Half-Skyrmions and Spike-Vortex Solutions of Two-Component Nonlinear Schrödinger Systems

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Abstract

Recently, skyrmions with integer topological charges have been observed numerically but have not yet been shown rigorously on two-component systems of nonlinear Schrödinger equations (NLSE) describing a binary mixture of Bose-Einstein condensates (cf. [2] and [25]). Besides, *half-skyrmions* characterized by half-integer topological charges can also be found in the nonlinear σ model which is a model of the Bose-Einstein condensate of the Schwinger bosons (cf. [18]). Here we prove rigorously the existence of *half-skyrmions* which may come from a new type of soliton solutions called spike-vortex solutions of two-component systems of NLSE on the entire plane $\mathbb{R}^2$. These spike-vortex solutions having spikes in one component and a vortex in the other component may form half-skyrmions. By Liapunov-Schmidt reduction process, we may find spike-vortex solutions of two-component systems of NLSE.

1 Introduction

Spikes and vortices are important phenomena in one-component nonlinear Schrödinger equations (NLSE) having applications in many physical problems, especially in Bose-Einstein condensation. In the last two decades, there have been many analytical works on both spikes and vortices, respectively. One may refer to [19] for a good survey on spikes, and [1], [11] and [21] for survey on vortices. Recently, a double condensate i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine states has been observed and described by two-component systems of NLSE (cf. [22]). It would be possible to find spike and vortex solutions from two-component systems of NLSE. However, until now, there is no result to deal with spike-vortex solutions having spikes in one component and a vortex in the other component. In this paper, we want to find such solutions and investigate the interaction of spikes and vortices.

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Ordinary skyrmions being of topological solitons resemble polyhedral shells which look like closed loops, possibly linked or knotted. Besides their intrinsic fundamental interest, skyrmions have important applications in nuclear physics (cf. [17]), and analogous structures are postulated for early universe cosmology (cf. [4]). To get a skyrmion, one may consider the multi-component wave function which may introduce the extra internal degrees of freedom and result in a nontrivial structure characterized by topological charges. For two-dimensional Skyrme model, skyrmions have been investigated by the method of concentration-compactness (cf. [13]). In a double condensate, skyrmions with integer topological charges have been observed by numerical simulations on two-component systems of NLSE (cf. [2] and [25]). Besides, *half-skyrmions* characterized by half-integer topological charges can also be found in the nonlinear σ model which may describe the Bose-Einstein condensate of the Schwinger bosons (cf. [18]). Here we prove rigorously the existence of *half-skyrmions* in a double condensate using spike-vortex solutions of two-component systems of NLSE.

To get spike-vortex solutions, we study soliton solutions of time-dependent two-component systems of NLSE as follows:

$$-\sqrt{-1}\,\frac{\partial \psi_j}{\partial t} = \triangle \psi_j + \sum_{i=1}^{2} \beta_{ij} |\psi_i|^2 \psi_j\,, \qquad \text{for } x \in \mathbb{R}^n\,, t > 0\,, j = 1, 2\,, \tag{1.1}$$

where the spatial dimension $n = 2$ and $\psi_j = \psi_j(x,t) \in \mathbb{C}$ for $j = 1, 2$. The system (1.1) is a standard model to describe a double condensate. Physically, ψ_j's are the corresponding complex-valued wave functions, and the coefficients $\beta_{ij} \sim -a_{ij}$ for $i, j = 1, 2$, where a_{jj}'s and $a_{12} = a_{21}$ are the intraspecies and interspecies scattering lengths, respectively. When the spatial dimension is one, i.e. $n = 1$, it is well-known that the system (1.1) is integrable, and there are many analytical and numerical results on soliton solutions of coupled nonlinear Schrödinger equations (e.g. [7], [8], [9]). Recently, from physical experiments (cf. [3]), even three-dimensional solitons have been observed in Bose-Einstein condensates. It is natural to believe that there are two-dimensional (i.e. $n = 2$) solitons in double condensates. However, when the spatial dimension is two, the system (1.1) becomes non-integrable and has only few results on two-dimensional solitons. This may lead us to study two-dimensional soliton solutions of the system (1.1) and find different types of solitons.

In the vicinity of a Feshbach resonance, scattering lengths a_{ij}'s depend sensitively on the magnitude of an externally applied magnetic field (cf. [23] and [24]), allowing the magnitude and sign of β_{ij}'s to be tuned to any value. Generically, when both β_{jj}'s are positive, the system (1.1) is of self-focusing and has bright soliton solutions on the associated two components. On the other hand, when both β_{jj}'s are negative, the system (1.1) is of self-defocusing and has dark soliton solutions on the associated two components. Here we have interest on the case that β_{11} and β_{22} have opposite signs which may result in a new type of soliton solutions called spike-vortex solutions of the system (1.1) i.e. one component has spikes and the other component has a vortex. Furthermore, we may obtain half-skyrmions by these spike-vortex solutions.

To obtain soliton solutions of the system (1.1), we set

$$\psi_j(x,t) = e^{\sqrt{-1}\,\lambda_j t} \cdot u_j(x)\,, \quad u_j \in \mathbb{C}\,,\; j = 1, 2\,. \tag{1.2}$$

Substituting (1.2) into (1.1), we may obtain a two-component system of semilinear elliptic partial differential equations given by

$$\begin{cases} \triangle u_1 - \lambda_1 u_1 + \beta_{11}\,|u_1|^2\,u_1 + \beta_{12}\,u_1\,|u_2|^2 = 0, \\ \triangle u_2 - \lambda_2 u_2 + \beta_{22}\,|u_2|^2\,u_2 + \beta_{12}|u_1|^2 u_2 = 0, \end{cases} \tag{1.3}$$

where β_{12} is the coupling constant. In [15]-[16], we studied the ground state solutions of (1.3) for the case that $\beta_{11} > 0$, $\beta_{22} > 0$ and u_j's are positive functions. Namely, we investigated the following problem:

$$\begin{cases} \triangle u - \lambda_1 u + \beta_{11} u^3 + \beta_{12} u v^2 = 0, \\ \triangle v - \lambda_2 v + \beta_{22} v^3 + \beta_{12} u^2 v = 0, \\ u, v > 0,\ \ u, v \in H^1(\mathbb{R}^n)\,. \end{cases} \tag{1.4}$$

Due to each $\lambda_j > 0$ and $\beta_{jj} > 0$, both u and v components have attractive self-interaction which may let spikes occur in these two components. One the other hand, when $n = 2$, $\lambda_j < 0$ and $\beta_{jj} < 0, j = 1, 2$, and u_j's are complex-valued solutions of (1.3), vortices may exist in both u_1 and u_2 components (cf. [12]).

In this paper, we study the case that $n = 2$, $\lambda_1, \beta_{11} > 0$, $\lambda_2, \beta_{22} < 0$, and u_1 is positive but u_2 is complex-valued function. Without loss of generality, we may assume that

$$\lambda_1 = -\lambda_2 = \beta_{11} = -\beta_{22} = 1\,.$$

Namely, we study the following system:

$$\begin{cases} \triangle u - u + u^3 + \beta u |v|^2 = 0 \quad \text{ in } \mathbb{R}^2\,, \\ \triangle v + v - |v|^2 v + \beta u^2 v = 0 \quad \text{ in } \mathbb{R}^2\,, \end{cases} \tag{1.5}$$

where $u > 0$ and $v \in \mathbb{C}$. To get skyrmions, a defining property of the skyrmion is that the atomic field approaches a constant value at spatial infinity (cf. [26]). Hence we may set boundary conditions of the system (1.5) as follows: $u(x) \to 0$ and $|v(x)| \to 1$ as $|x| \to \infty$.

As $\beta = 0$, the first equation of the system (1.5) becomes a standard nonlinear Schrödinger equation given by

$$\triangle u - u + u^3 = 0, \quad u \in H^1(\mathbb{R}^2) \tag{1.6}$$

which has a unique least-energy spike solution $w = w(r), r = |x|$ satisfying $w'(r) < 0$ for $r > 0$ and

$$w(r) = A_0 r^{-\frac{1}{2}} e^{-r}\Big[1 + O\Big(\frac{1}{r}\Big)\Big], \quad w'(r) = -A_0 r^{-\frac{1}{2}} e^{-r}\Big[1 + O\Big(\frac{1}{r}\Big)\Big]. \tag{1.7}$$

Actually, this is also a typical spike solution of nonlinear Schrödinger equations. On the other hand, as $\beta = 0$, the second equation of the system (1.5) can be written as

$$\triangle v + v - |v|^2 v = 0, \quad v = v(z) \in \mathbb{C} \quad \text{ for } z \in \mathbb{C} \cong \mathbb{R}^2\,, \tag{1.8}$$

which is of conventional Ginzburg-Landau equations (cf. [1]) having a symmetric vortex solution of degree $d \in \mathbb{Z}\backslash\{0\}$ with the following form

$$v_d(z) = S_d(r) e^{\sqrt{-1}\,d\theta}, \tag{1.9}$$

where $S_d(r)$ satisfies

$$\begin{cases} S_d''+\frac{1}{r}S_d'-\frac{d^2}{r^2}S_d+S_d-S_d^3=0, \\ S_d(0)=0,\ S_d(+\infty)=1, \end{cases} \tag{1.10}$$

and

$$S_d'(r)>0, \quad S_d(r)=1-\frac{d^2}{2r^2}+O\Big(\frac{1}{r^4}\Big) \quad \text{as } r\to+\infty\,. \tag{1.11}$$

Here we want to prove that when β increases or decreases, there exist spike-vortex solutions of the system (1.5). This may become the first paper to illustrate such solutions of two-component nonlinear Schrödinger systems.

The main purpose of this paper is to construct a spike-vortex solution (u,v) of the system (1.5) such that $u\sim w$ and $v\sim v_d$. Actually, the main difficulty of this paper is to study the interaction of typical spike and vortex solutions. Our first result shows that even a weak repulsive interaction ($\beta>0$ being small) can produce abundant bound states by solving the system (1.5). More precisely, we have

Theorem 1.1. *Let $n=2$, $d\in\mathbb{N}$ and $k\geq 2$ satisfy*

- *(i)* $k\geq 2$ *is any positive integer if* $d=1$,
- *(ii)* $2(d-1)\not\equiv 0 \mod k$ *i.e. there does not exist any integer μ such that $2(d-1)=k\mu$ if $d\geq 2$.*

Then for $\beta>0$ sufficiently small, the problem (1.5) has a solution (u_β,v_β) satisfying $u_\beta(z)>0$ for $z\in\mathbb{C}$, $v_\beta(0)=0$ with degree d, and $u_\beta(z)\to 0\,, |v_\beta(z)|\to 1$ as $|z|\to\infty$. Moreover, as $\beta\to 0+$, (u_β,v_β) has the following asymptotic form

$$\begin{cases} u_\beta(z)=\sum_{j=1}^{k} w\big(z-\xi_j^\beta\big)+O(|\beta|), \\ v_\beta(z)=S_d(r)e^{\sqrt{-1}\,d\theta}e^{\sqrt{-1}\psi_\beta(z)},\ \psi_\beta(z)=O(|\beta|)\in\mathbb{C}, \end{cases} \tag{1.12}$$

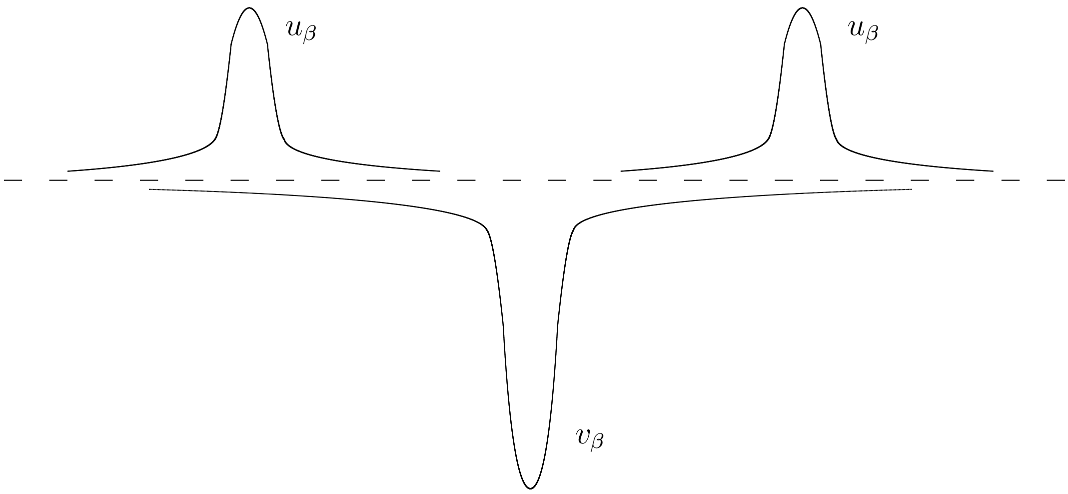
where w is the unique radial solution of (1.6), $\big\langle\xi_1^\beta,...,\xi_k^\beta\big\rangle$ forms a regular k-polygon with

$$\xi_j^\beta=l_\beta e^{\sqrt{-1}\frac{2\pi(j-1)}{k}},\ j=1,...,k, \tag{1.13}$$

and $l_\beta\to+\infty$ as $\beta\to 0+$ satisfying $l_\beta=\hat{l}_\beta+O(1)$, where $\hat{l}_\beta$ satisfies

$$\hat{l}_\beta^{\frac{5}{2}}e^{-2\hat{l}_\beta\sin\frac{\pi}{k}}=\beta. \tag{1.14}$$

A picture of (u_β,v_β) is given by



In [15]-[16], the positive sign of β may give inter-component **attraction** when the inter-component interaction is only for spikes. Conversely, the positive sign of β may contribute inter-component **repulsion** when the inter-component interaction is for spikes and a vortex. The inter-component repulsion between u and v components may balance with self-attraction in u-component so a new kind of soliton solutions called spike-vortex solutions of two-component nonlinear Schrödinger systems can be obtained in Theorem 1.1.

For $\beta < 0$, we may consider the radial solution of (1.5) given by

$$u = u(r)\,, \quad v = f(r)\, e^{\sqrt{-1}d\theta}\,, \tag{1.15}$$

satisfying

$$\begin{cases} u'' + \frac{1}{r}\, u' - u + u^3 \, + \beta f^2\, u & = 0\,, \quad \forall r > 0\,, \\ f'' + \frac{1}{r}\, f' - \frac{d^2}{r^2}\, f + f - f^3 + \beta u^2\, f & = 0\,, \quad \forall r > 0\,, \\ u'(0) = 0, u(+\infty) = 0, \\ f(0) = 0, f(+\infty) = 1\,, \end{cases} \tag{1.16}$$

where $d \in \mathbb{N}$ and (r, θ) is the polar coordinates in $\mathbb{R}^2$. Then we have the following existence theorem:

Theorem 1.2. *Assume that $n = 2$ and $\beta < 0$. Then the problem (1.5) has radially symmetric solutions of the following form:*

$$u = u(r)\,, \quad v = f(r)\, e^{\sqrt{-1}d\theta} \tag{1.17}$$

where $u(r)$ is strictly decreasing, $f(r)$ is strictly increasing and (u, f) satisfies (1.16).

Remarks:

1. It is easy to see that the solution $(u(r), f(r))$ is unique for the system (1.16) if β is small enough. It is an interesting question to study the uniqueness for general $\beta < 0$.

2. Note that the solution $(u(r), f(r)e^{\sqrt{-1}d\theta})$ is not a global minimizer for the corresponding energy functional of (1.5) since the equation of u is superlinear. It is conjectured that there

exists a least energy solution (see definitions in [16]) satisfying $E[u(r), f(r)e^{\sqrt{-1}d\theta}] \leq E[u,v]$ for any solution (u,v) of (1.5) with $\deg(v) = d$. Here the corresponding energy functional is defined as

$$E[u,v] = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + u^2) - \frac{1}{4}\int_{\mathbb{R}^2} u^4 + \frac{1}{2}\int_{\mathbb{R}^2}|\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^2}(1-|v|^2)^2 - \frac{\beta}{2}\int_{\mathbb{R}^2} u^2|v|^2. \quad (1.18)$$

To get half-skyrmions, we may define a S^2-valued map

$$\overrightarrow{n} \equiv \frac{1}{\sqrt{u^2+v_1^2+v_2^2}} \begin{pmatrix} v_1 \\ v_2 \\ u \end{pmatrix}, \quad (1.19)$$

where $(u, v_1+\sqrt{-1}\,v_2)$ is the spike-vortex solution obtained in Theorem 1.1 and 1.2. Generically, the topological charge of S^2-valued maps is defined by (cf. [6])

$$Q = \frac{1}{4\pi}\int_{\mathbb{R}^2} \overrightarrow{n} \cdot (\partial_x \overrightarrow{n} \wedge \partial_y \overrightarrow{n})\, dx\, dy\,. \quad (1.20)$$

Then we have

Theorem 1.3. *The S^2-map defined by (1.19) is of half-skyrmions with topological charge $\frac{d}{2}$.*

Throughout the rest of the paper, we assume that

$$\hat{l}_\beta - \gamma < l < \hat{l}_\beta + \gamma \quad (1.21)$$

for some suitable γ. Note that

$$\hat{l}_\beta = \frac{1}{2\sin\frac{\pi}{k}}\log\frac{1}{\beta} + c_k \log\log\frac{1}{\beta} + O(1), \quad (1.22)$$

where c_k is constant depending on k only. Besides, unless otherwise stated, the letter C will always denote various generic constants which are independent of β, especially for β sufficiently small. The constant $\alpha \in (0, \frac{1}{2})$ is a fixed small constant.

The rest of this paper is organized as follows: In Section 2, we introduce useful properties about the spike solution w and the symmetric vortex solution v_d. In Section 3, we define the approximate solutions of spike-vortex solutions and derive the associated error estimates. In Section 4, we use Liapunov-Schmidt reduction process to find spike-vortex solutions. Then we may complete the proof of Theorem 1.1 and 1.2 in Section 5 and 6, respectively. Finally, we give the proof of Theorem 1.3 in Section 7.

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2 Properties of Spikes and Vortex

We recall some properties of w and $S_d(r)e^{\sqrt{-1}\,d\theta}$, where $|d|=1$ or $|d|>1$ and $2(d-1)\not\equiv 0 \mod k$. Let

$$\begin{cases} L_1[\phi] = \triangle\phi - \phi + 3w^2\phi, \\ L_2[\psi] = \triangle\psi + \psi - S_d^2\psi - 2Re(S_d(r)e^{-\sqrt{-1}\,d\theta}\psi)S_d(r)e^{\sqrt{-1}\,d\theta}\,, \end{cases} \tag{2.1}$$

for ϕ is a real-valued function and ψ is a complex-valued function. For convenience, we may define the conjugate operator of L_2 by

$$\hat{L}_2 := e^{-\sqrt{-1}d\theta} L_2 e^{\sqrt{-1}d\theta} \tag{2.2}$$

Then by simple computations, it is easy to check that

$$\hat{L}_2[\psi_1 + \sqrt{-1}\psi_2] = \hat{L}_{2,1}[\psi_1,\psi_2] + \sqrt{-1}\,\hat{L}_{2,2}[\psi_1,\psi_2]\,, \tag{2.3}$$

for ψ_1 and ψ_2 are real-valued functions, where

$$\hat{L}_{2,1}[\psi_1,\psi_2] = \Delta\psi_1 + (1-3S_d^2)\psi_1 - \frac{d^2}{r^2}\psi_1 - \frac{2d}{r^2}\partial_\theta\psi_2\,,$$

$$\hat{L}_{2,2}[\psi_1,\psi_2] = \Delta\psi_2 + (1-S_d^2)\psi_2 - \frac{d^2}{r^2}\psi_2 + \frac{2d}{r^2}\partial_\theta\psi_1\,.$$

Set a function space

$$\Sigma = \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} \in \mathbb{R}\times\mathbb{C} \;\middle|\; \begin{array}{ll} \phi(ze^{\sqrt{-1}\,\frac{2\pi}{k}}) = \phi(z), & \phi(\bar{z}) = \phi(z), \\ \psi(ze^{\sqrt{-1}\,\frac{2\pi}{k}}) = e^{\sqrt{-1}\,\frac{2\pi}{k}}\psi(z), & \psi(\bar{z}) = \psi(z)^* \end{array} \right\}, \tag{2.4}$$

where $k\geq 2$ is an integer. Hereafter, both the over-bar and asterisk denote complex conjugate. We remark that the equation (1.5) is invariant under the following two maps

$$\begin{cases} T_1\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi(ze^{\sqrt{-1}\,\frac{2\pi}{k}}) \\ e^{-\sqrt{-1}\,\frac{2\pi}{k}}\psi(ze^{\sqrt{-1}\,\frac{2\pi}{k}}) \end{pmatrix}, \\ \\ T_2\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi(\bar{z}) \\ \psi(\bar{z})^* \end{pmatrix}. \end{cases} \tag{2.5}$$

Therefore, we may look for solutions of (1.5) in the space $\sum$. We first have

Lemma 2.1.

(1) Suppose $L_1[\phi]=0$, $\phi\in H^2(\mathbb{R}^2)$ and $\phi(\bar{z})=\phi(z)$. Then $\phi(z) = c\frac{\partial w}{\partial z_1}(z)$, where $z = z_1+\sqrt{-1}z_2$, $z_j\in\mathbb{R}$ and c is a constant.

(2) Suppose

$$L_2[\psi]=0,\ |\psi|\le C,\ \psi(z)^*=\psi(\bar z),\ \ \textit{and}\ \ \psi(ze^{\sqrt{-1}\frac{2\pi}{k}})=e^{\sqrt{-1}\frac{2\pi}{k}}\psi(z)\,, \tag{2.6}$$

where C is a positive constant. Then $\psi\equiv 0$, provided that $|d|=1$ or $|d|>1$ and $2(d-1)\not\equiv 0 \mod k$.

Proof. (1) is easy to show. See Appendix C of [20]. We only need to show (2). We firstly state the proof for the case when $|d|=1$. For simplicity, we may assume $d=1$. From [21](Theorem 3.2), we know that

$$\psi=c_0\big(\underbrace{\sqrt{-1}S(r)e^{\sqrt{-1}\theta}}_{\substack{\parallel\\ \psi_{0,0}}}\big)+\sum_{j=1}^{2}c_j\underbrace{\frac{\partial}{\partial z_j}\big(S(r)e^{\sqrt{-1}\theta}\big)}_{\substack{\parallel\\ \psi_{0,j}}}\,, \tag{2.7}$$

where c_j's are constants. It is easy to calculate that $\left(\sqrt{-1}S(r)e^{\sqrt{-1}\theta}\right)^*=-\sqrt{-1}S(r)e^{-\sqrt{-1}\theta}$, $\psi_{0,1}=\big(\frac{S(r)}{r}\big)'\frac{x^2}{r}+\frac{S(r)}{r}+\sqrt{-1}\,(\frac{S(r)}{r})'xy$, and $\psi_{0,2}=\big(\frac{S(r)}{r}\big)'\frac{xy}{r}+\sqrt{-1}\big[(\frac{S(r)}{r})'y^2+\frac{S(r)}{r}\big]$, where $x=z_1$ and $y=z_2$. Consequently, $(\psi_{0,0}(z))^*=-\psi_{0,0}(\bar z)$, $(\psi_{0,1}(z))^*=\psi_{0,1}(\bar z)$ and $(\psi_{0,2}(z))^*\neq\psi_{0,2}(\bar z)$. Moreover, due to $\psi(z)^*=\psi(\bar z)$, we have $c_0=c_2=0$. Hence we only have

$$\psi(z)=c_1\psi_{0,1}(z)\,.$$

However, it is obvious that $\psi_{0,1}$ doesn't satisfy $\psi_{0,1}\big(ze^{\sqrt{-1}\frac{2\pi}{k}}\big)=e^{\sqrt{-1}\frac{2\pi}{k}}\psi_{0,1}(z)$. Thus $c_1=0$ and $\psi\equiv 0$.

Now we give the proof for the case that $|d|>1$. From [14], the solution ψ locally may become a linear combination of $\psi_{d,0}(z)=\sqrt{-1}\,h(r)\,e^{\sqrt{-1}\,d\theta}$ and the following forms

$$\psi_{d,m}(z)=a(r)\,e^{\sqrt{-1}\,(d-m)\theta}+b(r)\,e^{\sqrt{-1}\,(d+m)\theta}\,, \tag{2.8}$$

for $m\in\mathbb{N}$, where $z=r\,e^{\sqrt{-1}\,\theta}$ and (r,θ) is the polar coordinate. Here h, a and b are real-valued functions. Actually, these forms are invariant to the operator L_2 so one may decompose the function space Σ as invariant subspaces having the forms like $\psi_{d,0}$ and $\psi_{d,m}$'s. Then the condition $\psi(z)^*=\psi(\overline{z})$ may imply $\psi_{d,0}(z)^*=\psi_{d,0}(\overline{z})$. However, since $\psi_{d,0}(z)=\sqrt{-1}\,h(r)\,e^{\sqrt{-1}\,d\theta}$, then $\psi_{d,0}(z)^*=-\psi_{d,0}(\overline{z})$ which gives $\psi_{d,0}(z)\equiv 0$. Besides, the condition $\psi(ze^{\sqrt{-1}\frac{2\pi}{k}})=e^{\sqrt{-1}\frac{2\pi}{k}}\psi(z)$ may give $\psi_{d,m}(ze^{\sqrt{-1}\frac{2\pi}{k}})=e^{\sqrt{-1}\frac{2\pi}{k}}\psi_{d,m}(z)$. Consequently,

$$a(r)\,e^{\sqrt{-1}\,(d-m)\theta}\left(e^{\sqrt{-1}\,(d-m)\frac{2\pi}{k}}-e^{\sqrt{-1}\,\frac{2\pi}{k}}\right)+b(r)\,e^{\sqrt{-1}\,(d+m)\theta}\left(e^{\sqrt{-1}\,(d+m)\frac{2\pi}{k}}-e^{\sqrt{-1}\,\frac{2\pi}{k}}\right)=0\,. \tag{2.9}$$

Hence $a\equiv 0$ or $b\equiv 0$ if $e^{\sqrt{-1}\,(d-m-1)\frac{2\pi}{k}}\neq 1$ or $e^{\sqrt{-1}\,(d+m-1)\frac{2\pi}{k}}\neq 1$. Due to $L_2[\psi_{d,m}]=0$, $a(r)$ and $b(r)$ satisfy

$$\left\{\begin{array}{ll} a''+\frac{1}{r}a'-\frac{(d-m)^2}{r^2}a+(1-S_d^2)\,a-(a+b)S_d^2 & =0\,,\\ b''+\frac{1}{r}b'-\frac{(d+m)^2}{r^2}b+(1-S_d^2)\,b-(a+b)S_d^2 & =0\,. \end{array}\right. \tag{2.10}$$

This implies $a \equiv b \equiv 0$ if $a \equiv 0$ or $b \equiv 0$. Thus $a \equiv b \equiv 0$ if $e^{\sqrt{-1}\,(d-m-1)\frac{2\pi}{k}} \neq 1$ or $e^{\sqrt{-1}\,(d+m-1)\frac{2\pi}{k}} \neq 1$. It is trivial that $e^{\sqrt{-1}\,(d-m-1)\frac{2\pi}{k}} \neq 1$ or $e^{\sqrt{-1}\,(d+m-1)\frac{2\pi}{k}} \neq 1$ if $2(d-1) \not\equiv 0 \mod k$. Therefore $a \equiv b \equiv 0$ i.e. $\psi_{d,m} \equiv 0$ if $2(d-1) \not\equiv 0 \mod k$, and we may complete the proof of Lemma 2.1. $\square$

To study the properties of L_2 (or $\hat{L}_2$), we introduce some Sobolev spaces. Let $\alpha \in (0, \frac{1}{2})$. We introduce Hilbert spaces X_α and Y_α as follows:

$$X_\alpha = \left\{ u = u_1 + \sqrt{-1}\,u_2 \in L^2_{loc}(\mathbb{R}^2; \mathbb{C}) \middle| \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})(|u_1| + |u_2|)^2 < +\infty \right\}$$

equipped with inner product $(u, v)_{X_\alpha} = \displaystyle\int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})(u_1 v_1 + u_2 v_2) dx$, and

$$Y_\alpha = \left\{ v = v_1 + \sqrt{-1}\,v_2 \in W^{2,2}_{loc}(\mathbb{R}^2; \mathbb{C}) \middle| \int_{\mathbb{R}^2} |\triangle v|^2 (1 + |x|^{2+\alpha}) dx + \int_{\mathbb{R}^2} \frac{|v|^2}{1 + |x|^{2+\alpha}} dx < +\infty \right\}$$

equipped with inner product $(u, v)_{Y_\alpha} = (\triangle u, \triangle v)_{X_\alpha} + \displaystyle\int_{\mathbb{R}^2} \frac{u \cdot v}{1 + |x|^{2+\alpha}} dx$, respectively. Thanks to the inequality

$$\int_{\mathbb{R}^2} |h| \leq \left(\int_{\mathbb{R}^2} \frac{1}{(1 + |z|)^{2+\alpha}} \right)^{\frac{1}{2}} \|h\|_{X_\alpha}.$$

Besides, we see that X_α has a compact embedding to $L^1(\mathbb{R}^2)$. Originally, these spaces are real-valued function spaces introduced in Chae and Imanuvilor (cf. [5]). Here we generalize X_α, Y_α as complex-valued function spaces, and regard the operator L_2 from the space Y_α to the space X_α. We list some properties of X_α and Y_α, whose proofs are exactly the same as in [5].

Lemma 2.2.

(1) Let $v \in Y_\alpha$ be a harmonic function. Then $v =$constant.

(2) $\forall v \in Y_\alpha$, we have $|v(z)| \leq C_1 ||v||_{Y_\alpha} (\ln(1 + |z|) + 1)$, $\forall z \in \mathbb{R}^2$.

(3) The image of L_2 (or $\hat{L}_2$) is closed in $X_\alpha \cap \Sigma_0$, where $\Sigma_0 \equiv \{\psi = \psi(z) \in \mathbb{C} : (0, \psi) \in \Sigma\}$.

Now we study the invertibility of L_2 (or $\hat{L}_2$) on the space $Y_\alpha \cap \Sigma_0$.

Lemma 2.3. *For $\alpha \in (0, \frac{1}{2})$. Then operator L_2 (or $\hat{L}_2$) from the space $Y_\alpha \cap \Sigma_0$ onto the space $X_\alpha \cap \Sigma_0$ is invertible. Furthermore,*

$$\|\psi\|_{Y_\alpha} \leq C \|L_2[\psi]\|_{X_\alpha}, \quad \|\psi\|_{Y_\alpha} \leq C \|\hat{L}_2[\psi]\|_{X_\alpha} \tag{2.11}$$

Proof. It suffices to consider the invertibility of $\hat{L}_2$. Then the invertibility of L_2 follows from a trivial transformation. To show the invertibility of $\hat{L}_2$, we claim that $(Im(\hat{L}_2))^\perp = \{0\}$. Suppose $\psi \in (Im(\hat{L}_2))^\perp$. Then it is easy to check that $\hat{L}_2[\psi] = 0, \psi \in X_\alpha \cap \Sigma_0$. By Lemma 2.1, we just need to show that ψ is bounded. To this end, we note that since $\psi \in \Sigma_0$, we have

$\psi(0) = e^{\sqrt{-1}\frac{2\pi}{k}}\psi(0)$ and $k \geq 2$ which may give $\psi(0) = 0$. Let $\phi = \psi\, e^{\sqrt{-1}\, d\theta}$. Then $\phi \in X_\alpha$ and $L_2[\phi] = 0$. Hence $\phi \in Y_\alpha$ and

$$\triangle\phi(z) = h(z)/\left(1 + |z|^{2+\alpha}\right)^{1/2}, \forall z \in \mathbb{R}^2,$$

where $h \in L^2(\mathbb{R}^2)$. Consequently, by Lemma 2.2 and Riesz representation formula, we may obtain

$$|\phi(z)| \leq C \log(1 + |z|), \quad \forall z \in \mathbb{R}^2,$$

and

$$|\partial_\theta\, \phi(z)| \leq C \log(1 + |z|), \quad \forall z \in \mathbb{R}^2.$$

Thus

$$|\psi(z)|\,,\ |\partial_\theta \psi(z)| \leq C \log(1 + |z|), \quad \forall z \in \mathbb{R}^2. \tag{2.12}$$

The equation $\hat{L}_2[\psi] = 0, \psi = \psi_1 + \sqrt{-1}\psi_2$ can be written as

$$\Delta\psi_1 + (1 - 3S_d^2)\psi_1 - \frac{d^2}{r^2}\psi_1 - \frac{2d}{r^2}\partial_\theta\psi_2 = 0, \tag{2.13}$$

$$\Delta\psi_2 + (1 - S_d^2)\psi_2 - \frac{d^2}{r^2}\psi_2 + \frac{2d}{r^2}\partial_\theta\psi_1 = 0. \tag{2.14}$$

From (2.12) and (2.13), ψ_1 satisfies

$$|\Delta\psi_1 + (1 - 3S_d^2)\psi_1 - \frac{d^2}{r^2}\psi_1| \leq \frac{C}{r^2}\log r \quad \text{for } r = |z| > 1.$$

Hence by comparison principle,

$$|\psi_1(z)| \leq \frac{C}{1 + |z|}. \tag{2.15}$$

Here we have used the fact that $S_d(r) \sim 1$ as $r \to \infty$. Similarly, we may have

$$|\partial_\theta\psi_1(z)| \leq \frac{C}{1 + |z|}. \tag{2.16}$$

It remains to show that ψ_2 is bounded. Now we can use (2.14) and (2.16) to get

$$|\Delta\psi_2 + (1 - S_d^2)\psi_2 - \frac{d^2}{r^2}\psi_2| \leq \frac{C}{r^2(1 + r)} \quad \text{for } r = |z| > 1.$$

In fact, ψ_2 satisfies

$$-\Delta\psi_2 = f(z), \text{ where } f(z) = (1 - S_d^2 - \frac{d^2}{r^2})\psi_2 + \frac{2d}{r^2}\partial_\theta\psi_1$$

Since $\psi \in \Sigma_0$, we deduce that $f(z) \in X_\alpha \cap \Sigma_0$ and $\int_{\mathbb{R}^2} f(z) = 0$. Since $\psi_2(0) = 0$, we have

$$\psi_2(z) = \psi_2(0) + \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|\tau|}{|z - \tau|} f(\tau) d\tau = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|\tau||z|}{|z - \tau|} f(\tau) d\tau \tag{2.17}$$

from which we obtain that

$$|\psi_2(z)| \leq C\|f\|_{X_\alpha} < +\infty \quad \text{for } z \in \mathbb{R}^2. \tag{2.18}$$

Therefore by (2.15) and (2.18), we may complete the proof of Lemma 2.3. □

To study the vortex solutions, we perform the following key transformation:

$$v=v_d(z)e^{\sqrt{-1}\psi(z)}\,,\quad v_d(z)=S_d(r)\,e^{\sqrt{-1}\,d\theta}\,, \tag{2.19}$$

where $\psi=\psi_1+\sqrt{-1}\psi_2$ and $\psi_j\in\mathbb{R}\,,j=1,2\,.$ (Here we have *assumed* that the vortex occurs at the center.) Now we define

$$S_1[u,\psi]:=\Delta u-u+u^3+\beta uS_d^2e^{-2\psi_2}\,,$$

$$S_2[u,\psi]:=\Delta\psi+\frac{2\nabla v_d}{v_d}\cdot\nabla\psi-\sqrt{-1}S_d^2(1-e^{-2\psi_2})+\sqrt{-1}|\nabla\psi|^2-\sqrt{-1}\beta u^2\,.$$

We also write

$$S_2[u,\psi]=S_2[u,0]+\tilde{L}_2[\psi]+N[u,\psi] \tag{2.20}$$

where

$$\begin{aligned}\tilde{L}_2[\psi] &= \Delta\psi+\frac{2\nabla v_d}{v_d}\nabla\psi-2\sqrt{-1}S_d^2\psi_2 &(2.21)\\ &= \left[\Delta\psi_1+2\left(\frac{S_d'(|z|)}{S_d(|z|)}\frac{z}{|z|}\right)\nabla\psi_1-d\nabla\theta\cdot\nabla\psi_2\right]\\ &\quad+\sqrt{-1}\left[\Delta\psi_2-2S_d^2\psi_2+2\left(\frac{S_d'}{S_d}\frac{z}{|z|}\right)\nabla\psi_2+d\nabla\theta\cdot\nabla\psi_1\right]\,, &(2.22)\end{aligned}$$

$$N[u,\psi]=\sqrt{-1}|\nabla\psi|^2-\sqrt{-1}S_d^2(1-e^{-2\psi_2}-2\psi_2)\,,$$

where $\nabla\theta=\frac{1}{r}(-\sin\theta,\cos\theta)$. Then it is easy to see that solving (1.5) is equivalent to solving

$$S_1[u,\psi]=0,\quad S_2[u,\psi]=0\,. \tag{2.23}$$

Let $\psi=\psi_1+\sqrt{-1}\psi_2, h=h_1+\sqrt{-1}h_2$. We may define two norms as follows:

$$\|\psi\|_*=\sup_{z\in\mathbb{R}^2}[|\psi_1|+(1+|z|)|\nabla\psi_1|+(1+|z|)^{1+\alpha}|\psi_2|+(1+|z|)^{1+\alpha}|\nabla\psi_2|]\,, \tag{2.24}$$

and

$$\|h\|_{**}=\sup_{z\in\mathbb{R}^2}[(1+|z|)^{2+\alpha}(|h_1|+|h_2|)]\,.$$

Then we may show the following key lemma

Lemma 2.4. *For any $h\in X_\alpha\cap\Sigma_0$ with $\|h\|_{**}<+\infty$, there exists a unique $\tilde{\psi}\in Y_\alpha\cap\Sigma_0$ such that*

$$\tilde{L}_2[\psi]=h \tag{2.25}$$

Furthermore, we have

$$\|\psi\|_*\le C\|h\|_{**} \tag{2.26}$$

Proof: Let $\hat{h} = -\sqrt{-1}\, v_d h$ and $\hat{\phi} = -\sqrt{-1}\, v_d \psi$. Then (2.25) is equivalent to

$$L_2[\hat{\phi}] = \hat{h} \tag{2.27}$$

where $\hat{h}$ satisfies $\hat{h} \in \Sigma_0$ and

$$\|\hat{h}\|_{**} \leq \|h\|_{**} < +\infty\,. \tag{2.28}$$

Consequently, (2.28) implies that $\int_{\mathbb{R}^2} |\hat{h}|^2(1+|z|^{2+\alpha}) < +\infty$ and hence $\hat{h} \in X_\alpha \cap \Sigma_0$. By Lemma 2.3, (2.27) has a unique solution ϕ. Furthermore, as for (2.17), we obtain

$$\hat{\phi}(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{|\tau||z|}{|z-\tau|} \hat{f}(\tau) d\tau \tag{2.29}$$

where $|\hat{f}(z)|\,(1+|z|)^{2+\alpha} \leq C\|h\|_{**}$. From (2.29), we may deduce that $|\hat{\phi}|+(1+|z|)|\nabla\hat{\phi}| \leq C\|h\|_{**}$ which implies that $|\psi| + (1+|z|)|\nabla\psi| \leq C\|h\|_{**}$ i.e. $|\psi_j| + (1+|z|)|\nabla\psi_j| \leq C\|h\|_{**}\,, j = 1, 2\,.$

To obtain better estimates for ψ_2, we may use the equation for ψ_2 and (2.22). Then

$$\Delta\psi_2 - 2S_d^2\psi_2 + O\left(\frac{1}{|z|}|\nabla\psi|\right) = O\left(\|h\|_{**}\,(1+|z|)^{-2-\alpha}\right)\,,$$

which gives

$$-\frac{C\,\|h\|_{**}}{(1+|z|)^2} \leq -\Delta\psi_2 + 2S_d^2\psi_2 \leq \frac{C\,\|h\|_{**}}{(1+|z|)^2}\,.$$

Hence we may use a barrier and elliptic estimates to get

$$|\psi_2| + |\nabla\psi_2| \leq \frac{C\,\|h\|_{**}}{(1+|z|)^{1+\alpha}}\,.$$

Here we have used the fact that $S_d(r) \sim 1$ as $r \to \infty$. Therefore we may complete the proof of Lemma 2.4. $\square$

3 Approximate Solutions and Error Estimates

In this section, we introduce some approximate solutions and derive some useful estimates. Let

$$S[u,\psi] = \begin{pmatrix} S_1[u,\psi] \\ S_2[u,\psi] \end{pmatrix}.$$

Let

$$\xi_j = le^{\sqrt{-1}\frac{2\pi}{k}(j-1)},\ j = 1, ..., k,\ w_j(z) := w(z - \xi_j),$$
$$u_l(z) = \sum_{j=1}^{k} w_j(z),$$

Then we have

Lemma 3.1. *For l large enough, we have*

$$\left\|S_1[u_l,0]\right\|_{L^2(\mathbb{R}^2)} + \left\|S_2[u_l,0]\right\|_{**} \le C\left(|\beta| l^{2+\alpha} + e^{-2l\sin\frac{\pi}{k}}\right). \tag{3.1}$$

Proof. It is easy to check that

$$\begin{aligned} S_1[u_l,0] =& \triangle u_l - u_l + u_l^3 + \beta u_l |v_d|^2 \\ =& \Big(\sum_j w_j(z)\Big)^3 - \sum_j w_j^3 + \beta \sum_j w_j S_d^2 \\ =& O\Big(\sum_{i\neq j} w_i^2 w_j + |\beta| \sum_j |w_j|\Big). \end{aligned}$$

Due to

$$\int_{\mathbb{R}^2} w_i^4 w_j^2 \le C e^{-2|\xi_i-\xi_j|} \le C e^{-4l\sin\frac{\pi}{k}}, \quad \forall i \neq j, \tag{3.2}$$

we may obtain

$$\left\|S_1[u_l,0]\right\|_{L^2} \le \left[e^{-2l\sin\frac{\pi}{k}} + |\beta|\right]. \tag{3.3}$$

Here we have used the fact that ξ_j's are vertices of regular k-polygon with side length $2l\sin\frac{\pi}{k}$. On the other hand, we have

$$|S_2[u_l,0]| = |\beta| u_l^2 \le C \sum_{j=1}^k |\beta| e^{-2|z-\xi_j|}$$

and so by (1.21)

$$\|S_2[u_l,0]\|_{**} \le C|\beta| \sup_{z\in\mathbb{R}^2} \left(\sum_{j=1}^k |z|^{2+\alpha} e^{-2|z-\xi_j|}\right) \le C|\beta| \sum_{j=1}^k |\xi_j|^{2+\alpha} \le C|\beta||l|^{2+\alpha}. \tag{3.4}$$

Therefore by (3.3) and (3.4), we may obtain (3.1) and complete the proof of Lemma 3.1. □

4 Liapunov-Schmidt reduction process

Let $\widetilde{X} := (L^2(\mathbb{R}^2) \times X_\alpha) \cap \Sigma$, $\widetilde{Y} := (H^2(\mathbb{R}^2) \times Y_\alpha) \cap \Sigma$ and $L := \begin{pmatrix} \widetilde{L}_1 \\ \widetilde{L}_2 \end{pmatrix} : \widetilde{Y} \to \widetilde{X}$, where $\widetilde{L}_1[\phi] \equiv \triangle\phi - \phi + 3u_l^2\,\phi$ and $\widetilde{L}_2$ is defined by (2.21). To solve (2.23), we first consider the following linear problem: Given $f \in L^2(\mathbb{R}^2) \cap \Sigma_1$, find (ϕ, c) such that

$$\begin{cases} \widetilde{L}_1 \phi = f + c\dfrac{\partial u_l}{\partial l}, \quad \phi \in H^2(\mathbb{R}^2) \cap \Sigma_1, \\ \displaystyle\int_{\mathbb{R}^2} \phi \frac{\partial u_l}{\partial l} = 0, \end{cases} \tag{4.1}$$

where $\Sigma_1 \equiv \{\phi = \phi(z) \in \mathbb{R} : (\phi, 0) \in \Sigma\}$.

Lemma 4.1. *For each $f \in L^2(\mathbb{R}^2) \cap \Sigma_1$, there exists a unique pair (ϕ, c) satisfying (4.1) such that*

$$\|\phi\|_{H^2} \le C\|f\|_{L^2}. \tag{4.2}$$

Proof. The existence and uniqueness may follow from Fredholm's alternatives. Now we prove (4.2) by contradiction. Suppose (4.2) is not true. Then there exist $\beta_n,\ l_n \in \mathbb{R}$, $\phi_n \in H^2(\mathbb{R}^2) \cap \Sigma_1$, $f_n \in L^2(\mathbb{R}^2) \cap \Sigma_1$ and c_n such that $\beta_n \to 0,\ l_n \to +\infty$,

$$\|f_n\|_{L^2} \to 0, \quad \|\phi_n\|_{H^2} = 1\,, \tag{4.3}$$

as $n \to \infty$,

$$\widetilde{L}_1\phi_n = f_n + c_n\frac{\partial u_l}{\partial l}\,, \tag{4.4}$$

and

$$\int_{\mathbb{R}^2}\phi_n\frac{\partial u_l}{\partial l} = 0\,. \tag{4.5}$$

Let $\widetilde{\phi}_n(z) := \phi_n(z+\xi_1)$. Then by (4.4), $\widetilde{\phi}_n$ satisfies

$$\begin{aligned}
&\triangle\widetilde{\phi}_n(z) - \widetilde{\phi}_n(z) + 3w^2(z)\widetilde{\phi}_n(z)\\
&+3\left[2\sum_{j=2}^{k} w(z)w(z+\xi_1-\xi_j) + \left(\sum_{j=2}^{k} w(z+\xi_1-\xi_j)\right)^2\right]\widetilde{\phi}_n(z)\\
&= f_n(z+\xi_1) + c_n\left[-\tfrac{\partial w}{\partial z_1}(z) + \sum_{j=2}^{k}\frac{\partial w_j}{\partial l}(z+\xi_1)\right].
\end{aligned} \tag{4.6}$$

We may multiply (4.4) by $\dfrac{\partial u_l}{\partial l}$ and integrate over the whole space $\mathbb{R}^2$. Then by (4.3), it is obvious that

$$\begin{aligned}
c_n\int_{\mathbb{R}^2}\left(\frac{\partial u_l}{\partial l}\right)^2 &= \int_{\mathbb{R}^2}(\triangle\phi_n - \phi_n + 3u_l^2\phi_n)\frac{\partial u_l}{\partial l} - \int_{\mathbb{R}^2} f_n\frac{\partial u_l}{\partial l}\\
&= \int_{\mathbb{R}^2}\left(\triangle\frac{\partial u_l}{\partial l} - \frac{\partial u_l}{\partial l} + 3u_l^2\frac{\partial u_l}{\partial l}\right)\phi_n - \int_{\mathbb{R}^2} f_n\frac{\partial u_l}{\partial l}\\
&\xrightarrow{n\to+\infty} 0.
\end{aligned}$$

So $c_n \to 0$ as $n \to +\infty$. Hence by (4.6), as $n \to \infty$, $\widetilde{\phi}_n \to \widetilde{\phi}_0$ which satisfies $\triangle\widetilde{\phi}_0 - \widetilde{\phi}_0 + 3w^2\widetilde{\phi}_0 = 0$ in $\mathbb{R}^2$. Thus by [20], $\widetilde{\phi}_0 = \displaystyle\sum_{j=1}^{2} a_j\frac{\partial w}{\partial z_j}$ for some constants a_1 and a_2. Moreover, due to $\widetilde{\phi}_n(\bar{z}) = \widetilde{\phi}_n(z)$, we have $\widetilde{\phi}_0(\bar{z}) = \widetilde{\phi}_0(z)$. Consequently, $a_2 = 0$ and $\widetilde{\phi}_0 = a_1\dfrac{\partial w}{\partial z_1}$. On the other hand, by

(4.5),

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^2} \phi_n \frac{\partial u_l}{\partial l} \\
&= \int_{\mathbb{R}^2} \widetilde{\phi}_n(z) \left[-\frac{\partial w}{\partial z_1}(z) + \sum_{j=2}^{k} \frac{\partial w_j}{\partial l}(z+\xi_1) \right] \\
&\xrightarrow{n\to+\infty} -a_1 \int_{\mathbb{R}^2} \left(\frac{\partial w}{\partial z_1} \right)^2 .
\end{aligned}
$$

Therefore $a_1 = 0$, $\widetilde{\phi}_0 = 0$, and $\widetilde{\phi}_n \to 0$ in $L^2_{loc}(\mathbb{R}^2)$. Then we have $3u_l^2 \widetilde{\phi}_n \to 0$ in L^2 and hence $\|\phi_n\|_{H^2} \to 0$ which may give a contradiction and complete the proof. □

From Lemma 2.4 and 4.1, we may obtain that

Lemma 4.2. *For* $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \widetilde{X}$, *there exists a unique* $\left(\begin{pmatrix} \phi \\ \psi \end{pmatrix}, c \right) \in \widetilde{Y} \times \mathbb{R}$, *such that*

$$
\widetilde{L} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + c \begin{pmatrix} \frac{\partial u_l}{\partial l} \\ 0 \end{pmatrix}, \quad \int_{\mathbb{R}^2} \phi \frac{\partial u_l}{\partial l} = 0. \tag{4.7}
$$

Moreover, we have $\|\phi\|_{H^2} + \|\psi\|_* \le C\left(\|f_1\|_{L^2} + \|f_2\|_{**}\right)$.

We may denote A as the inverse operator for Lemma 4.2, i.e. $A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$. Finally, we have

Lemma 4.3. *For l large satisfying (1.21), there exists a unique* $\begin{pmatrix} \phi_l \\ \psi_l \end{pmatrix} \in \widetilde{Y}$ *such that*

$$
S[u_l + \phi_l, \psi_l] = c(l) \begin{pmatrix} \frac{\partial u_l}{\partial l} \\ 0 \end{pmatrix}, \quad \int_{\mathbb{R}^2} \phi_l \frac{\partial u_l}{\partial l} = 0. \tag{4.8}
$$

Furthermore,

$$
\|\phi_l\|_{H^2} + \|\psi_l\|_* \le C \left(e^{-2l \sin \frac{\pi}{k}} + |\beta| l^{2+\alpha} \right).
$$

Proof. This may follow from standard contraction mapping principle. We choose $(\phi, \psi) \in B$, where

$$
B = \left\{ (\phi, \psi) \in \widetilde{Y} : \|\phi\|_{H^2} + \|\psi\|_* \le C\left(e^{-2l \sin \frac{\pi}{k}} + |\beta| l^{2+\alpha} \right) \right\},
$$

and then expand

$$
\begin{aligned}
S_1[u_l + \phi, \psi] &= S_1[u_l, 0] + \tilde{L}_1[\phi] + N_1[\phi, \psi], \\
S_2[u_l + \phi, \psi] &= S_2[u_l, 0] + \tilde{L}_2[\psi] + N_2[\phi, \psi],
\end{aligned}
$$

where

$$N_1[\phi, \psi] = 3u_l\phi^2 + \phi^3 + \beta\left[\phi\, S_d^2\, e^{-2\psi_2} + u_l\, S_d^2\left(e^{-2\psi_2} - 1\right)\right], \tag{4.9}$$

and

$$N_2[\phi, \psi] = -\sqrt{-1}\,\beta(2u_l\phi + \phi^2) + \sqrt{-1}\,|\nabla\psi|^2 - \sqrt{-1}S_d^2(1 - e^{-2\psi_2} - 2\psi_2)\,. \tag{4.10}$$

By (1.21) and $(\phi, \psi) \in B$, we have

$$\|\phi\|_{H^2} + \|\psi\|_* \leq |\beta|^{1-\sigma}, \quad \sigma \in \left(0\,, \frac{1}{100}\right),$$

as $\beta > 0$ sufficiently small i.e. l sufficiently large. Moreover, by (4.9), (4.10) and the norms defined at (2.24), we see that

$$\|N_1\|_{L^2(\mathbb{R}^2)} \leq C|\beta|\|\phi\|_{L^2} + \|\phi\|_{L^2}^2 + |\beta|\,, \tag{4.11}$$

and

$$\|N_2\|_{**} \leq C\|\psi\|_*^2 + C|\beta|\,. \tag{4.12}$$

Now we can write (4.8) as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = A\begin{pmatrix} S_1[u_l, 0] + N_1[\phi, \psi] \\ S_2[u_l, 0] + N_2[\phi, \psi] \end{pmatrix}.$$

Then as for the proof of Proposition 1 in [16], we may use a contraction mapping argument to obtain the desired result. Here we also need to use (4.11), (4.12), Lemma 3.1, 4.1 and 4.2. $\square$

5 Finding zero of $c(l)$

To prove Theorem 1.1, it is enough to find a zero of $c(l)$ in (4.8). We multiply the first equation of (4.8) by $\frac{\partial u_l}{\partial l}$ and integrate it over the whole space $\mathbb{R}^2$. Then we obtain

$$c(l)\int_{\mathbb{R}^2}\left(\frac{\partial u_l}{\partial l}\right)^2 = \int_{\mathbb{R}^2}\left[\triangle(u_l + \phi_l) - (u_l + \phi_l) + (u_l + \phi_l)^3\right]\frac{\partial u_l}{\partial l} \tag{5.1}$$

$$+\,\beta\int_{\mathbb{R}^2}(u_l + \phi_l)|v_d + \psi_l|^2\frac{\partial u_l}{\partial l}$$

$$=: I_1 + I_2, \tag{5.2}$$

where

$$\begin{aligned}
I_2 =&\beta\int_{\mathbb{R}^2} u_l|v_d|^2\frac{\partial u_l}{\partial l}+\beta\int_{\mathbb{R}^2} O\left(|\phi_l|+|u_l+\phi_l||\psi_l|\right)\left|\frac{\partial u_l}{\partial l}\right|\\
=&\beta\int_{\mathbb{R}^2} u_l S_d^2\frac{\partial u_l}{\partial l}+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)\\
=&\beta\int_{\mathbb{R}^2} u_l\left(1-\frac{d^2}{2r^2}+O\Big(\frac{1}{r^4}\Big)\right)^2\frac{\partial u_l}{\partial l}+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)\\
=&\beta\int_{\mathbb{R}^2} u_l\frac{\partial u_l}{\partial l}+O\Big(|\beta|l^{-4}+|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)-\beta\int_{\mathbb{R}^2} u_l\frac{d^2}{r^2}\frac{\partial u_l}{\partial l}.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\mathbb{R}^2} u_l\frac{\partial u_l}{\partial l}=&\int_{\mathbb{R}^2}\sum_{j=1}^k w_j\sum_{i\neq j}\frac{\partial w_i}{\partial l}\\
=&O\Big(e^{-2l\sin\frac{\pi}{k}}\Big),
\end{aligned}\tag{5.3}$$

$$\begin{aligned}
-\beta\int_{\mathbb{R}^2} u_l\frac{d^2}{r^2}\frac{\partial u_l}{\partial l}=&\beta\left[\int_{\mathbb{R}^2} w\frac{d^2}{|z+\xi_1|^2}\frac{\partial w}{\partial z_1}+O\Big(e^{-2l\sin\frac{\pi}{k}}\Big)\right]\\
=&-c_2\,\beta l^{-3}\int_{\mathbb{R}^2} w^2+O\Big(|\beta|l^{-5}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big),
\end{aligned}\tag{5.4}$$

where c_2 is a positive constant independent of β and l. So

$$I_2=-c_2\,\beta l^{-3}\int_{\mathbb{R}^2} w^2+O\Big(|\beta|l^{-4}+|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big).\tag{5.5}$$

For I_1, we have

$$\begin{aligned}
I_1=&\int_{\mathbb{R}^2}\left(\triangle u_l-u_l+u_l^3\right)\frac{\partial u_l}{\partial l}+\int_{\mathbb{R}^2}\left(\triangle\phi_l-\phi_l+3u_l^2\phi_l\right)\frac{\partial u_l}{\partial l}+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)\\
=&\int_{\mathbb{R}^2}\left[\Big(\sum_j w_j\Big)^3-\sum_j w_j^3\right]\frac{\partial u_l}{\partial l}+\int_{\mathbb{R}^2}\left[\triangle\frac{\partial u_l}{\partial l}-\frac{\partial u_l}{\partial l}+3u_l^2\frac{\partial u_l}{\partial l}\right]\phi_l+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)\\
=&6k\int_{\mathbb{R}^2} w_1^2 w_2\Big(-\frac{\partial w_1}{\partial z_1}\Big)+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)\\
=&-6k\int_{\mathbb{R}^2} w^2(|z-\xi_1|)w'(|z-\xi_1|)w(|z-\xi_2|)\frac{z_1-l}{|z-\xi_1|}\,dz+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)\\
=&-6k\int_{\mathbb{R}^2} w^2(|z|)w'(|z|)\,w(|z+\xi_1-\xi_2|)\frac{z_1}{|z|}\,dz\;+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)\\
=&c_0\cdot w(|\xi_1-\xi_2|)+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big)\\
=&c_0\left(2\sin\frac{\pi}{k}\right)^{-1/2}\cdot l^{-1/2}e^{-2l\sin\frac{\pi}{k}}+O\Big(|\beta|^2 l^{2+\alpha}+|\beta|e^{-2l\sin\frac{\pi}{k}}\Big),
\end{aligned}\tag{5.6}$$

where c_0 is a positive constant. Here we have used the fact that (1.7) holds.

Combining (5.5) and (5.6), we see that

$$c(l) \approx c_1 l^{-1/2} e^{-2l\sin\frac{\pi}{k}} - c_2 \beta l^{-3} + O\big(\beta l^{-5}\big).$$

Moreover, we may choose $\hat{l}_\beta$ such that

$$\hat{l}_\beta^{-1/2} e^{-2\hat{l}_\beta \sin\frac{\pi}{k}} = \beta \hat{l}_\beta^{-3}. \tag{5.7}$$

Then

$$\hat{l}_\beta = \frac{1}{2\sin\frac{\pi}{k}} \log\frac{1}{\beta} + c_3 \log\log\frac{1}{\beta} + c_4.$$

Now we want to choose l such that $l \in (\hat{l}_\beta - \gamma, \hat{l}_\beta + \gamma)$ and $c(l) = 0$ for some suitable $\gamma > 0$. It is remarkable that

$$c\big(\hat{l}_\beta - \gamma\big) \approx c_1 \big(\hat{l}_\beta - \gamma\big)^{-1/2} e^{-2\hat{l}_\beta \sin\frac{\pi}{k} + 2\gamma \sin\frac{\pi}{k}} - c_2 \beta \big(\hat{l}_\beta - \gamma\big)^{-3} + O\bigg(\beta\Big(\log\frac{1}{\beta}\Big)^{-5}\bigg) > 0\,,$$

if γ is large enough. Similarly, $c\big(\hat{l}_\beta + \gamma\big) < 0$ if γ is large enough. Since $c(l)$ is continuous in l, then by the mean-value theorem, there exists $l_\beta \in (\hat{l}_\beta - \gamma, \hat{l}_\beta + \gamma)$ such that $c(l_\beta) = 0$. Consequently, the function $\begin{pmatrix} u_{l_\beta} + \phi_{l_\beta} \\ v_d e^{\sqrt{-1}\psi_{l_\beta}} \end{pmatrix} =: \begin{pmatrix} u_\beta \\ v_\beta \end{pmatrix}$ is a solution of (1.5). Furthermore, it is easy to check that (u_β, v_β) satisfies all the properties of Theorem 1.1. Therefore we may complete the proof of Theorem 1.1.

6 Proof of Theorem 1.2

We first consider problem (1.16) on a ball B_R:

$$\begin{cases} \Delta u - u + u^3 + \beta u S^2 = 0, u = u(r), r < R \\ \Delta S - \frac{d^2}{r^2} S + S(1 - S^2) + \beta u^2 S = 0, S = S(r), r < R \\ S(0) = 0, S(R) = 1, u(R) = 0, u > 0, 0 < S < 1 \end{cases} \tag{6.1}$$

Our idea is to find a solution of (6.1), and then let $R \to +\infty$. To this end, we consider the associated energy functional

$$\begin{aligned} E_R[u, S] \;\; &= \;\; \frac{1}{2}\int_{B_R} (|\nabla u|^2 + u^2) + \frac{1}{2}\int_{B_R} (|\nabla S|^2 + \frac{d^2}{r^2} S^2) \\ &\quad + \frac{1}{4}\int_{B_R} (1 - S^2)^2 - \frac{\beta}{2}\int_{B_R} S^2 u^2 - \frac{1}{4}\int_{B_R} u^4\,, \end{aligned} \tag{6.2}$$

for $u \in H_0^1(B_R)$ and $S \in I_R \equiv \{S \in H^1(B_R) : S(z) = S(|z|), S(0) = 0, S(R) = 1\}$.

Let the Nehari manifold be

$$\mathcal{N} = \left\{ (u, S) \in H_0^1(B_R) \times I_R, u \geq 0, u \not\equiv 0 : \int_{B_R} (|\nabla u|^2 + u^2) = \int_{B_R} (u^4 + \beta u^2 S^2) \right\}.$$

Then we consider the following energy minimization problem

$$c_R = \inf_{(u,S)\in\mathcal{N}} E_R[u,S]\,, \tag{6.3}$$

and we have

Lemma 6.1. *If $\beta < 0$, then c_R is obtained by some radially symmetric function (u_R, S_R). Furthermore, $u_R^{'}(r) < 0$ and $S_R^{'}(r) > 0$ for $r > 0$.*

Proof: We follow the proof of (1) Theorem 3.3 of [15]. To this end, we define another energy functional

$$\begin{aligned} E_R'[u,S] &= \frac{1}{4}\int_{B_R}(|\nabla u|^2+u^2)+\frac{1}{2}\int_{B_R}(|\nabla S|^2+\frac{d^2}{r^2}S^2) \\ &\quad +\frac{1}{4}\int_{B_R}(1-S^2)^2-\frac{\beta}{4}\int_{B_R}S^2u^2 \end{aligned} \tag{6.4}$$

and another solution manifold

$$\mathcal{N}' = \left\{(u,S)\in H_0^1(B_R)\times I_R, u\geq 0, u\not\equiv 0 : \int_{B_R}(|\nabla u|^2+u^2)\leq \int_{B_R}(u^4+\beta u^2S^2)\right\}.$$

We consider another minimization problem:

$$c_R' = \inf_{(u,S)\in\mathcal{N}'} E_R'[u,S]. \tag{6.5}$$

Certainly, we have

$$c_R^{'} \leq c_R. \tag{6.6}$$

Let (u_n, S_n) be a minimizing sequence of c_R' on $\mathcal{N}'$. Replacing S_n by $\min(S_n, 1)$, we may assume that $S_n \leq 1$. We may denote u_n^* and S_n^* as the Schwartz symmetrization of u_n and S_n, respectively. Then $(1-S_n)^* = 1-S_n^*$. By Theorem 3.4 of [10],

$$\int_{B_R}(1-S_n^2)u_n^2 \leq \int_{B_R}(1-S_n^2)^*(u_n^*)^2 \tag{6.7}$$

and hence due to $\beta < 0$,

$$-\beta\int_{B_R}(u_n^*)^2(S_n^*)^2 \leq -\beta\int_{B_R}u_n^2S_n^2 \tag{6.8}$$

On the other hand, we also have

$$\int_{B_R}(\frac{1}{2}(u_n^*)^2+\frac{1}{2}\frac{d^2}{r^2}(S_n^*)^2-\frac{1}{4}(u_n^*)^4) = \int_{B_R}(\frac{1}{2}u_n^2+\frac{1}{2}\frac{d^2}{r^2}S_n^2-\frac{1}{4}u_n^4)\,,$$

$$\int_{B_R}(|\nabla u_n^*|^2+|\nabla S_n^*|^2)\leq \int_{B_R}(|\nabla u_n|^2+|\nabla S_n|^2).$$

Hence we obtain

$$E_R^{'}[u_n^*, S_n^*] \leq E_R^{'}[u_n, S_n]$$

and

$$\int_{B_R} (|\nabla u_n^*|^2 + (u_n^*)^2) \leq \int_{B_R} ((u_n^*)^4 + \beta (u_n^*)^2 (S_n^*)^2). \tag{6.9}$$

Thus we may replace (u_n, S_n) by its symmetrization (u_n^*, S_n^*). Since $S_n \leq 1$ and $H^1(B_R)$ is a compact embedding to $L^4(B_R)$, we see that $(u_n S_n) \to (u_R, S_R)$ weakly in $H^1(B_R)$ and strongly in $L^4(B_R)$, where $(u_R, S_R) \in \mathcal{N}'$ attains $c_R^{'}$. So $c_R^{'}$ is attained. If $(u_R, S_R) \in (\mathcal{N}')^\circ$−the interior of $\mathcal{N}'$, then (u_R, S_R) is a local minimizer of E_R' and hence we have

$$\Delta u_R - u_R + \beta u_R S_R = 0, \quad \forall 0 < r < R, \quad \text{ and } \; u_R(R) = 0\,,$$

which implies $u_R \equiv 0$ since $\beta < 0$. This is impossible since from (6.9) and Sobolev embedding, we infer that $\int_{B_R} u_R^4 \geq C > 0$. Therefore $(u_R, S_R) \in \partial(\mathcal{N}') = \mathcal{N}$ and hence

$$c_R \leq E_R[u_R, S_R] = E_R^{'}[u_R, S_R] = c_R^{'}. \tag{6.10}$$

Combining (6.6) and (6.10), we conclude that c_R is attained by (u_R, S_R). Then we have the following equality:

$$G_R[u_R, S_R] = \int_{B_R} (|\nabla u_R|^2 + u_R^2 - \beta u_R^2 S_R^2 - u_R^4) = 0\,. \tag{6.11}$$

Hence there exists a Lagrange multiplier λ_R such that

$$\nabla E_R + \lambda_R \nabla G_R = 0\,. \tag{6.12}$$

Acting (6.12) on $(u_R, 0)$, we may obtain

$$\int_{B_R} (|\nabla u_R|^2 + u_R^2 - \beta u_R S_R^2 - u_R^4) + 2\lambda_R \int_{B_R} (|\nabla u_R|^2 + u_R^2 - \beta u_R^2 S_R^2 - 2u_R^4) = 0\,,$$

and hence by (6.11),

$$-\lambda_R \int_{B_R} u_R^4 = 0\,, \quad \text{ i.e. } \quad \lambda_R = 0\,.$$

Therefore, we may complete the proof of Lemma 6.1.

□

Theorem 1.2 is proved by the following lemma

LEMMA 6.2. *As $R \to +\infty$, $(u_R, S_R) \to (u_\infty, S_\infty)$ and (u_∞, S_∞) is a solution of (1.16).*

Proof: Since $S_R \leq 1, S_R(0) = 0$, we first show that u_R is uniformly bounded, independent of $R > 1$. Actually, it is sufficient to show that $\int_{B_R} (|\nabla u_R|^2 + u_R^2) \leq C$, where C is a positive constant independent of $R > 1$. Let

$$GL_{B_R}[S] = \int_{B_R} \left(\frac{1}{2} |\nabla S|^2 + \frac{d^2}{2r^2} S^2 + \frac{1}{4} (1 - S^2)^2 \right) \,.$$

It is remarkable that GL_{B_R} may come from the conventional Ginzburg-Landau functional.

From [21], we may set $\bar{S}_R$ as the unique minimizer of GL_R. Then for any $u \in H_0^1(B_R)$, there exists t_R such that $(\sqrt{t_R}u, \bar{S}_R) \in \mathcal{N}$, where t_R is simply given by

$$t_R = \frac{\int_{B_R}(|\nabla u|^2 + u^2 - \beta \bar{S}_R^2 u^2)}{\int_{B_R} u^4} . \tag{6.13}$$

Thus by Lemma 6.1 and $0 \le \bar{S}_R < 1$,

$$\begin{aligned} c_R \le E_R[\sqrt{t_R}u, \bar{S}_R] &= GL_{B_R}(\bar{S}_R) + \frac{1}{4}\left[\frac{\int_{B_R}|\nabla u|^2 + u^2 - \beta \bar{S}_R^2 u^2}{(\int_{B_R} u^4)^{\frac{1}{2}}}\right]^2 \\ &\le GL_{B_R}(\bar{S}_R) + \left[\frac{\int_{B_R}|\nabla u|^2 + (1-\beta)u^2}{(\int_{B_R} u^4)^{\frac{1}{2}}}\right]^2 , \end{aligned}$$

for all $u \in H_0^1(B_R)$. Consequently, due to $\beta < 0$,

$$c_R \le GL_{B_R}(\bar{S}_R) + C_0 , \tag{6.14}$$

where C_0 is a positive constant independent of $R > 1$. Here we have used the fact that $\lim\limits_{R\to\infty} \inf\limits_{u\in H_0^1(B_R)} \frac{\int_{B_R}|\nabla u|^2 + (1-\beta)u^2}{(\int_{B_R} u^4)^{\frac{1}{2}}} < \infty$.

By standard theory of Ginzburg-Landau equation (cf. [1]), we have

$$GL_{B_R}[S_R] \ge GL_{B_R}[\bar{S}_R] . \tag{6.15}$$

Combining (6.14) and (6.15), we see that

$$\frac{1}{2}\int_{B_R}(|\nabla u_R|^2 + u_R^2 - \beta S_R^2 u_R^2) - \frac{1}{4}\int_{B_R} u_R^4 \le C_3$$

and hence by the equation of u_R,

$$\int_{B_R}(|\nabla u_R|^2 + u_R^2) \le C , \tag{6.16}$$

from which standard elliptic regularity theory gives that $u_R \le C$. Thus we may obtain that $(u_R, S_R) \to (u_\infty, S_\infty)$ which solves $\Delta u_\infty - u_\infty + u_\infty^3 + \beta u_\infty S_\infty^2 = 0$. Note that $u_R(0) \ge 1$ and hence $u_\infty \not\equiv 0$. By the Maximum Principle, $u_\infty > 0$. Similarly, $0 < S_\infty(r) < 1$ for $r > 0$. Therefore we may complete the proof of Theorem 1.2.

□

7 Proof of Theorem 1.3

In this section, we want to construct S^2-valued map to get half-skyrmions by spike-vortex solutions obtained in Theorem 1.1 and 1.2. For simplicity, we firstly use spike-vortex solutions in Theorem 1.2 to find half-skyrmions. Let (u, v) be the radial spike-vortex solution in Theorem 1.2. We may define a S^2-valued map by

$$\overrightarrow{n} \equiv \frac{1}{\sqrt{u^2+v_1^2+v_2^2}} \begin{pmatrix} v_1 \\ v_2 \\ u \end{pmatrix} = \begin{pmatrix} \cos(\phi(r))\,\cos(d\theta) \\ \cos(\phi(r))\,\sin(d\theta) \\ \sin(\phi(r)) \end{pmatrix}, \tag{7.1}$$

where $v = v_1 + \sqrt{-1}\, v_2$,

$$\cos(\phi(r)) = \frac{f(r)}{\sqrt{u^2+f^2}}, \tag{7.2}$$

and

$$\sin(\phi(r)) = \frac{u(r)}{\sqrt{u^2+f^2}}. \tag{7.3}$$

Since both u and f are positive everywhere, the function ϕ is well-defined and single-valued. The map $\overrightarrow{n}$ can be decomposed into

$$\overrightarrow{n} = \cos(\phi(r)) \begin{pmatrix} \cos(d\theta) \\ \sin(d\theta) \\ 0 \end{pmatrix} + \sin(\phi(r)) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then it is easy to check that

$$\begin{aligned} \int_{\mathbb{R}^2} \overrightarrow{n} \cdot (\partial_x \overrightarrow{n} \wedge \partial_y \overrightarrow{n})\, dx\, dy &= \int_{\mathbb{R}^2} -\frac{d}{r}\phi'(r)\cos(\phi(r))\, dx\, dy \\ &= -2\pi\, d \sin(\phi(r))|_{r=0}^{+\infty} = 2\pi\, d, \end{aligned}$$

i.e. the topological charge

$$Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} \overrightarrow{n} \cdot (\partial_x \overrightarrow{n} \wedge \partial_y \overrightarrow{n})\, dx\, dy = \frac{d}{2}. \tag{7.4}$$

Here we have used (7.3) and the fact that $u(0) > 0$, $u(+\infty) = f(0) = 0$ and $f(+\infty) = 1$.

For the spike-vortex solution (u, v) in Theorem 1.1, since $\beta > 0$ sufficiently small, the associated map $\overrightarrow{n}$ has the following form

$$\overrightarrow{n} \equiv \frac{1}{\sqrt{u^2+v_1^2+v_2^2}} \begin{pmatrix} v_1 \\ v_2 \\ u \end{pmatrix} = \begin{pmatrix} \cos(\phi)\,\cos(d\psi) \\ \cos(\phi)\,\sin(d\psi) \\ \sin(\phi) \end{pmatrix}, \tag{7.5}$$

where $\phi = \phi(r, \theta)$ and $\psi = \psi(r, \theta)$ satisfying

$$\cos(\phi(r,\theta)) = \frac{|v|(r,\theta)}{\sqrt{u^2+|v|^2}}, \tag{7.6}$$

$$\sin(\phi(r,\theta)) = \frac{u(r,\theta)}{\sqrt{u^2+|v|^2}}\,, \tag{7.7}$$

and $\psi = \theta + h$, where h is a single-valued regular function satisfying $h = O(\beta)$ as $\beta \to 0+$. Here both u and $|v|$ may not have radial symmetry. Due to $\beta > 0$, we may apply the standard maximum principle on the first equation of the system (1.5) i.e. the equation of u. Then the solution u is positive everywhere so the function ϕ is well-defined and single-valued.

Now we want to calculate the topological charge Q as for (7.4). By (7.5), it is easy to check that

$$\overrightarrow{n}\cdot(\partial_x\overrightarrow{n}\wedge\partial_y\overrightarrow{n}) = -\frac{d}{r}\,\phi_r\cos\phi + \frac{d}{r}\,(\phi_\theta h_r - \phi_r h_\theta)\cos\phi\,.$$

Hence by (7.6), (7.7) and using integration by part, we may obtain

$$Q = \frac{1}{4\pi}\int_{\mathbb{R}^2}\overrightarrow{n}\cdot(\partial_x\overrightarrow{n}\wedge\partial_y\overrightarrow{n}) = \frac{d}{2}\,.$$

Here we have used the fact that $u(0) > 0$, $u(\infty) = 0$, $v(0) = 0$ and $|v(\infty)| = 1$. Therefore we may complete the proof of Theorem 1.3.

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