sqrt{M_n}} = C_4 e^{-\frac{\varepsilon^2}{4}\sqrt{|\beta_n|}}.$$

For n large enough such that $\sqrt{|\beta_n|} \ge \frac{8\tau}{\varepsilon^2} \log |\beta_n|$, we conclude

$$v_n(x_0) \le C_4 |\beta_n|^{-2\tau},$$

where the constant C_4 does not depend on x_0. Hence $\sup_K |\beta_n|^\tau v \to 0$ as $n \to \infty$, as claimed.
(ii) For $\varphi \in C_0^\infty(P(u))$ we have

$$\begin{aligned} \int_{P(u)} u \Delta\varphi \, dx &= \lim_{n\to\infty} \int_{P(u)} u_n \Delta\varphi \, dx = \lim_{n\to\infty} \int_{P(u)} \Delta u_n \varphi \, dx \\ &= \lim_{n\to\infty} \int_{P(u)} (u_n - u_n^3 - \beta_n v_n^2 u_n)\varphi \, dx = \int_{P(u)} (u - u^3)\varphi \, dx \end{aligned}$$

as a consequence of (i) and (4.6). Hence u is a distributional solution of $-\Delta u + u = u^3$ in $P(u)$. Since we already know that u is continuous, classical elliptic regularity shows that u is in fact a classical solution. The statement for v is proved in the same way. □

Corollary 4.2.
(i) If $0 < r_1 < r_2 \le 1$ are such that u is positive in $\mathcal{A} := \{x \in \mathbb{B} \,:\, r_1 < |x| < r_2\}$ and $u|_{\partial\mathcal{A}} = 0$, then

$$\int_{\mathcal{A}} (|\nabla u|^2 + u^2 - u^4)\, dx = 0 \tag{4.7}$$

(ii) If $0 < r \le 1$ is such that u is positive in $\mathcal{B} := \{x \in \mathbb{B} \,:\, |x| < r\}$ and $u|_{\partial\mathcal{B}} = 0$, then

$$\int_{\mathcal{B}} (|\nabla u|^2 + u^2 - u^4)\, dx = 0 \tag{4.8}$$

Remark 4.3. *The same statements are true for v in place of u.*

Proof. (i) Since u is differentiable in $\mathcal{A} \subset \mathcal{P}(u)$ by Lemma 4.1(ii), we may pick $r_1 < s_n < t_n < r_2$ such that $s_n \to r_1$, $t_n \to r_2$ as $n \to \infty$ and $u'(s_n) \ge 0$, $u'(t_n) \le 0$ for all n. Then $\varepsilon_n := \max\{u(s_n), u(t_n)\} \to 0$ as $n \to \infty$. Now Lemma 4.1(ii) implies that

$$\begin{aligned}
\Big|\int_{s_n<|x|<t_n} (|\nabla u|^2 + u^2 - u^4)\, dx\Big| &= \Big|\int_{|x|=t_n} u\frac{\partial u}{\partial r}\, d\sigma - \int_{|x|=s_n} u\frac{\partial u}{\partial r}\, d\sigma\Big| \\
&\le \varepsilon_n\Big|\int_{|x|=t_n} \frac{\partial u}{\partial r}\, d\sigma - \int_{|x|=s_n} \frac{\partial u}{\partial r}\, d\sigma\Big| = \varepsilon_n\Big|\int_{s_n<|x|<t_n} \Delta u\, dx\Big| \\
&\le \varepsilon_n \int_{\mathcal{A}} |u - u^3|\, dx \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Hence (4.7) follows. The proof of (ii) is similar. □

Lemma 4.4.

(i) $u_n v_n \to uv = 0$ *uniformly in* $\mathbb{B}$.
(ii) $\beta_n \int_{\mathbb{B}} u_n^2 v_n^2\, dx \to 0$ *as* $n \to \infty$.
(iii) $\max\{u(0), v(0)\} \ge \sqrt{2}$.

Proof. (i) follows immediately from (4.6) and Lemma 4.1(i).
(ii) Since

$$0 \le -\int_{\partial\mathbb{B}} \frac{\partial u_n}{\partial r}\, d\sigma = -\int_{\mathbb{B}} \Delta u_n\, dx = \int_{\mathbb{B}} (u_n^3 - u_n + \beta_n v_n^2 u_n)\, dx \le C_5 - |\beta_n| \int_{\mathbb{B}} v_n^2 u_n\, dx,$$

we have $|\beta_n| \int_{\mathbb{B}} v_n^2 u_n dx \le C_5$ and similarly $|\beta_n| \int_{\mathbb{B}} u_n^2 v_n dx \le C_5$. From (i) we therefore deduce

$$|\beta_n| \int_{\mathbb{B}} u_n^2 v_n^2 dx \le |\beta_n| \sqrt{|u_n v_n|_\infty} \int_{\mathbb{B}} u_n v_n (u_n + v_n) dx \le 2C_5 \sqrt{|u_n v_n|_\infty} \to 0 \quad \text{as } n \to \infty.$$

(iii) Since $u_n'(0) = v_n'(0) = 0$ and $\beta_n < 0$,

$$0 < H_n(0) \le u_n^2(0)[\frac{u_n^2(0)}{2} - 1] + v_n^2(0)[\frac{v_n^2(0)}{2} - 1],$$

and hence $\max\{u_n(0), v_n(0)\} > \sqrt{2}$ for all n. Since $u_n(0) \to u(0)$ and $v_n(0) \to v(0)$ by (4.6), we conclude that $\max\{u(0), v(0)\} \ge \sqrt{2}$. □

Lemma 4.5. *Let* $0 < r_1 < r_2 < 1$.
(i) If $u \equiv 0$ *on* $[r_1, r_2]$*, then* $u_n' \to 0$ *uniformly on every closed interval contained in* (r_1, r_2).
(ii) If $v \equiv 0$ *on* $[r_1, r_2]$*, then* $v_n' \to 0$ *uniformly on every closed interval contained in* (r_1, r_2).

Proof. (i) By assumption and uniform convergence, $u_n < 1$ on $[r_1, r_2]$ for n large, hence

$$(r^{N-1}u_n')' = r^{N-1}(u_n - u_n^3 - \beta_n v_n^2 u_n) > 0 \qquad \text{on } [r_1, r_2].$$

For $r \in [r_1, r_2]$ we therefore have

$$u_n(r_2) > u_n(r_2) - u_n(r) = \int_r^{r_2} u_n'(s)\,ds \geq \int_r^{r_2} s^{N-1}u_n'(s)\,ds \geq (r_2 - r)r_1^{N-1}u_n'(r)$$

and

$$-u_n(r_1) < u_n(r) - u_n(r_1) = \int_{r_1}^{r} u_n'(s)ds \leq r_1^{1-N}\int_{r_1}^{r} s^{N-1}u_n'(s)ds \leq \Big(\frac{r_2}{r_1}\Big)^{N-1}(r - r_1)u_n'(r).$$

Now consider points $r_1 < s_1 < s_2 < r_2$. Then, for every $r \in [s_1, s_2]$,

$$-\frac{r_1^{N-1}u_n(r_1)}{r_2^{N-1}(s_1 - r_1)} \leq -\frac{r_1^{N-1}u_n(r_1)}{r_2^{N-1}(r - r_1)} \leq u_n'(r) \leq \frac{u_n(r_2)}{(r_2 - r)r_1^{N-1}} \leq \frac{u_n(r_2)}{(r_2 - s_2)r_1^{N-1}}.$$

Consequently,

$$\max_{[s_1,s_2]} |u_n'| \leq C_6 \max\{u_n(r_1), u_n(r_2)\} \to 0 \qquad \text{as } n \to \infty.$$

Thus (i) is true. The proof of (ii) is similar. □

Next we introduce the bounded nonnegative nonincreasing function

$$h_\infty : [0,1] \to \mathbb{R}, \qquad h_\infty(r) := \liminf_{n\to\infty} H_n(r) \qquad \text{for } 0 \leq r \leq 1.$$

Lemma 4.6. *(i) If* $N = 1$*, then* h_∞ *equals a positive constant in* $[0, 1]$.
(ii) If $N \geq 2$ *and* $\max\{u(r), v(r)\} > 0$ *for some* $r < 1$*, then* $h_\infty(r) > 0$.

Proof. (i) If $N = 1$, then the functions H_n are constant by (4.2), hence h_∞ is also constant. By integration and Lemma 4.4(ii), we get

$$\begin{aligned} h_\infty(r) &= \liminf_{n\to\infty} \int_0^1 H_n(s)\,ds \\ &= \liminf_{n\to\infty} \int_0^1 \Big(|u_n'|^2 + |v_n'|^2 - (u_n^2 + v_n^2) + \frac{1}{2}(u_n^4 + v_n^4)\Big)\,ds \\ &\geq \int_0^1 \Big(|u'|^2 + |v'|^2 - (u^2 + v^2) + \frac{1}{2}(u^4 + v^4)\Big)\,ds = \int_0^1 (H_u + H_v)\,ds, \end{aligned}$$

where $H_u = |u'|^2 - u^2 + \frac{u^4}{2}$ and $H_v = |v'|^2 - v^2 + \frac{v^4}{2}$. Let $I \subset P(u)$ be a maximal open subinterval. Since $H_u' = 2u'(u'' - u + u^3) = 0$ in $P(u)$ by Lemma 4.1(ii), H_u is constant in I. An elementary phase plane analysis shows that if $H_u \leq 0$ in I, then u is bounded away from zero in I (since I is bounded), which contradicts the maximality of I. Hence $H_u > 0$ in I, and therefore $H_u > 0$ in $P(u)$. In the same way we deduce that $H_v > 0$ in $P(v)$. Since $H_u = 0$ a.e. on the zero set of u and $H_v = 0$ a.e. on the zero set of v, we conclude that

$$h_\infty(r) \geq \int_0^1 (H_u(s) + H_w(s))\, ds > 0,$$

as claimed.

(ii) We may assume that $u(r) > 0$. Since $H_n(1) = (u_n'(1))^2 + (v_n'(1))^2 > 0$, (4.2) implies

$$H_n(r) \geq -\int_r^1 H_n'(s)\, ds = \int_r^1 \frac{N-1}{s}[|u_n'|^2 + |v_n'|^2]\, ds \geq \int_r^1 |u_n'|^2\, ds,$$

so that by weak convergence $u_n \rightharpoonup u$ in $H^1(\mathbb{B})$,

$$h_\infty(r) \geq \int_r^1 |u'|^2\, ds \geq \frac{1}{1-r}\Big(\int_r^1 u'\, ds\Big)^2 = \frac{u^2(r)}{1-r} > 0.$$

□

We now have all the tools to study the intersection properties of u_n and v_n resp. u and v.

Lemma 4.7. *Suppose that $0 < r_0 < 1$ are such that $u(r_0) > 0$, $u(r) \geq 0$ and $v(r) = 0$ for $r_0 \leq r \leq 1$. Then $u_n \geq v_n$ on $[r_0, 1]$ for n sufficiently large.*

Remark 4.8. *The analoguous statement is true with the roles of u and v (resp. of u_n and v_n) exchanged.*

Proof. By uniform convergence we have $v_n < \min\{1, u(r_0)\}$ on $[r_0, 1]$ for n large, so that $\Delta v_n > 0$ on $[r_0, 1]$ and therefore

$$v_n(r) \leq \max\{v_n(r_0), v_n(1)\} = v_n(r_0) = o(|\beta_n|^{-1}) \qquad \text{for } r_0 \leq r \leq 1$$

by Lemma 4.1(i). Hence a short calculation shows that $w_n = u_n - v_n$ satisfies

$$w_n^3 = -\Delta w_n + [1 + (\beta_n - 3)u_n v_n]w_n = -\Delta w_n + [1 + o(1)]w_n \qquad \text{in } (r_0, 1). \tag{4.9}$$

Suppose by contradiction that, for a subsequence, there are points $r_0 < r_1^n < r_2^n \leq 1$ such that $w_n(r_1^n) = 0 = w_n(r_2^n)$ and $w_n(r) < 0$ for $r_1^n < r < r_2^n$. Then, multiplying (4.9) with w_n and integrating by parts, we obtain

$$\int_{r_1^n}^{r_2^n} r^{N-1} w_n^4\, dx = \int_{r_1^n}^{r_2^n} r^{N-1}(|w_n'|^2 + [1 + o(1)]w_n^2)\, dr \geq \int_{r_1^n}^{r_2^n} r^{N-1}|w_n'|^2\, dr$$
$$\geq C_7\Big(\int_{r_1^n}^{r_2^n} r^{N-1} w_n^4\, dr\Big)^{\frac{1}{2}}$$

for n large, so that $\int_{r_0}^1 r^{N-1}|w_n^-|^4 dr \geq \int_{r_1^n}^{r_2^n} r^{N-1} w_n^4 dr \geq C_7^2$. This however contradicts the fact that $w_n^- \to 0$ uniformly on $[r_0, 1]$ by assumption. □

Lemma 4.9. *Suppose that $0 < r_1 < r_2 < r_3 < 1$ are such that $u(r_1) > 0$, $u(r_2) = 0$, and $u(r_3) > 0$. Then there exists $r \in (r_1, r_3)$ with $v(r) > 0$.*

Remark 4.10. *Again, the analoguous statement is true with the roles of u and v exchanged.*

Proof. By uniform convergence $u_n \to u$, the asumptions on u imply that there exists $\varepsilon_0 > 0$ and, for large n, $\tau_n \in [r_1 + \varepsilon_0, r_3 - \varepsilon_0]$ with $u_n'(\tau_n) = 0$ and $u_n(\tau_n) \to 0$. Now suppose by contradiction that $v \equiv 0$ on $[r_1, r_3]$. Then $v_n \to 0$ and $v_n' \to 0$ uniformly on $[r_1 + \varepsilon_0, r_3 - \varepsilon_0]$ by Lemma 4.5, and therefore

$$H_n(r_3) \leq H_n(\tau_n) \leq |u_n'(\tau_n)|^2 + |v_n'(\tau_n)|^2 + \frac{1}{2}(u_n^4(\tau_n) + v_n^4(\tau_n)) = o(1).$$

This contradicts Lemma 4.6. Hence there exists $r \in (r_1, r_3)$ with $v(r) > 0$. □

Lemma 4.11. *Suppose that $0 < r_1 < r_2 < r_3 < 1$ are such that $u(r_1) > 0$, $v(r_3) > 0$, $v \equiv 0$ in $[r_1, r_2]$ and $u \equiv 0$ in $[r_2, r_3]$. Then, for n sufficiently large, $u_n - v_n$ has precisely one zero in (r_1, r_3).*

Remark 4.12. *Again, the analoguous statement is true with the roles of u and v (resp. of u_n and v_n) exchanged.*

Proof. Since $h_\infty(r_3) > 0$ by Lemma 4.6, we may choose $0 < \varepsilon < \min\{1, u(r_1), v(r_3)\}$ such that

$$\varepsilon^4 + 2\varepsilon^2 < h_\infty(r_3). \tag{4.10}$$

Let $s_1 \in (r_1, r_2]$, $s_2 \in [r_2, r_3)$ be such that

$$u(s_1) = \varepsilon,\ u(r) < \varepsilon \text{ for } s_1 < r \leq r_3 \qquad \text{and} \qquad v(s_2) = \varepsilon,\ v(r) < \varepsilon \text{ for } r_1 \leq r < s_2.$$

By assumption and Lemma 4.9 we have $u > 0$ on $[r_1, s_1]$ and $v > 0$ on $[s_2, r_3]$. Thus $s_1 < s_2$ and

$$v_n < u_n \quad \text{on } [r_1, s_1], \qquad u_n < v_n \quad \text{on } [s_2, r_3] \qquad \text{for } n \text{ large.} \tag{4.11}$$

Since, by Lemma 4.4(i), $v \equiv 0$ in a neighborhood of r_1 and $u \equiv 0$ in a neighborhood of r_3, Lemma 4.5 implies that

$$u_n'(r_3) < \left(\frac{s_1}{r_3}\right)^{N-1} \varepsilon \quad \text{and} \quad v_n'(r_1) > -\varepsilon \qquad \text{for } n \text{ large.} \tag{4.12}$$

For n large we also have $u_n < 1$ on $[s_1, r_3]$, therefore

$$(r^{N-1}u_n')' = r^{N-1}\Delta u_n > 0,$$

so that $r^{N-1}u_n'$ is increasing in $[s_1, r_3]$. Similarly, $r^{N-1}v_n'$ is increasing in $[r_1, s_2]$. So (4.12) implies that

$$u_n' < \varepsilon \quad \text{on } [s_1, r_3] \quad \text{and} \quad v_n' > -\varepsilon \quad \text{on } [r_1, s_2] \qquad \text{for } n \text{ large.} \tag{4.13}$$

Now suppose by contradiction that, for a subsequence, the functions $u_n - v_n$ have at least two zeros in (r_1, r_3). By (4.11) these points must lie in (s_1, s_2) for large n. Hence there is a point $\tau_n \in (s_1, s_2)$ with $u_n'(\tau_n) = v_n'(\tau_n)$, so that $|u_n'(\tau_n)| = |v_n'(\tau_n)| < \varepsilon$ by (4.13). Hence

$$\begin{aligned} H_n(\tau_n) &\le |u_n'(\tau_n)|^2 + |v_n(\tau_n)'|^2 + \frac{1}{2}(u_n^4(\tau_n) + v_n^4(\tau_n)) \\ &\le 2\varepsilon^2 + \varepsilon^4 + o(1). \end{aligned} \tag{4.14}$$

We conclude that

$$h_\infty(r_3) = \liminf_{n\to\infty} H_n(r_3) \le \liminf_{n\to\infty} H_n(\tau_n) \le 2\varepsilon^2 + \varepsilon^4,$$

which contradicts (4.10). The proof is finished. □

Corollary 4.13. *The function $w = u - v$ is a radial solution of (1.4) with $E_S(w) = c_k$ which has precisely $k-1$ interior zeros. Moreover, $u_n \to u$ and $v_n \to v$ in $H^1(\mathbb{B})$.*

Proof. Since $w_n := u_n - v_n$ changes sign precisely $k-1$ times in $(0,1)$ for every n and $w_n \to w$ uniformly in $[0,1]$, the function w changes sign at most $k-1$ times. On the other hand, since $u \cdot v = 0$ in $[0,1]$, Lemma 4.11 implies that in every subinterval where w changes sign precisely once, w_n also changes sign precisely once for large n. Together with Lemmas 4.4(iii), 4.7 and 4.9 this implies that w changes sign precisely $k-1$ times in $[0,1]$. Moreover, by weak convergence and Lemma 4.4(ii),

$$\begin{aligned} E_S(w) &= E_S(u) + E_S(v) \le \liminf_{n\to\infty} (E_S(u_n) + E_S(v_n)) \\ &= \liminf_{n\to\infty} \left(E(u_n, v_n) + \frac{\beta}{2} \int_{\mathbb{B}} u_n^2 v_n^2 \right) = \liminf_{n\to\infty} E(u_n, v_n) \le c_k. \end{aligned} \tag{4.15}$$

Corollary 4.2 implies that w is contained in the set Γ_k defined in Section 2, so that w is a minimizer of the minimization problem (2.2). Thus $E_S(w) = c_k$, and w is a radial solution of (1.4) having precisely $k-1$ interior zeros by Proposition 2.1. A posteriori we conclude that equality holds in all steps in (4.15), and therefore

$$\int_{\mathbb{B}} |\nabla u_n|^2\,dx \to \int_{\mathbb{B}} |\nabla u|^2\,dx, \quad \int_{\mathbb{B}} |\nabla v_n|^2\,dx \to \int_{\mathbb{B}} |\nabla v|^2\,dx \qquad \text{as } n \to \infty.$$

Hence $u_n \to u$ and $v_n \to v$ in $H^1(\mathbb{B})$, as claimed. □

Theorem 1.2 is a direct consequence of (4.6), Lemma 4.4(i) and Corollary 4.13.

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