STABILITY OF NEW SPIKE SOLUTIONS IN A SUPPLY CHAIN MODEL

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Abstract. We study a supply chain model which is based on Schnakenberg type kinetics. This model is realistic in a predator-prey context if cooperation of predators is prevalent in the system and if there is practically an unlimited pool of prey, i.e. the predators will not feel the impact of the limited amount of prey due to its large quantity. The system also serves as a model for a sequence of irreversible autocatalytic reactions in a container which is in contact with a well-stirred reservoir. It is an extension of the Schnakenberg model suggested in [12, 28] for which there is only one prey and one predator. In this supply chain model there is one predator feeding on the prey and a second predator feeding on the first predator. This means that the first predator plays a hybrid role: it acts as both predator and prey. It is assumed that both the prey and the second predator diffuse much faster than the first predator.

We construct new single spike solutions on an interval for which the profile of the first predator (middle component) is that of a spike. The profile of the prey and the second predator only varies on a large spatial scale which comes from the faster diffusion of these components. They both interact with the middle component in a novel way. It is shown that there exist two different single spike solutions if the feedrates are small enough, a large-amplitude and a small-amplitude spike.

We study the stability properties of this solution in terms of the system parameters. We use a rigorous analysis for the linearized operator around single spike solutions based on nonlocal eigenvalue problems. The following result is established: The large-amplitude spike solution is stable if the time-relaxation constants for both predators are small enough. The small-amplitude spike solution is always unstable.

Dedicated to the memory of Professor Klaus Kirchgässner, with deep gratitude.

Pattern Formation, Supply Chain Model, Predator-Prey Model, Reaction-Diffusion System, Spiky Solutions, Stability.
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1. Introduction

We consider a supply chain model with diffusion which is an extension of the Schnakenberg model [12, 28] to three components. This generalized model considers the interactions of

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two predators and one prey, whereas the Schnakenberg model only takes into account the interaction of one predator and one prey.

This model is realistic in a predator-prey context if cooperation of predators occurs in the system. For background on predator-prey models we refer to [22]. We also refer to the recent work in which the stability of food chains was analyzed [18]. It has recently been confirmed empirically by considering large amounts of worldwide data that in cities typical per capita quantities like wages, GDP or number of educational institutions but also crime, traffic congestions or certain diseases grow at a superlinear rate with population size [2]. This underpins the assumption in our model that the interaction terms are superlinear and emphasizes its relevance and validity as a simple socio-economic model for the resource allocation in cities.

The system also serves as a model for a sequence of irreversible autocatalytic reactions in a container which is in contact with a well-stirred reservoir. Similar systems have been suggested to model chains of chemical reactions, see e.g. Chapter 8 of [31] and the references therein.

In the supply chain model under current investigation there is one predator feeding on the prey and a second predator feeding on the first predator. This means that the first predator plays a hybrid role: it acts both as predator and prey. It is assumed that the prey diffuses and the second predator both diffuse much faster than the first predator.

The system can be written as follows:

$$\begin{cases} \tau \dfrac{\partial S}{\partial t} = D_1 \Delta S + 1 - \dfrac{a_1}{\epsilon} S u_1^2, \quad x \in \Omega,\, t > 0, \\ \dfrac{\partial u_1}{\partial t} = \epsilon^2 \Delta u_1 - u_1 + S u_1^2 - a_2 u_1 u_2^2, \quad x \in \Omega,\, t > 0, \\ \tau_1 \dfrac{\partial u_2}{\partial t} = D_2 \Delta u_2 - u_2 + \dfrac{1}{\epsilon} u_1 u_2^2, \quad x \in \Omega,\, t > 0, \end{cases} \tag{1.1}$$

where S and u_i denote the concentrations of the prey (food source) and the two predators, respectively. Here $0 < \epsilon^2 \ll 1$ and $0 < D_1$, $0 < D_2$ are three positive diffusion constants. The constants a_1, a_2 (positive) for the feed rates and τ, τ_1 (nonnegative) for the time relaxation constants will be treated as parameters and their choices will distinguish between stability and instability of steady-state solutions.

We choose as domain the interval $\Omega = (-1, 1)$ and consider Neumann boundary conditions

$$\frac{dS}{dx}(-1,0) = \frac{dS}{dx}(1,0) = 0, \quad \frac{du_1}{dx}(-1,0) = \frac{du_1}{dx}(1,0) = 0, \quad \frac{du_2}{dx}(-1,0) = \frac{du_2}{dx}(1,0) = 0. \tag{1.2}$$

We first prove the existence of single spike solutions in an interval for which the profile of the first predator is that of a spike. The prey and second predator have a profile on the $O(1)$ scale. It is shown that such patterns exist if the feed rates are small enough. We will see that there are spike solutions with a large and with a small amplitude.

We study the stability properties of this solution in terms of the system parameters. We use a rigorous analysis for the linearized operator around single spike solutions based on nonlocal eigenvalue problems. The following result is established: The large-amplitude middle component spike solution is stable if the time-relaxation constants for both predators are small enough, the first feedrate is much larger than the second and the diffusivity of the prey is much smaller than the diffusivity of the second predator.

These results are generalizations of similar statements for the Schnakenberg model. Let us briefly recall some previous results in this subject area: In [15, 32] the existence and stability of spiky patterns on bounded intervals is established. In [43] similar results are shown for two-dimensional domains. In [1] it is shown how the degeneracy of the Turing bifurcation can be lifted using spatially varying diffusion coefficients. In [23, 24, 25] spikes are considered rigorously for the shadow system.

For a closely related system, the Gray-Scott model introduced in [13, 14], some of the results are the following. In [4, 5, 6, 7] the existence and stability of spike patterns on the real line is proved. In [16, 17] different regimes for the Gray-Scott systems are considered and the existence and stability of spike patterns in an interval is shown. In [26, 27] a skeleton structure and separators for the Gray-Scott model are established.

Other "large" reaction diffusion systems (more than two components) with concentrated patterns include the hypercycle of Eigen and Schuster [8, 9, 10, 11, 37, 39], and Meinhardt and Gierer's model of mutual exclusion and segmentation [21, 44].

In a previous study [46] on the same model we considered a completely different scaling: The diffusivity of the middle component is small and the diffusivity of the third component is very small. Thus we have a small scale (much smaller than order 1) and a very small scale (much smaller than the small scale). Existence and stability of a new type of spiky pattern has been established which consists of a spike on the very small scale for the third component and two parts of spikes pasted together (continuous but with a jump in the derivative) on the small scale for the second component.

The structure of this paper is as follows:

In Section 2 we state and explain the main theorems on existence and stability.

In Sections 3 and 4, we will prove the main existence result, Theorem 2.1. In Section 3, we compute the amplitudes of the spikes. In Section 4, we give a rigorous existence proof.

In Sections 5 and 6, we will prove the main stability result, Theorem 2.2. In Section 5, we derive a nonlocal eigenvalue problem (NLEP) and determine the stability of the $O(1)$ eigenvalues. In Section 6, we study the stability of the $o(1)$ eigenvalues.

Throughout this paper, the letter C will denote various generic constants which are independent of ϵ, for ϵ sufficiently small. The notation $A \sim B$ means that $\lim_{\epsilon\to 0} \frac{A}{B} = 1$ and $A = O(B)$ is defined as $|A| \le C|B|$ for some $C > 0$.

2. Main Results: Existence and stability of a single spike solution

We now state the main results of this paper on existence and stability. We first construct stationary spike solutions to (1.1), i.e. spike solutions to the system

$$\begin{cases} D_1\Delta S + 1 - \dfrac{a_1}{\epsilon} S u_1^2 = 0, \quad x \in \Omega,\, t > 0, \\ \epsilon^2 \Delta u_1 - u_1 + S u_1^2 - a_2 u_1 u_2^2 = 0, \quad x \in \Omega,\, t > 0, \\ D_2 \Delta u_2 - u_2 + \dfrac{1}{\epsilon} u_1 u_2^2 = 0, \quad x \in \Omega,\, t > 0, \end{cases} \tag{2.1}$$

with the Neumann boundary conditions given in (1.2).

We will construct solutions of (2.1) which are even:

$$S = S(|x|) \in H_N^2(\Omega),$$
$$u_1 = u_1(|x|) \in H_N^2(\Omega_\epsilon),$$
$$u_2 = u_2(|x|) \in H_N^2(\Omega),$$

where

$$H_N^2(\Omega) = \left\{ v \in H^2(\Omega) \,:\, v'(-1) = v'(1) = 0 \right\},$$
$$\Omega_\epsilon = \left(-\frac{1}{\epsilon}, \frac{1}{\epsilon} \right),$$
$$H_N^2(\Omega_\epsilon) = \left\{ v \in H^2(\Omega_\epsilon) \,:\, v'\left(-\frac{1}{\epsilon} \right) = v'\left(\frac{1}{\epsilon} \right) = 0 \right\}.$$

Before stating the main results, we introduce some necessary notations and assumptions. Let w be the unique solution of the problem

$$\begin{cases} w_{yy} - w + w^2 = 0, \quad w > 0 \text{ in } \mathbb{R}, \\ w(0) = \max_{y\in\mathbb{R}} w(y), \quad w(y) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{2.2}$$

The ODE problem (2.2) can be solved explicitly and w can be written as

$$w(y) = \frac{3}{2\cosh^2 \frac{y}{2}}. \tag{2.3}$$

We now state the main existence result.

Theorem 2.1. *Assume that*

$$D_1 = const., \quad \epsilon \ll 1, \quad D_2 = const. \tag{2.4}$$

and

$$a_1^2 a_2 < \frac{|\Omega|^2}{4} G_{D_2}(0,0) - \delta_0. \tag{2.5}$$

(Expressed more precisely, (2.4) means that ϵ is small enough; (2.5) means the following: there are positive numbers δ_0 and ϵ_0 such that (2.5) is valid for all ϵ with $0 < \epsilon < \epsilon_0$.)

Then problem (2.1) admits two "single-spike" solutions $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ and $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$ with the following properties:

(1) all components are even functions.

(2)

$$S_\epsilon(x) = c_1^\epsilon G_{D_1}(x,0) + O(\epsilon), \tag{2.6}$$

$$u_{1,\epsilon}(x) = \xi_\epsilon w\left(\frac{|x|\sqrt{1+\alpha_\epsilon}}{\epsilon}\right)\chi(|x|) + O(\epsilon), \tag{2.7}$$

$$u_{2,\epsilon}(x) = c_2^\epsilon G_{D_2}(x,0) + O(\epsilon), \tag{2.8}$$

where w is the unique solution of (2.2), χ is a smooth cutoff function whose properties are stated in (3.2),

$$(\xi_\epsilon^l)^2 = \frac{|\Omega|^2 + \sqrt{|\Omega|^4 - 4a_1^2 a_2 |\Omega|^2 G_{D_2}^{-2}(0,0)}}{72a_1^2} + O(\epsilon), \tag{2.9}$$

$$(\xi_\epsilon^s)^2 = \frac{|\Omega|^2 + \sqrt{|\Omega|^4 - 4a_1^2 a_2 |\Omega|^2 G_{D_2}^{-2}(0,0)}}{72a_1^2} + O(\epsilon), \tag{2.10}$$

$$c_1^\epsilon = \frac{1+\alpha_\epsilon}{\xi_\epsilon G_{D_1}(0,0)} + O(\epsilon), \quad c_2^\epsilon = \frac{\sqrt{1+\alpha_\epsilon}}{6\xi_\epsilon G_{D_2}^2(0,0)} + O(\epsilon), \tag{2.11}$$

where α_ϵ is given by (3.5).

(3) If ϵ is small enough and

$$a_1^2 a_2 > \frac{|\Omega|^2}{4} G_{D_2}(0,0) + \delta_0.$$

for some $\delta_0 > 0$ independent of ϵ (in the same sense as in (2.5)) then there are no single-spike solutions which satisfy (1) – (2).

Remark. *We choose to keep the factor $|\Omega|$ in the estimate (2.5) although of course in our scaling we have $|\Omega| = 2$.*

Theorem 2.1 will be proved in Sections 3 and 4.

The second main goal of this paper is to study the stability properties of the single-spike solution constructed in Theorem 2.1. We now state our main result on stability.

Theorem 2.2. *Assume that (2.4) and (2.5) are satisfied.*

Let $(S_\epsilon, u_{1,\epsilon}, u_{2,\epsilon})$ be one of the single-spike solutions constructed in Theorem 2.1.

Suppose that $0 \le \tau < \tau_0$, and $0 \le \tau_1 < \tau_{1,0}$, where $\tau_0 > 0$ and $\tau_{1,0} > 0$ are suitable constants which may be chosen independently of ϵ.

Then we have the following:

(1) (Stability) The solution $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$ is linearly stable. There is a small eigenvalue which is given in (6.21).

(2) (Instability) The solution $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ is linearly unstable.

This result can be interpreted as follows: To have this type of spiky solution, the feed rates for both a_1 and a_2, in particular their combination $a_1^2 a_2$ must be small enough. Otherwise the food source S and the hybrid u_1 will not be able to sustain u_1 and u_2, respectively. Instead, among others, one of the following three behaviors can happen:

(i) The predator u_2 dies out and a spike for the Schnakenberg model remains which involves only the components S and u_1 with $u_2 = 0$ (equivalent to $\alpha = 0$ in the previous result) which has been described in [15].

(ii) The component u_2 dies out and u_1, S will both approach positive constants. It can easily be seen that

$$S = \frac{a_1}{\epsilon}, \quad u_1 = \frac{\epsilon}{a_1}.$$

(iii) The components approach the positive homogeneous steady state

$$S = \frac{\epsilon}{a_1 u_1^2}, \quad u_1^2 - \frac{\epsilon}{a_1} u_1 + a_2 \epsilon^2 = 0, \quad u_2 = \frac{\epsilon}{u_1}.$$

We do not rigorously study the dynamics of this model. Instead we analyze the stability or instability of the steady states in combination with numerical simulations of the dynamics.

Figure 1 shows the spatial profiles of the steady states S, u_1, u_2. These have been reached in the simulations as long-time limits of the time-dependent problem.

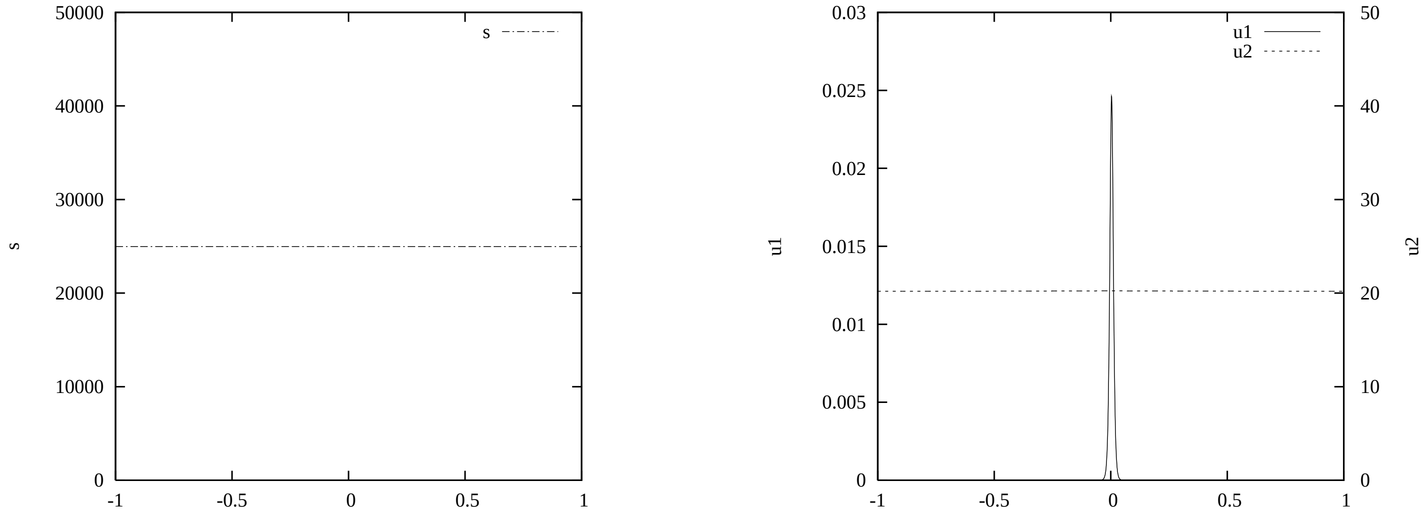

Figure 1. The spatial profiles of the steady states S_ϵ, $u_{1,\epsilon}$, $u_{2,\epsilon}$.

To elucidate these issues, we will rigorously derive the existence result Theorem 2.1 in Sections 3 and 4. The stability result Theorem 2.2 will be proved in Sections 5 and 6.

3. Existence I: Formal computation of the amplitudes

In this section and the next, we will show the existence of spike solutions to (2.1) and prove Theorem 2.1. We begin by a formal approach and give a rigorous proof in the next section.

Proof of Theorem 2.1: We will show the existence of spike solutions to (2.1) for which (2.7), (2.8) are valid. More precisely, we choose the approximate solution as follows:

$$u_{1,\epsilon}(x) = \xi_\epsilon \, w\left(\frac{|x|\sqrt{1+\alpha_\epsilon}}{\epsilon}\right) \chi(|x|) \tag{3.1}$$

for some positive constants ξ_ϵ and α_ϵ. (From now on we will drop the superscripts ϵ where doing so will not cause confusion.) Further, S_ϵ is determined by the Green's functions G_{D_1} which is defined by

$$D_1 \Delta G_{D_1} - \delta = 0, \quad G'_{D_1}(-1) = G'_{D_1}(1) = 0$$

and $u_{2,\epsilon}$ is given by the Green's function G_{D_2} which solves

$$D_2 \Delta G_{D_2} - G_{D_2} - \delta = 0, \quad G'_{D_2}(-1) = G'_{D_2}(1) = 0.$$

Here

$$\chi \in C_0^\infty(-1,1), \quad \chi = 1 \text{ for } |x| \le \frac{5}{8}, \quad \chi = 0 \text{ for } |x| \ge \frac{3}{4} \tag{3.2}$$

is a smooth cutoff function. We set

$$y = \frac{x}{\epsilon}, \quad i = 1$$

and consider the limit

$$\epsilon \to 0.$$

For the rest of this section, we drop the subscript ϵ if this does not cause confusion.

Substituting (2.7) into the second equation of (2.1) and using (2.2), we get considering the y scale:

$$\alpha_\epsilon = a_2 u_2^2(0), \tag{3.3}$$

$$\xi_\epsilon = \frac{\alpha_\epsilon + 1}{S(0)} \tag{3.4}$$

since $w\left(y\sqrt{1+\alpha}\right)$ satisfies

$$w_{yy} - (1+\alpha)w + (1+\alpha)w^2 = 0.$$

Substituting (2.7) into the third equation of (2.1) and using (2.2), we get

$$u_2(x) = G_{D_2}(x,0)u_2^2(0)\xi \int w(y)\,dy \,\frac{1}{\sqrt{1+\alpha}} + O(\epsilon)$$

which implies

$$u_2(0) = \frac{\sqrt{1+\alpha}}{G_{D_2}(0,0)\xi \int w(y)\,dy} + O(\epsilon),$$

$$u_2(x) = \frac{G_{D_2}(x,0)\sqrt{1+\alpha}}{G_{D_2}^2(0,0)\xi \int w(y)\,dy} + O(\epsilon).$$

Next we will derive two conditions, by substituting (2.7), (2.8) with (3.4) in (2.1). Then we will solve these two conditions to determine α and ξ.

Integrating the first equation in (2.1), using the Neumann boundary condition and balancing the last two terms, we get the first condition

$$|\Omega| = a_1 S(0)\xi^2 \int_{\mathbb{R}} w^2(y)\,dy \,\frac{1}{\sqrt{1+\alpha}} + O(\epsilon).$$

From (3.3), we compute

$$\alpha = \frac{a_2(1+\alpha)}{\xi^2 G_{D_2}^2(0,0)(\int_{\mathbb{R}} w(y)\,dy)^2} + O(\epsilon).$$

Summarizing these results, (α, ξ) solve the system

$$\alpha = \frac{a_2}{\xi^2 G_{D_2}^2(0,0)(\int_{\mathbb{R}} w(y)\,dy)^2 - a_2} + O(\epsilon), \tag{3.5}$$

$$|\Omega| = a_1 \xi \int_{\mathbb{R}} w^2(y)\, dy \sqrt{1+\alpha} + O(\epsilon). \tag{3.6}$$

Using

$$\int_{\mathbb{R}} w(y)^2\, dy = \int_{\mathbb{R}} w(y)\, dy = 6,$$

the system (3.5), (3.6) can be rewritten as a quadratic equation in ξ^2

$$36^2 a_1^2 G_{D_2}^2 \xi^4 - 36 G_{D_2}^2 \xi^2 |\Omega|^2 + a_2 |\Omega|^2 = O(\epsilon)$$

which has the two solutions

$$\xi^2 = \frac{|\Omega|^2 \pm \sqrt{|\Omega|^4 - 4a_1^2 a_2 |\Omega|^2 G_{D_2}^{-2}}}{72 a_1^2}$$

under the condition

$$a_1^2 a_2 < \frac{|\Omega|^2}{4} G_{D_2}^2.$$

The last condition states that, the rest being equal, the combination $a_2^2 a_2$ must be small enough.

This implies that under the condition

$$a_1^2\, a_2 < \frac{|\Omega|^2}{4} G_{D_2}^2 - \delta_0,$$

there are two solutions for ξ which satisfy

$$0 < \xi^s < \frac{|\Omega|^2}{72 a_1^2} < \xi^l.$$

If

$$a_1^2\, a_2 > \frac{|\Omega|^2}{4} G_{D_2}^2 + \delta_0,$$

there are no such solutions.

4. Existence II: Rigorous proofs

We linearize (2.1) around the approximate spike solution introduced in (3.1):

$$\tilde{S}_\epsilon - \tilde{S}_\epsilon(0) = -(G_{D_1}(x,0) - G_{D_1}(0,0)) a_1 \tilde{S}_\epsilon(0) 6 \xi_\epsilon^2 + O(\epsilon),$$

$$\tilde{u}_{1,\epsilon}(x) = \xi_\epsilon\, w\left(\frac{|x|\sqrt{1+\alpha_\epsilon}}{\epsilon}\right) \chi(|x|) + O(\epsilon),$$

$$\tilde{u}_{2,\epsilon} = G_{D_2}(x,0) 6 \xi_\epsilon \tilde{u}_{2,\epsilon}^2(0) + O(\epsilon),$$

where the amplitude $\xi_\epsilon = \xi + O(\epsilon)$ has been computed in the previous section to leading order and χ has been introduced in (3.2). Note that $\tilde{S}_\epsilon$ and $\tilde{u}_{2,\epsilon}$ each solve a partial differential equation exactly which depends on $\tilde{u}_{1,\epsilon}$ only. Therefore we denote $\tilde{S}_\epsilon = T_1[\tilde{u}_{1,\epsilon}]$ and $\tilde{u}_{2,\epsilon} = T_2[\tilde{u}_{1,\epsilon}]$, respectively.

We first determine how good the approximate spike solution solves the system by computing the error up to which it is a solution.

By definition, the first and third equations of (2.1) are solved exactly.

The second equation of (2.1) at $(\tilde{S}_\epsilon,\, \tilde{u}_{1,\epsilon},\, \tilde{u}_{2,\epsilon}) = (T_1[\tilde{u}_{1,\epsilon}],\, \tilde{u}_{1,\epsilon},\, T_2[\tilde{u}_{1,\epsilon}])$ is calculated as follows:

$$\begin{aligned}
&(\tilde{u}_{1,\epsilon})'' - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon \tilde{u}_{1,\epsilon}^2 - a_2 \tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}^2\\
&= \tilde{u}_{1,\epsilon}'' - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon(0)\tilde{u}_{1,\epsilon}^2 - a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}^2(0)\\
&\quad +[\tilde{S}_\epsilon - \tilde{S}_\epsilon(0)]\tilde{u}_{1,\epsilon}^2\\
&\quad -a_2\tilde{u}_{1,\epsilon}2(\tilde{u}_{2,\epsilon} - \tilde{u}_{2,\epsilon}(0))\tilde{u}_{2,\epsilon}(0) + O(\epsilon^2)\\
&=: E_1 + E_2 + E_3 + O(\epsilon^2)
\end{aligned}$$

in $L^2(\Omega_\epsilon)$. We compute

$$E_1 = 0$$

by the definition of ξ_ϵ. Computing $\tilde{S}_\epsilon(x)$ using the Green's function G_{D_1} introduced in the appendix, we derive the following estimate:

$$\begin{aligned}
E_2 &= [\tilde{S}_\epsilon(\epsilon y) - \tilde{S}_\epsilon(0)]\tilde{u}_{1,\epsilon}^2(\epsilon y)\\
&= \tilde{u}_{1,\epsilon}^2(\epsilon y) a_1 \int_{-1/\epsilon}^{1/\epsilon} [G_{D_1}(\epsilon y, \epsilon z) - G_D(0,\epsilon z)]\tilde{S}_\epsilon(z)\tilde{u}_{1,\epsilon}^2(z)\, dz\, (1+O(\epsilon))\\
&= a_1 \frac{\tilde{u}_{1,\epsilon}^2(\epsilon y)}{\tilde{S}_\epsilon(0)}\epsilon(1+\alpha_\epsilon)\int_{\mathbb{R}} \left(\frac{1}{2D_1}|y-z| - \frac{1}{2D_1}|z|\right) w^2(z\sqrt{1+\alpha_\epsilon})\, dz\, (1+O(\epsilon|y|))\\
&\quad + a_1(1+\alpha_\epsilon)^{3/2}\frac{\tilde{u}_{1,\epsilon}^2(\epsilon y)}{\tilde{S}_\epsilon(0)}\epsilon^2 y^2 \nabla^2 H_D(0,0) 6\,(1+O(\epsilon|y|))\\
&= O(\epsilon|y|)\tilde{u}_{1,\epsilon}^2 = O(\epsilon) \quad \text{ in } L^2(\Omega_\epsilon).
\end{aligned}$$

Note that $\nabla H_{D_1}(0,0) = 0$ by symmetry. (See the computation of H_{D_1} in the appendix).

Similarly, we compute

$$\begin{aligned}
E_3 &= -a_2\tilde{u}_{1,\epsilon}(\epsilon y)2(\tilde{u}_{2,\epsilon}(\epsilon y) - \tilde{u}_{2,\epsilon}(0))\tilde{u}_{2,\epsilon}(0)\\
&= -2a_2\tilde{u}_{1,\epsilon}(\epsilon y)\tilde{u}_{2,\epsilon}^3(0)\int_{-1/\epsilon}^{1/\epsilon}[G_{D_2}(\epsilon y, \epsilon z) - G_D(0,\epsilon z)]\tilde{u}_{1,\epsilon}(\epsilon z)\, dz\, (1+O(\epsilon))\\
&= -2\alpha_\epsilon(1+\alpha_\epsilon)\tilde{u}_{1,\epsilon}(\epsilon y)\frac{\tilde{u}_{2,\epsilon}(0)}{\tilde{S}_\epsilon(0)}\int_{\mathbb{R}}\left(\frac{1}{2D_2}|y-z| - \frac{1}{2D_2}|z|\right) w(z\sqrt{1+\alpha_\epsilon})\, dz\, (1+O(\epsilon|y|))\\
&\quad +2\alpha_\epsilon\sqrt{1+\alpha_\epsilon}\tilde{u}_{1,\epsilon}(\epsilon y)\tilde{u}_{2,\epsilon}(0)\epsilon^2 y^2\nabla^2 G_{D_2}(0,0)\left(\int_{\mathbb{R}} w\, dy\right)(1+O(\epsilon|y|))\\
&= -2\alpha_\epsilon(1+\alpha_\epsilon)\tilde{u}_{1,\epsilon}(\epsilon y)\frac{\tilde{u}_{2,\epsilon}(0)}{\tilde{S}_\epsilon(0)}\int_{\mathbb{R}}\left(\frac{1}{2D_2}|y-z| - \frac{1}{2D_2}|z|\right) w(z\sqrt{1+\alpha_\epsilon})\, dz\, (1+O(\epsilon|y|))\\
&\quad +2\alpha_\epsilon\sqrt{1+\alpha_\epsilon}\tilde{u}_{1,\epsilon}(\epsilon y)\tilde{u}_{2,\epsilon}(0)\epsilon^2 y^2\nabla^2 G_{D_2}(0,0) 6\,(1+O(\epsilon|y|)).\\
&= O(\epsilon|y|)\tilde{u}_{1,\epsilon}.\\
&= O(\epsilon) \quad \text{ in } L^2(\Omega_\epsilon).
\end{aligned}$$

Writing the system (2.1) as $R_\epsilon(S, u_1, u_2) = 0$, we have now shown the estimate

$$\||R_\epsilon(T_1[\tilde{u}_{1,\epsilon}], \tilde{u}_{1,\epsilon}, T_2[\tilde{u}_{1,\epsilon}])\||_{L^2(\Omega_\epsilon)} = O(\epsilon)\,. \tag{4.1}$$

(Note that the first and third equations are solved exactly and they do not contribute to the definition of this norm.)

Next, we study the linearized operator $\tilde{\mathcal{L}}_\epsilon$ around the approximate solution $(\tilde{S}_\epsilon,\ \tilde{u}_{\epsilon,1},\ \tilde{u}_{\epsilon,2})$. It is defined as follows

$$\tilde{\mathcal{L}}_\epsilon\ : (H^2_N(\Omega))^3 \to (L^2(\Omega))^3, \quad \tilde{\mathcal{L}}_\epsilon \begin{pmatrix} \psi_{1,\epsilon} \\ \phi_\epsilon \\ \psi_{2,\epsilon} \end{pmatrix} = \tag{4.2}$$

$$= \begin{pmatrix} D_1\Delta\psi_{1,\epsilon} - 2\frac{a_1}{\epsilon}\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_\epsilon - \frac{a_1}{\epsilon}\psi_{1,\epsilon}\tilde{u}^2_{1,\epsilon} \\ \epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + 2\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_\epsilon + \psi_{1,\epsilon}\tilde{u}^2_{1,\epsilon} - a_2\phi_\epsilon\tilde{u}^2_{2,\epsilon} - 2a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\psi_{2,\epsilon} \\ \Delta_2\psi_{2,\epsilon} - \psi_{2,\epsilon} + \phi_\epsilon\tilde{u}^2_{2,\epsilon} + 2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\psi_{2,\epsilon} \end{pmatrix}.$$

When discussing the kernel of $\tilde{\mathcal{L}}_\epsilon$ we may solve the first and third components for $\psi_1 = T_1'[\tilde{u}_{1,\epsilon}]\phi$ and $\psi_2 = T_2'[\tilde{u}_{1,\epsilon}]\phi$ using the Green's functions G_{D_1} and G_{D_2}, respectively. Therefore we may instead study the following operator $\bar{\mathcal{L}}_\epsilon$ which is applied to the second component only. Further, to have uniform invertibility we have to introduce suitable approximate kernel and co-kernel given by

$$\mathcal{K}_\epsilon = \text{span}\{\tilde{u}'_{1,\epsilon}\} \subset H^2_N(\Omega_\epsilon),$$
$$\mathcal{C}_\epsilon = \text{span}\{\tilde{u}'_{1,\epsilon}\} \subset L^2(\Omega_\epsilon).$$

Then the linear operator $\bar{\mathcal{L}}_\epsilon$ is defined by

$$\bar{\mathcal{L}}_\epsilon\ : \mathcal{K}^\perp_\epsilon \to \mathcal{C}^\perp_\epsilon, \tag{4.3}$$

$$\bar{\mathcal{L}}_\epsilon(\phi_\epsilon) = \Delta_y\phi_\epsilon - \phi_\epsilon + 2\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_\epsilon + (T_1^{'}[\tilde{u}_{1,\epsilon}]\phi_\epsilon)\tilde{u}^2_{1,\epsilon} - a_2\phi_\epsilon\tilde{u}^2_{2,\epsilon} - 2a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}(T_2^{'}[\tilde{u}_{2,\epsilon}]\phi_\epsilon),$$

where $\perp$ means perpendicular in the sense of $L^2(\Omega_\epsilon)$. This operator is uniformly invertible for ϵ small enough. In fact, we have the following result:

Proposition 4.1. *There exist positive constants $\bar{\epsilon}$, λ such that for all $\epsilon \in (0, \bar{\epsilon})$,*

$$\|\tilde{\mathcal{L}}_\epsilon\phi\|_{L_2(\Omega_\epsilon)} \geq \lambda\|\phi\|_{H^2(\Omega_\epsilon)} \quad \text{for all } \phi \in \mathcal{K}^\perp_\epsilon. \tag{4.4}$$

Further, the linear operator $\bar{\mathcal{L}}_\epsilon$ is surjective.

Proof of Proposition 4.1. We give an indirect proof. Suppose (4.4) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\phi^k\}$ with $\epsilon_k \to 0$, $\phi^k = \phi_{\epsilon_k}$, $k = 1, 2, \ldots$ such that

$$\|\bar{\mathcal{L}}_{\epsilon_k}\phi^k\|_{L^2} \to 0, \quad \text{as } k \to \infty, \tag{4.5}$$
$$\|\phi^k\|_{H^2} = 1, \qquad k = 1,\ 2,\ \ldots\ . \tag{4.6}$$

By using the cut-off function defined in (3.2), we define the following functions:

$$\phi_{1,\epsilon}(y) = \phi_\epsilon(y)\chi(x), \quad y \in \Omega_\epsilon. \tag{4.7}$$

$$\phi_{2,\epsilon}(y) = \phi_\epsilon(y)\left(1 - \chi(x)\right), \quad y \in \Omega_\epsilon.$$

At first the functions $\phi_{1,\epsilon}$, $\phi_{2,\epsilon}$ are only defined in Ω_ϵ. However, by a standard extension result, $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ can be extended to $\mathbb{R}$ such that the norms of $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ in $H^2(\mathbb{R})$ are bounded by a constant independent of ϵ for all ϵ small enough. In the following we shall study this extension. For simplicity we keep the same notation for the extension. Since for $i = 1, 2$ each sequence $\{\phi_i^k\} := \{\phi_{i,\epsilon_k}\}$ $(k = 1, 2, \ldots)$ is bounded in $H^2_{loc}(\mathbb{R})$ it has a weak limit in $H^2_{loc}(\mathbb{R})$, and therefore also a strong limit in $L^2_{loc}(\mathbb{R})$ and $L^\infty_{loc}(\mathbb{R})$. Call these limits ϕ_i. Then $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ satisfies

$$\int_{\mathbb{R}} \phi_1 w_y \, dy = 0$$

and it solves the system

$$\mathcal{L}\phi_1 = 0, \tag{4.8}$$

where the operator $\mathcal{L}$ is defined as follows

$$\mathcal{L}\phi_1 = \Delta_y \phi_1 - (1+\alpha)\phi_1 + 2(1+\alpha)w\phi_1 - 2(1+\alpha)\frac{\int_{\mathbb{R}} w\phi_1 \, dy}{\int_{\mathbb{R}} w^2 \, dy} w^2 + 2\alpha \frac{\int_{\mathbb{R}} \phi_1 \, dy}{\int_{\mathbb{R}} w \, dy} w.$$

The system (4.8) is derived by taking the limit $\epsilon \to 0$ in (4.3). It will be computed in (5.3) below.

By the analysis in Section 5 based on Theorem 5.2, from (4.8) it follows that $\phi_1 = 0$.

Further, trivially, $\phi_2 = 0$ in $\mathbb{R}$.

By elliptic estimates we get $\|\phi_{i,\epsilon_k}\|_{H^2(\mathbb{R})} \to 0$ for $i = 1, 2$ as $k \to \infty$.

This contradicts $\|\phi^k\|_{H^2} = 1$. To complete the proof of Proposition 4.1, we need to show that the operator which is conjugate to L_ϵ (denoted by L_ϵ^*) is injective from H^2 to L^2.

The limiting process as $\epsilon \to 0$ for the adjoint operator L_ϵ^* follows exactly along the same lines as the proof for L_ϵ and is therefore omitted. In Section 5, we will show that the limiting adjoint operator $\mathcal{L}^*$ has only the trivial kernel. □

Finally, we solve the system (2.1), which we write as

$$R_\epsilon(\tilde{S}_\epsilon + \psi_1, \tilde{u}_{1,\epsilon} + \phi, \tilde{u}_{2,\epsilon} + \psi_2) = R_\epsilon(U_\epsilon + \Phi) = 0, \tag{4.9}$$

using the notation $U_\epsilon = \left(\tilde{S}_\epsilon, \tilde{u}_{1,\epsilon}, \tilde{u}_{2,\epsilon}\right)^T$, $\Phi = (\psi_1, \phi, \psi_2)^T$. Since $\mathcal{L}_\epsilon$ is uniformly invertible if ϵ is small enough and calling the inverse $\mathcal{L}_\epsilon^{-1}$, we can write (4.9) as follows:

$$\Phi = -\mathcal{L}_\epsilon^{-1} R_\epsilon(U_\epsilon) - \mathcal{L}_\epsilon^{-1} N_\epsilon(\Phi) =: M_\epsilon(\Phi), \tag{4.10}$$

where

$$N_\epsilon(\Phi) = R_\epsilon(U_\epsilon + \Phi) - R_\epsilon(U_\epsilon) - R'_\epsilon(U_\epsilon)\Phi \tag{4.11}$$

and the operator M_ϵ defined by (4.10) is a mapping from H^2 into itself. We are going to show that the operator M_ϵ is a contraction on

$$B_{\epsilon,\delta} \equiv \{\phi \in H^2 \, : \, \|\phi\|_{H^2} < \delta\}$$

if δ and ϵ are suitably chosen. We have by (4.1) and Proposition 4.1 that

$$\|M_\epsilon(\Phi)\|_{H^2} \le \lambda^{-1}\Big(\|N_\epsilon(\Phi)\|_{L^2} + \|R_\epsilon(U_\epsilon)\|_{L^2}\Big)$$
$$\le \lambda^{-1} C_0(c(\delta)\delta + \epsilon),$$

where $\lambda > 0$ is independent of $\delta > 0$, $\epsilon > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly, we show

$$\|M_\epsilon(\Phi_1) - M_\epsilon(\Phi_2)\|_{H^2} \le \lambda^{-1} C_0(c(\delta)\delta)\|\Phi_1 - \Phi_2\|_{H^2},$$

where $c(\delta) \to 0$ as $\delta \to 0$. Choosing $\delta = C_1\epsilon$ for $\lambda^{-1}C_0 < C_1$ and ϵ small enough, then $M_{\epsilon,\mathbf{t}}$ maps from $B_{\epsilon,\delta}$ into $B_{\epsilon,\delta}$ and it is a contraction mapping in $B_{\epsilon,\delta}$. The existence of a fixed point Φ_ϵ now follows from the standard contraction mapping principle, and Φ_ϵ is a solution of (4.10).

We have thus proved

Lemma 4.1. *There exists $\bar{\epsilon} > 0$ such that for every ϵ with $0 < \epsilon < \bar{\epsilon}$ there is a unique $\Phi_\epsilon \in H^2_N(\Omega) \otimes \mathcal{K}^\perp_\epsilon \otimes H^2_N(\Omega)$ satisfying $R_\epsilon(U_\epsilon + \Phi_\epsilon) = 0$. Furthermore, we have the estimate*

$$\|\Phi_\epsilon\|_{H^2} \le C\epsilon. \tag{4.12}$$

In this section we have constructed an exact spike solution of the form $U_\epsilon + \Phi_\epsilon = (S_\epsilon, u_{\epsilon,1}, u_{\epsilon,2})$. We are now going to study its stability.

5. Stability I: Derivation, rigorous deduction and analysis of a NLEP

We linearize (1.1) around the single-spike solution $S_\epsilon + \psi_{1,\epsilon}e^{\lambda t}$, $u_{\epsilon,1} + \phi_\epsilon e^{\lambda t}$, $u_{\epsilon,2} + \psi_{2,\epsilon}e^{\lambda t}$. Then we study the eigenvalue problem of the linearized operator around the steady state $(S_\epsilon,\ u_{\epsilon,1},\ u_{\epsilon,2})$.

The eigenvalue problem becomes

$$\mathcal{L}_\epsilon \begin{pmatrix} \psi_{1,\epsilon} \\ \phi_\epsilon \\ \psi_{2,\epsilon} \end{pmatrix} = \begin{pmatrix} \tau\lambda_\epsilon\psi_{1,\epsilon} \\ \lambda_\epsilon\phi_\epsilon \\ \tau_1\lambda_\epsilon\psi_{2,\epsilon} \end{pmatrix}, \tag{5.1}$$

where $\mathcal{L}_\epsilon$ denotes the operator linearized around the steady state steady state $(S_\epsilon,\ u_{\epsilon,1},\ u_{\epsilon,2})$.

We assume that the domain of the operator $\mathcal{L}_\epsilon$ is $H^2_N(\Omega) \times H^2_N(\Omega_\epsilon) \times H^2_N(\Omega)$ and that $\lambda_\epsilon \in \mathbb{C}$, the set of complex numbers.

We say that a spike solution is **linearly stable** if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in a left half plane $\{\lambda \in \mathbb{C} :\ \mathrm{Re}(\lambda) \le -c_0\}$ for some $c_0 > 0$. A spike solution is called **linearly unstable** if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with $\mathrm{Re}(\lambda_\epsilon) > 0$.

We first assume that $\tau = 0$ and $\tau_1 = 0$, and at the end of the proof we will explain how to generalize the argument to the case of $\tau \ge 0$ small and $\tau_1 \ge 0$ small. Writing down $\mathcal{L}_\epsilon$ explicitly and expressing $\psi_{i,\epsilon} = T_i'[u_{i,\epsilon}]\phi_\epsilon$, $i = 1, 2$, using Green's functions G_{D_i}, we can rewrite (5.1) as

$$\epsilon^2\phi_{\epsilon.xx} - \phi_\epsilon + 2S_\epsilon u_{1,\epsilon}\phi_\epsilon + (T_1'[u_{1,\epsilon}]\phi_\epsilon)u_{1,\epsilon}^2 - a_2\phi_\epsilon u_{2,\epsilon}^2 - 2a_2u_{1,\epsilon}u_{2,\epsilon}(T_2'[u_{2,\epsilon}]\phi_\epsilon) = \lambda_\epsilon\phi_\epsilon. \tag{5.2}$$

Then, arguing as in the proof of Proposition 4.1, the sequence ϕ_ϵ has a converging subsequence. We derive an eigenvalue problem for its limit. (Since we consider even eigenfunctions, it is enough to restrict our attention to the positive real axis $y > 0$.)

Now we derive the limiting eigenvalue problem for ϕ.

Integrating the first equation of (5.2), we get

$$\psi_{1,\epsilon}(0)\int_{-1}^{1} u_{1,\epsilon}^2\,dx = -S_\epsilon(0)\int_{-1}^{1} u_{1,\epsilon}\phi_\epsilon\,dx$$

which implies

$$\psi_{1,\epsilon}(0) = -\frac{S_\epsilon(0)}{\xi_\epsilon}\frac{\int_{\mathbb{R}} w\phi\,dy}{\int_{\mathbb{R}} w^2\,dy}(1+O(\epsilon))$$

Putting everything together, we compute

$$\psi_{1,\epsilon}(0)u_{1,\epsilon}^2 = -2\frac{S_\epsilon(0)}{\xi_\epsilon}\frac{\int_{\mathbb{R}} w\phi\,dy}{\int_{\mathbb{R}} w^2\,dy}\xi_\epsilon^2 w^2\,(1+O(\epsilon))$$

$$= -2(1+\alpha)\frac{\int_{\mathbb{R}} w\phi\,dy}{\int_{\mathbb{R}} w^2\,dy}w^2\,(1+O(\epsilon)).$$

We also derive from (2.11) that

$$u_{2,\epsilon}(0) = \frac{\sqrt{1+\alpha_\epsilon}}{G_{D_2}(0,0)6\xi_\epsilon} + O(\epsilon)$$

and compute

$$\psi_{2,\epsilon}(0) = G_{D_2}(0,0)\left[u_{2,\epsilon}^2(0)\frac{1}{\sqrt{1+\alpha_\epsilon}}\int_{\mathbb{R}}\phi\,dy + 2\psi_{2,\epsilon}(0)u_{2,\epsilon}(0)\frac{\xi_\epsilon}{\sqrt{1+\alpha_\epsilon}}\int_{\mathbb{R}} w\,dy\right](1+O(\epsilon))$$

$$= u_{2,\epsilon}(0)G_{D_2}(0,0)\frac{1}{\sqrt{1+\alpha_\epsilon}}\left[u_{2,\epsilon}(0)\int_{\mathbb{R}}\phi\,dy + 2\psi_{2,\epsilon}(0)\xi_\epsilon\int_{\mathbb{R}} w\,dy\right]$$

which implies

$$\psi_{2,\epsilon}(0) = -\frac{u_{2,\epsilon}(0)}{\xi_\epsilon}\frac{\int_{\mathbb{R}}\phi\,dy}{\int_{\mathbb{R}} w\,dy}\,(1+O(\epsilon))$$

and finally we get

$$\psi_{2,\epsilon}(0) = -\frac{\sqrt{1+\alpha_\epsilon}}{G_{D_2}(0,0)\xi_\epsilon^2}\frac{\int_{\mathbb{R}}\phi\,dy}{\int_{\mathbb{R}} w\,dy}\,(1+O(\epsilon)).$$

Therefore, we compute

$$\begin{aligned}&-a_2u_{1,\epsilon}2u_{2,\epsilon}\psi_{2,\epsilon}\\&= -a_2u_{1,\epsilon}2u_{2,\epsilon}(0)\psi_{2,\epsilon}(0)\,(1+O(\epsilon))\\&= -2\alpha_\epsilon w\frac{\xi_\epsilon\psi_{2,\epsilon}(0)}{u_{2,\epsilon}(0)}\,(1+O(\epsilon))\\&= +2\alpha_\epsilon\frac{\int_{\mathbb{R}}\phi\,dy}{\int_{\mathbb{R}} w\,dy}w\,(1+O(\epsilon))\end{aligned}$$

in $H^2(\Omega_\epsilon)$.

Putting all these expressions into (5.2) and taking the limit $\epsilon\to 0$, we derive the NLEP

$$\Delta_y\phi - (1+\alpha)\phi + 2(1+\alpha)w\phi - 2(1+\alpha)\frac{\int_{\mathbb{R}} w\phi\,dy}{\int_{\mathbb{R}} w^2\,dy}w^2 + 2\alpha\frac{\int_{\mathbb{R}}\phi\,dy}{\int_{\mathbb{R}} w\,dy}w = \lambda\phi. \tag{5.3}$$

Although the derivations given above are formal, we can rigorously prove the following separation of eigenvalues.

Theorem 5.1. *Let λ_ϵ be an eigenvalue of (5.2) for which $Re(\lambda_\epsilon) > -a_0$ for some suitable constant a_0 fixed independent of ϵ.*

(1) Suppose that (for suitable sequences $\epsilon_n \to 0$) we have $\lambda_{\epsilon_n} \to \lambda_0 \neq 0$. Then λ_0 is an eigenvalue of the NLEP given in (5.3).

(2) Let $\lambda_0 \neq 0$ be an eigenvalue of the NLEP given in (5.3). Then for ϵ sufficiently small, there is an eigenvalue λ_ϵ of (5.2) with $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$.

Remark. *From Theorem 5.1 we see rigorously that the eigenvalue problem (5.2) is reduced to the study of the NLEP (5.3).*

Now we prove Theorem 5.1.

Proof of Theorem 5.1:

Part (1) follows by an asymptotic analysis combined with passing to the limit as $\epsilon \to 0$ which is similar to the proof of Proposition 4.1 in Section.

Part (2) follows from a compactness argument by Dancer introduced in Section 2 of [3]. It was applied in [42] to a related situation, therefore we omit the details.

The stability or instability of the large eigenvalues follows from the following results:

Theorem 5.2. [34]*: Consider the following nonlocal eigenvalue problem*

$$\phi'' - \phi + 2w\phi - \gamma \frac{\int_{\mathbb{R}} w\phi}{\int_{\mathbb{R}} w^2} w^2 = \alpha\phi. \tag{5.4}$$

(1) If $\gamma < 1$, then there is a positive eigenvalue to (5.4).

(2) If $\gamma > 1$, then for any nonzero eigenvalue λ of (5.4), we have

$$Re(\lambda) \leq -c_0 < 0.$$

(3) If $\gamma \neq 1$ and $\lambda = 0$, then $\phi = c_0 w'$ for some constant c_0.

In our applications to the case when $\tau > 0$ or $\tau_1 > 0$, we need to handle the situation when the coefficient γ is a complex function of $\tau\lambda$. Let us suppose that

$$\gamma(0) \in \mathbb{R}, \quad |\gamma(\tau\lambda)| \leq C \ \text{ for } \lambda_R \geq 0,\, \tau \geq 0, \tag{5.5}$$

where C is a generic constant independent of τ, λ.

Now we have

Theorem 5.3. *(Theorem 3.2 of* [42]*.)*

Consider the following nonlocal eigenvalue problem

$$\phi'' - \phi + 2w\phi - \gamma(\tau\lambda) \frac{\int_{\mathbb{R}} w\phi}{\int_{\mathbb{R}} w^2} w^2 = \lambda\phi, \tag{5.6}$$

where $\gamma(\tau\lambda)$ satisfies (5.5). Then there is a small number $\tau_0 > 0$ such that for $\tau < \tau_0$,

(1) if $\gamma(0) < 1$, then there is a positive eigenvalue to (5.4);

(2) if $\gamma(0) > 1$, then for any nonzero eigenvalue λ of (5.6), we have

$$Re(\lambda) \leq -c_0 < 0.$$

First we consider the eigenvalue problem (5.3) which covers the case $\tau = \tau_1 = 0$. Later we will explain that by a perturbation argument the result can be extended to the case when τ and τ_1 are both small enough.

Integrating (5.3), we derive

$$(\lambda + 1 - \alpha) \int_{\mathbb{R}} \phi \, dy = 0.$$

Suppose that $\int_{\mathbb{R}} \phi \, dy \neq 0$. Then for all eigenvalues we have (i) $\lambda + 1 - \alpha = 0$ or (ii) $\int_{\mathbb{R}} \phi \, dy = 0$.

Let us first consider the case (i). If $\alpha < 1$ then (i) implies $\lambda < 0$ and the problem is stable for all eigenfuctions. If $\alpha > 1$, then we can choose the eigenfuntion ϕ with eigenvalue $\lambda = \alpha - 1 > 0$ as follows and the eigenvalue problem is unstable: We set

$$\phi = (L + 1 - \alpha)^{-1} \left[c_1 w^2 + c_2 w \right], \tag{5.7}$$

where

$$L : K^{\perp} \to C^{\perp}, \quad L\phi := \Delta\phi - (1 + \alpha)\phi + 2(1 + \alpha) w\phi,$$

$$K^{\perp} = \left\{ v \in H^2(\mathbb{R}) : \int v w_y \, dy = 0 \right\}, \quad C^{\perp} = \left\{ v \in L^2(\mathbb{R}) : \int v w_y \, dy = 0 \right\},$$

$$c_1 = \frac{2(1 + \alpha) \int_{\mathbb{R}} w\phi \, dy}{\int_{\mathbb{R}} w^2 \, dy}, \quad c_2 = -\frac{2\alpha \int_{\mathbb{R}} \phi \, dy}{\int_{\mathbb{R}} w \, dy}.$$

Multiplying (5.7) by w and 1, respectively, and integrating we get a linear system for the coefficients $(\int_{\mathbb{R}} w\phi \, dy, \int_{\mathbb{R}} \phi \, dy)$ which has a nontrivial solution. Solving this system using the identities

$$Lw = (1 + \alpha) w^2, \quad L \left(\frac{y\sqrt{\alpha + 1}}{2} w_y + w \right) = (1 + \alpha) w,$$

we finally get

$$c_1 = \int_{\mathbb{R}} w (L + 1 - \alpha)^{-1} w \, dy,$$

$$c_2 = -\int_{\mathbb{R}} w (L + 1 - \alpha)^{-1} w^2 \, dy + \frac{3}{1 - \alpha}.$$

Next we consider the case (ii). The NLEP (5.3) reduces to the familiar NLEP considered in Theorem 5.2 which implies that (the real parts of) all eigenvalues are strictly negative. Therefore we can have instability only through case (i) above.

We now consider the adjoint operator $\mathcal{L}_\epsilon^*$ to the linear operator $\mathcal{L}_\epsilon$. Expressing $\mathcal{L}_\epsilon^*$ explicitly, we can rewrite the adjoint eigenvalue problem as follows

$$\begin{cases} D_1 \Delta \psi_{1,\epsilon} + \dfrac{1}{\epsilon} (\phi_\epsilon - a_1 \psi_{1,\epsilon}) u_{1,\epsilon}^2 = \tau \lambda_\epsilon \psi_{1,\epsilon}, \\ \epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + 2 S_\epsilon u_{1,\epsilon} (\phi_\epsilon - a_1 \psi_{1,\epsilon}) + (\psi_{2,\epsilon} - a_2 \phi_\epsilon) u_{2,\epsilon}^2 = \lambda_\epsilon \phi_\epsilon, \\ D_2 \Delta \psi_{2,\epsilon} - \psi_{2,\epsilon} + \dfrac{2}{\epsilon} u_{1,\epsilon} u_{2,\epsilon} (\psi_{2,\epsilon} - a_2 \phi_\epsilon) = \tau_1 \lambda_\epsilon \psi_{2,\epsilon}. \end{cases} \tag{5.8}$$

We need to consider the kernel of this adjoint eigenvalue problem. (In the proof of Proposition 4.1 we need the result that this kernel is trivial.) Taking the limit $\epsilon \to 0$, we derive the following problem for the kernel of the adjoint problem in the same way as (5.3):

$$\Delta_y \phi - (1+\alpha)\phi + 2(1+\alpha)w\phi - 2(1+\alpha)\frac{\int_{\mathbb{R}} w^2\phi\, dy}{\int_{\mathbb{R}} w^2\, dy} w + 2\alpha \frac{\int_{\mathbb{R}} w\phi\, dy}{\int_{\mathbb{R}} w\, dy} = 0. \tag{5.9}$$

We are now going to show that this limit of the adjoint operator has only the trivial kernel.

Multiplying (5.9) by 1 and integrating, we derive $\int_{\mathbb{R}} w\phi\, dy = 0$ since otherwise there is an unbounded term. Further, we get the relation

$$\int_{\mathbb{R}} \phi\, dy + 2\int_{\mathbb{R}} w^2\phi\, dy = 0. \tag{5.10}$$

Multiplying (5.9) by w and integrating, we derive

$$\int_{\mathbb{R}} w^2\phi\, dy = 0. \tag{5.11}$$

Then from (5.10) we get $\int_{\mathbb{R}} \phi\, dy = 0$. Finally, going back to (5.9), all nonlocal terms vanish and by Theorem 5.2 in the special case $\gamma = 0$ we derive $\phi = 0$.

□

Now we continue to consider the stability problem for the linearized operator.

We extend this approach to $\tau \ge 0$ small and $\tau_1 \ge 0$ small.

First note that $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ are both continuous in $\tau\lambda_\epsilon$ which follows by using Green's functions to solve for $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$, respectively, and the change due to positive τ, τ_1 is of order $O((\tau+\tau_1)|\lambda_\epsilon|)$ in $H^2(\Omega)$.

This implies that in the second equation the factors of the nonlocal terms change by an amount which can be estimated by $O((\tau+\tau_1)|\lambda_\epsilon|)$.

By a regular perturbation argument, the eigenfunction ϕ_ϵ and the eigenvalue λ_ϵ changes only by $O((\tau+\tau_1)|\lambda_\epsilon|)$ in $H^2(\Omega_\epsilon)$.

Multiplying the eigenvalue problem by the eigenfunction and using quadratic forms, it can be shown that $|\lambda_\epsilon|$ is bounded for τ and τ_1 small enough. (This argument is a straightforward extension of a similar result in [38] and is therefore omitted.)

Therefore in fact for the errors mentioned above, we have $O((\tau+\tau_1)|\lambda_\epsilon|) = O(\tau+\tau_1)$.

This implies that all results on stability and instability of the eigenvalues of order $O(1)$ derived in this section are valid for sufficiently small values of the constants τ and τ_1, i.e. there exists positive numbers τ_0 and $\tau_{1,0}$ such that for $0 < \tau < \tau_0$ and $0 < \tau_1 < \tau_{1,0}$ the results hold unchanged.

6. Stability II: Computation of the small eigenvalues

We now compute the small eigenvalues of the eigenvalue problem (5.2), i.e. we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. We will prove that they satisfy $\lambda_\epsilon = O(\epsilon^2)$. Let us define

$$\tilde{u}_{1,\epsilon}(x) = \chi(|x|)u_{1,\epsilon}(x). \tag{6.1}$$

Then it follows easily that

$$u_{1,\epsilon}(x) = \tilde{u}_{1,\epsilon}(x) + \text{e.s.t.} \quad \text{in } H^2(\Omega_\epsilon). \tag{6.2}$$

Taking the derivative of the system (2.1) w.r.t. y, we compute

$$\tilde{u}_{1,\epsilon}''' - \tilde{u}_{1,\epsilon}' + 2S_\epsilon u_{1,\epsilon}\tilde{u}_{1,\epsilon}' + \epsilon S_\epsilon' u_{1,\epsilon}^2 - a_2\tilde{u}_{1,\epsilon}' u_{2,\epsilon}^2 - 2\epsilon a_2 u_{1,\epsilon} u_{2,\epsilon}\tilde{u}_{2,\epsilon}' = \text{e.s.t.}. \qquad (6.3)$$

Here $'$ denotes derivative w.r.t. the variable of the corresponding function, i.e. it means derivative w.r.t. x for S_ϵ for $u_{2,\epsilon}$ and w.r.t. y for $u_{1,\epsilon}$.

Let us now decompose the eigenfunction $(\psi_{1,\epsilon}, \phi_\epsilon, \psi_{2,\epsilon})$ as follows:

$$\phi_\epsilon = a^\epsilon \tilde{u}_{1,\epsilon}' + \phi_\epsilon^\perp \qquad (6.4)$$

where a^ϵ is a complex number to be determined and

$$\phi_\epsilon^\perp \perp \mathcal{K}_\epsilon = \text{span}\,\{\tilde{u}_{1,\epsilon}'\} \subset H_N^2\left(-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right).$$

We decompose the eigenfunction $\psi_{1,\epsilon}$ as follows:

$$\psi_{1,\epsilon} = a^\epsilon \psi_{1,\epsilon}^0 + \psi_{1,\epsilon}^\perp,$$

where $\psi_{1,\epsilon}^0$ satisfies

$$\begin{cases} D_1\Delta\psi_{1,\epsilon}^0 - \frac{a_1}{\epsilon}\psi_{1,\epsilon}^0 u_{1,\epsilon}^2 - 2\frac{a_1}{\epsilon} S_\epsilon u_{1,\epsilon}\frac{1}{\epsilon}\tilde{u}_{1,\epsilon}' = \tau\lambda_\epsilon\psi_{1,\epsilon}^0, \\ {\psi_{1,\epsilon}^0}'(\pm 1) = 0 \end{cases} \qquad (6.5)$$

and $\psi_{1,\epsilon}^\perp$ is given by

$$\begin{cases} D_1\Delta\psi_{1,\epsilon}^\perp - \frac{a_1}{\epsilon}\psi_{1,\epsilon}^\perp u_{1,\epsilon}^2 - 2\frac{a_1}{\epsilon} S_\epsilon u_{1,\epsilon}\phi_\epsilon^\perp = \tau\lambda_\epsilon\psi_{1,\epsilon}^\perp, \\ {\psi_{1,\epsilon}^\perp}'(\pm 1) = 0. \end{cases} \qquad (6.6)$$

Similarly, we decompose the eigenfunction $\psi_{2,\epsilon}$ as follows:

$$\psi_{2,\epsilon} = a^\epsilon \psi_{2,\epsilon}^0 + \psi_{2,\epsilon}^\perp,$$

where $\psi_{2,\epsilon}^0$ satisfies

$$\begin{cases} D_2\Delta\psi_{2,\epsilon}^0 - \psi_{2,\epsilon}^0 + \frac{2}{\epsilon}u_{1,\epsilon}u_{2,\epsilon}\psi_{2,\epsilon}^0 + \frac{1}{\epsilon}\tilde{u}_{1,\epsilon}' u_{2,\epsilon}^2 = \tau_1\lambda_\epsilon\psi_{2,\epsilon}^0, \\ {\psi_{2,\epsilon}^0}'(\pm 1) = 0 \end{cases} \qquad (6.7)$$

and $\psi_{2,\epsilon}^\perp$ is given by

$$\begin{cases} D_2\Delta\psi_{2,\epsilon}^\perp - \psi_{2,\epsilon}^\perp + \frac{2}{\epsilon}u_{1,\epsilon}u_{2,\epsilon}\psi_{2,\epsilon}^\perp + \frac{1}{\epsilon}\phi_\epsilon^\perp u_{2,\epsilon}^2 = \tau\lambda_\epsilon\psi_{2,\epsilon}^\perp, \\ {\psi_{2,\epsilon}^\perp}'(\pm 1) = 0. \end{cases} \qquad (6.8)$$

Note that $\psi_{1,\epsilon}$ and $\psi_{3,\epsilon}$ can be uniquely expressed in terms of ϕ_ϵ by solving the first and third equation using the Green's function $G_{D_1,\tau\lambda_\epsilon}$ and $G_{D_2,\tau\lambda_\epsilon}$, respectively, given in the appendix:

$$\psi_{1,\epsilon} = a^\epsilon\psi_{1,\epsilon}^0 + \psi_{1,\epsilon}^\perp = a^\epsilon T_{1,\tau\lambda_\epsilon}'[\tilde{u}_{1,\epsilon}'] + T_{1,\tau\lambda_\epsilon}'[\phi_\epsilon^\perp]. \qquad (6.9)$$

$$\psi_{2,\epsilon} = a^\epsilon\psi_{2,\epsilon}^0 + \psi_{2,\epsilon}^\perp = a^\epsilon T_{2,\tau\lambda_\epsilon}'[\tilde{u}_{1,\epsilon}'] + T_{2,\tau\lambda_\epsilon}'[\phi_\epsilon^\perp]. \qquad (6.10)$$

Using the Green's function G_{D_1} (see Appendix) we compute S_ϵ' near zero. We get

$$S_\epsilon'(\epsilon y) - S_\epsilon'(0)$$

$$= a_1\epsilon \int_{-1/\epsilon}^{1/\epsilon} \left[\frac{1}{2D_1}(h(y-z)-h(-z)) + H_{D_1,x}(\epsilon y,\epsilon z) - H_{D_1,x}(0,\epsilon z)\right] S_\epsilon(\epsilon z)u_{1,\epsilon}^2(\epsilon z)\,dz + O(\epsilon^3|y|^2)$$

$$= a_1\frac{\epsilon}{D_1}\int_0^y S_\epsilon(\epsilon z)u_{1,\epsilon}^2(\epsilon z)\,dz + a_1\epsilon^2 y\int_{-1/\epsilon}^{1/\epsilon} H_{D_1,xx}(0,0)S_\epsilon(\epsilon z)u_{1,\epsilon}^2(\epsilon z)\,dz + O(\epsilon^2|y|^2)$$

$$= \frac{(1+\alpha_\epsilon)^2}{S_\epsilon(0)}\,\frac{\epsilon}{D_1}\left[\int_0^y w^2(z)\,dz - \frac{\epsilon y}{2}\int_{\mathbb{R}} w^2(z)\,dz\right] + O(\epsilon^3|y|^2)$$

$$= \frac{a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}\frac{\epsilon}{D_1}\left[\int_0^y w^2(z)\,dz - 3\epsilon y\right] + O(\epsilon^3|y|^2), \tag{6.11}$$

where h is the Heaviside function ($h(x)=1$ if $x>0$, $h(0)=0$, $h(x)=-1$ if $x<0$.)

Similarly, we compute using the Green's function $G_{D_1,\tau\lambda_\epsilon}$ (see appendix) that

$$\psi_{1,\epsilon}^0(\epsilon y) - \psi_{1,\epsilon}^0(0)$$

$$= a_1\epsilon\int_{\Omega_\epsilon}[G_{D_1,\tau\lambda_\epsilon}(\epsilon y,\epsilon z) - G_{D_1,\tau\lambda_\epsilon}(0,\epsilon z)]2S_\epsilon u_{1,\epsilon}(z)\frac{1}{\epsilon}\tilde{u}_{1,\epsilon}'(\epsilon z)\,dz + O(\epsilon^3|y|^2)$$

$$= \frac{\epsilon a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}\left[\int_{-1/\epsilon}^{1/\epsilon}\frac{1}{D_1}\epsilon(|y-z|-|z|)S_\epsilon u_{1,\epsilon}\frac{1}{\epsilon}\tilde{u}_{1,\epsilon}'(\epsilon z)\,dz\right.$$

$$\left.+2\underbrace{H_{D_1,xz}(x,z)|_{x=y=0}}_{=0}\frac{\epsilon}{D_1}y\int_{\mathbb{R}} zww'\,dz\right](1+O((\tau+\tau_1)|\lambda_\epsilon|)+O(\epsilon|y|)). \tag{6.12}$$

Note that from (6.5), we derive

$$\psi_{1,\epsilon}^0(0) = O(\epsilon+\tau|\lambda_\epsilon|). \tag{6.13}$$

Adding the contributions from (6.11) and (6.12), we get

$$\frac{d}{dx}\left[S_\epsilon(\epsilon y)-S_\epsilon(0)\right] - \left[\psi_{1,\epsilon}(\epsilon y)-\psi_{1,\epsilon}(0)\right]$$

$$= \epsilon^2\left(H_{D_1,xx}(0,0)+H_{D_1,xz}(0,0)\right)\frac{6a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}y\left(1+O(\epsilon|y|+(\tau+\tau_1)|\lambda_\epsilon|)\right)$$

$$= -\frac{\epsilon^2}{D_1}\,\frac{3a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}y\left(1+O(\epsilon|y|+(\tau+\tau_1)|\lambda_\epsilon|)\right). \tag{6.14}$$

Similarly, we from (6.7) we get

$$\psi_{2,\epsilon}^0(0) = O(\epsilon+\tau_1|\lambda_\epsilon|), \tag{6.15}$$

Using G_{D_2}, we compute that

$$\frac{d}{dx}\left[u_{2,\epsilon}(\epsilon y)-u_{2,\epsilon}(0)\right] - \left[\psi_{2,\epsilon}(\epsilon y)-\psi_{2,\epsilon}(0)\right]$$

$$= \epsilon^2\left(H_{D_2,xx}(0,0)+H_{D_2,xz}(0,0)\right)\frac{6u_{2,\epsilon}^2(0)(1+\alpha_\epsilon)}{S_\epsilon(0)}y\left(1+O(\epsilon|y|+(\tau+\tau_1)|\lambda_\epsilon|)\right)$$

$$= -\frac{\epsilon^2}{D_2}\,\frac{3(1+\alpha_\epsilon)}{S_\epsilon(0)}y\,\theta_2(\coth\theta_2-\tanh\theta_2)(1+O(\epsilon|y|+(\tau+\tau_1)|\lambda_\epsilon|)).,\tag{6.16}$$

where $\theta_i = \frac{1}{\sqrt{D_i}}$, $i=1,2$.

Suppose that ϕ_ϵ satisfies $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)}=1$. Then $|a^\epsilon|\le C$.

Substituting the decompositions of $\psi_{1,\epsilon}$, ϕ_ϵ and $\psi_{2,\epsilon}$ into (5.2) and subtracting (6.3), we have

$$\begin{aligned} & a^\epsilon u_{1,\epsilon}^2 \left(\psi_{1,\epsilon} - \epsilon S_\epsilon^{'}\right) \\ & -a^\epsilon 2a_2 u_{1,\epsilon} u_{2,\epsilon} \left(\psi_{2,\epsilon} - \epsilon u_{2,\epsilon}^{'}\right) \\ +(\phi_\epsilon^\perp)'' - \phi_\epsilon^\perp + 2u_{1,\epsilon}S_\epsilon\phi_\epsilon^\perp + u_{1,\epsilon}^2\psi_{1,\epsilon}^\perp - 2a_2u_{1,\epsilon}u_{2,\epsilon}\psi_{2,\epsilon}^\perp - 2a_2\phi_\epsilon^\perp u_{2\,\epsilon}^2 - \lambda_\epsilon\phi_\epsilon^\perp & \\ = \lambda_\epsilon a^\epsilon \tilde{u}_{1,\epsilon}^{'}. \end{aligned} \tag{6.17}$$

Let us first compute, using (6.13) and (6.14),

$$I_1 := a^\epsilon u_{1,\epsilon}^2 \left(\psi_{1,\epsilon} - \epsilon S_\epsilon^{'}\right)$$

$$= \epsilon^2 a^\epsilon \frac{a_1(1+\alpha_\epsilon)}{D_1}(\xi_\epsilon)^3 y w^2(y) 3\left(1 + O(\epsilon|y| + (\tau+\tau_1)|\lambda_\epsilon|)\right) \tag{6.18}$$

Similarly, we compute from (6.15) and (6.16),

$$I_2 := -a^\epsilon 2a_2 u_{1,\epsilon} u_{2,\epsilon} \left(\psi_{2,\epsilon} - \epsilon u_{2,\epsilon}^{'}\right)$$

$$= \epsilon^2 a^\epsilon \frac{2a_2}{D_2}(\xi_\epsilon)^2 u_{2,\epsilon}^3(0)\theta_2(\coth\theta_2 - \tanh\theta_2) y w(y) 3\left(1 + O(\epsilon|y| + (\tau+\tau_1)|\lambda_\epsilon|)\right) \tag{6.19}$$

We now estimate the orthogonal part of the eigenfunction which is given by $(T_1'[\phi_\epsilon^\perp], \phi_\epsilon^\perp, T_3'[\phi_\epsilon^\perp])$. Expanding we get

$$\mathcal{L}_\epsilon \phi_\epsilon^\perp = g_{1,\epsilon} + g_{2,\epsilon}$$

where

$$\|g_{1,\epsilon}\|_{L^2(\Omega_\epsilon)} = O(\epsilon^3 + \epsilon(\tau+\tau_1)|\lambda_\epsilon|).$$

and

$$g_{2,\epsilon} \perp \mathcal{C}_\epsilon^\perp.$$

By Proposition 4.1 we conclude that

$$\|\phi_\epsilon^\perp\|_{L^2(\Omega_\epsilon)} = O(\epsilon^3 + \epsilon(\tau+\tau_1)|\lambda_\epsilon|). \tag{6.20}$$

Multiplying the eigenvalue problem (5.2) by w' and integrating, we get

$$\begin{aligned} \text{l.h.s.} &= \int_{\mathbb{R}} (I_1 + I_2) w' \, dy \\ &= \epsilon^2 a^\epsilon \frac{a_1(1+\alpha_\epsilon)}{D_1}(\xi_\epsilon)^3 3\left(1 + O(\epsilon + (\tau+\tau_1)|\lambda_\epsilon|)\right)\int_{\mathbb{R}} y w^2(y) w'(y)\, dy \\ &+\epsilon^2 a^\epsilon \frac{2a_2}{D_2}(\xi_\epsilon)^2 u_{2,\epsilon}^3(0)\theta_2(\coth\theta_2 - \tanh\theta_2) 3\left(1 + O(\epsilon + (\tau+\tau_1)|\lambda_\epsilon|)\right)\int_{\mathbb{R}} y w(y) w'(y)\, dy \\ &= -\epsilon^2 a^\epsilon (\xi_\epsilon)^2 \left[\frac{7.2 a_1(1+\alpha_\epsilon)}{D_1}\xi_\epsilon + \frac{18 a_2}{D_2} u_{2,\epsilon}^3(0)\theta_2(\coth\theta_2 - \tanh\theta_2)\right](1 + O(\epsilon)) \end{aligned}$$

since

$$\int_{\mathbb{R}} y w^2(y) w'(y)\, dy = -\int_{\mathbb{R}} \frac{w^3}{3}\, dy_2 = -2.4,$$

$$\int_{\mathbb{R}} y w(y) w'(y)\, dy = -\int_{\mathbb{R}} \frac{w^2}{2}\, dy_2 = -3$$

and the contribution to l.h.s. coming from $\phi_\epsilon^\perp$ can be estimated by $\|g_{1,\epsilon}\|_{L^2(\Omega_\epsilon)} = O(\epsilon^3 + \epsilon(\tau+\tau_1)|\lambda_\epsilon|)$.

Further, we compute

$$\begin{aligned}\text{r.h.s.} &= \lambda_\epsilon a^\epsilon \int_{\mathbb{R}} (w')^2\, dy\, (1+O(\epsilon)) \\ &= 1.2 a^\epsilon \lambda_\epsilon (1+o(1))\,.\end{aligned}$$

Note that in the previous calculation

$$(\tau+\tau_1)|\lambda_\epsilon| = O(\epsilon)$$

and thus the error terms involving τ or τ_1 are not written. Therefore

$$\lambda_\epsilon = -\epsilon^2\xi_\epsilon^2 \left[\frac{6a_1(1+\alpha_\epsilon)}{D_1}\xi_\epsilon + \frac{15a_2}{D_2}u_{2,\epsilon}^3(0)\theta_2(\coth\theta_2 - \tanh\theta_2)\right] + o(\epsilon^2).$$

We summarize our result on the small eigenvalues in the following theorem.

Theorem 6.1. *The eigenvalues of (5.1) with $\lambda_\epsilon \to 0$ satisfy*

$$\lambda_\epsilon = -\epsilon^2\xi_\epsilon^2 \left[\frac{6a_1(1+\alpha_\epsilon)}{D_1}\xi_\epsilon + \frac{15a_2}{D_2}u_{2,\epsilon}^3(0)\theta_2(\coth\theta_2 - \tanh\theta_2)\right] + o(\epsilon^2). \tag{6.21}$$

In particular these eigenvalues are stable.

7. Appendix: Two Green's functions

Let $G_{D_1}(x,z)$ be the Green's function of the Laplace operator with Neumann boundary conditions:

$$\begin{cases} D_1 G''_{D_1}(x,z) + \frac{1}{2} - \delta_z = 0 & \text{in } (-1,1), \\ \int_{-1}^1 G_{D_1}(x,z)\,dx = 0, G'_{D_1}(-1,z) = G'_{D_1}(1,z) = 0. \end{cases} \tag{7.1}$$

We can decompose $G_{D_1}(x,z)$ as follows

$$G_{D_1}(x,z) = \frac{1}{2D_1}|x-z| + H_{D_1}(x,z) \tag{7.2}$$

where H_{D_1} is the regular part of G_{D_1}.

Written explicitly, we have

$$G_{D_1}(x,z) = \begin{cases} \frac{1}{D_1}\left[\frac{1}{3} - \frac{(x+1)^2}{4} - \frac{(1-z)^2}{4}\right], & -1 < x \le z, \\ \frac{1}{D_1}\left[\frac{1}{3} - \frac{(z+1)^2}{4} - \frac{(1-x)^2}{4}\right], & z \le x < 1. \end{cases} \tag{7.3}$$

By simple computations,

$$H_{D_1}(x,z) = \frac{1}{2D_1}\left[-\frac{1}{3} - \frac{x^2}{2} - \frac{z^2}{2}\right]. \tag{7.4}$$

For $x \ne z$ we calculate

$$\nabla_x\nabla_z G_{D_1}(x,z) = 0, \quad \nabla_x G_{D_1}(x,z) = \begin{cases} -\frac{x+1}{2D_1} & \text{if } x \le z \\ -\frac{x-1}{2D_1} & \text{if } z \le x. \end{cases} \tag{7.5}$$

We further have

$$< \nabla_x G_{D_1}(x,z)|_{x=z} >= \nabla_x H_{D_1}(x,z)|_{x=z} = -\frac{z}{2D_1}, \tag{7.6}$$

where $< \cdot >$ means average of lefthand and righthand limit at a jump point.

This implies

$$G_{D_1,xx}(0,0) = -\frac{1}{2D_1},$$
$$G_{D_1,xz}(0,0) = 0.$$

Note that in particular

$$G_{D_1,xx}(0,0) + G_{D_1,xz}(0,0) = -\frac{1}{2D_1} < 0.$$

Closely related, let $G_{D_1,\tau\lambda}(x,z)$ be Green's function of

$$\begin{cases} D_1 G''_{D_1,\tau\lambda}(x,z) - \tau\lambda G_{D_1,\tau\lambda}(x,z) - \delta_z = 0 \quad \text{in } (-1,1), \\ G'_{D_1,\tau\lambda}(-1,z) = G'_{D_1,\tau\lambda}(1,z) = 0. \end{cases} \tag{7.7}$$

We can decompose $G_{D_1,\tau\lambda}(x,z)$ as follows

$$G_{D_1,\tau\lambda}(x,z) = \frac{1}{2D_1}|x-z| + H_{D_1,\tau\lambda}(x,z) \tag{7.8}$$

where $H_{D_1,\tau\lambda}$ is the regular part of $G_{D_1,\tau\lambda}$.

An elementary computation shows that

$$|H_{D_1}(x,z) - H_{D_1,\tau\lambda}(x,z)| \le C|\tau\lambda|$$

uniformly for all $(x,z) \in \Omega \times \Omega$.

Further, let $G_{D_2}(x,z)$ be the following Green's function:

$$\begin{cases} D_2 G''_{D_2}(x,z) - G_{D_2}(x,z) - \delta_z(x) = 0 \quad \text{in } (-1,1), \\ G'_{D_2}(-1,z) = G'_{D_2}(1,z) = 0. \end{cases} \tag{7.9}$$

We can calculate explicitly

$$G_{D_2}(x,z) = \begin{cases} -\frac{\theta}{\sinh(2\theta)} \cosh[\theta(1+x)]\cosh[\theta(1-z)], \quad -1 < x < z < 1, \\ -\dfrac{\theta}{\sinh(2\theta)} \cosh[\theta(1-x)]\cosh[\theta(1+z)], \quad -1 < z < x < 1, \end{cases} \tag{7.10}$$

where

$$\theta_2 = \frac{1}{\sqrt{D_2}}. \tag{7.11}$$

We set

$$K_{D_2}(|x-z|) = -\frac{\theta_2}{2} e^{-\theta_2|x-z|} \tag{7.12}$$

to be the non-smooth part of $G_{D_2}(x,z)$, and we define the regular part H_{D_2} of G_{D_2} by $H_{D_2} = G_{D_2} - K_{D_2}$. Note that G_{D_2} is C^∞ for $(x,z) \in \Omega \times \Omega \setminus \{x=z\}$ and H_{D_2} is C^∞ for all $(x,z) \in \Omega \times \Omega$. Explicitly, we calculate

$$H_{D_2,xx}(0,0) = -\frac{\theta_2^3}{2} \coth\theta_2,$$

$$H_{D_2,xz}(0,0) = \frac{\theta_2^3}{2}\tanh\theta_2.$$

Note that in particular

$$G_{D_2,xx}(0,0) + G_{D_2,xz}(0,0) = -\frac{\theta_2^3}{2}(\coth\theta_2 - \tanh\theta_2) < 0.$$

Closely related, let $G_{D_2,\tau\lambda}(x,z)$ be Green's function of

$$\begin{cases} D_2 G''_{D_2,\tau\lambda}(x,z) - (1+\tau\lambda)G_{D_2,\tau\lambda}(x,z) - \delta_z = 0 \quad \text{in } (-1,1), \\ G'_{D_2,\tau\lambda}(-1,z) = G'_{D_2,\tau\lambda}(1,z) = 0. \end{cases} \tag{7.13}$$

We can decompose $G_{D_2,\tau\lambda}(x,z)$ as follows

$$G_{D_2,\tau\lambda}(x,z) = K_{D_2}(|x-z|) - H_{D_2,\tau\lambda}(x,z), \tag{7.14}$$

where $H_{D_2,\tau\lambda}$ is the regular part of $G_{D_2,\tau\lambda}$.

An elementary computation shows that

$$|H_{D_2}(x,z) - H_{D_2,\tau\lambda}(x,z)| \leq C|\tau\lambda|$$

uniformly for all $(x,z) \in \Omega \times \Omega$.

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