STABILITY OF THE SADDLE SOLUTIONS FOR THE ALLEN-CAHN EQUATION

YONG LIU, KELEI WANG, AND JUNCHENG WEI

Abstract. We are concerned with the saddle solutions of the Allen-Cahn equation constructed by Cabré and Terra [5, 6] in $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$. These solutions vanish precisely on the Simons cone. The existence and uniqueness of saddle solution are shown in [5, 6, 7]. Regarding the stability, Schatzman [31] proved that the saddle solution is unstable for $m = 1$, Cabré [7] showed the instability for $m = 2, 3$ and stability for $m \geq 7$. This has left open the case of $m = 4, 5, 6$. In this paper we show that the saddle solutions are stable when $m = 4, 5, 6$, thereby confirming Cabré's conjecture in [7]. The conjecture that saddle solutions in dimensions $2m \geq 8$ should be global minimizers of the energy functional remains open.

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1. Introduction and statement of the main result

Allen-Cahn type equation is a model arising from the phase transition theory. In this paper, we will investigate the stability of the saddle solutions to the following Allen-Cahn equation:

$$-\Delta u = u - u^3, \text{ in } \mathbb{R}^n. \quad (1)$$

Properties of solutions for this equation have delicate dependence on the dimension n. In the simplest case $n = 1$, we know all the solutions, thanks to the phase plane analysis technique. In this case, (1) has a heteroclinic solution $H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$. It is monotone increasing and plays an important role in the De Giorgi conjecture. As we will see, this function also plays a role in our later analysis on the stability in higher dimensions. Recall that the De Giorgi conjecture states that monotone bounded solutions of (1) have to be one-dimensional if $n \leq 8$. This conjecture has been proved to be true in dimension $n = 2$(Ghoussoub-Gui [19]), $n = 3$ (Ambrosio-Cabré [3]). In dimension $4 \leq n \leq 8$, Savin [30] proved it under an additional limiting condition:

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

Counter examples in dimension $n \geq 9$ have been constructed by del Pino-Kowalczyk-Wei in [12] using Lyapunov-Schmidt reduction and by us in [26] using Jerison-Monneau program [22]. We also refer to [11, 13, 14, 15, 16, 17, 18] and the references therein for related results on this subject.

It is commonly accepted that the theory of Allen-Cahn equation has deep relations with the minimal surface theory. The above mentioned De Giorgi conjecture is such an example. Our result in this paper, which will be stated below, indeed also has analogy in the minimal surface theory. To explain this, let us recall some basic facts from the minimal surface theory. In $\mathbb{R}^8$, there is a famous minimal cone with one singularity at the origin which minimizes the area, called Simons cone. It is given explicitly by:

$$\left\{x_1^2+...+x_4^2=x_5^2+...+x_8^2\right\}.$$

The minimality of this cone is proved in [4] and this property is related to the regularity theory of minimal surfaces. More generally, if we consider the so-called Lawson's cone $(2\leq i\leq j)$

$$C_{i,j}:=\left\{(x,y)\in\mathbb{R}^i\oplus\mathbb{R}^j:|x|^2=\frac{i-1}{j-1}|y|^2\right\},$$

then for $i+j\leq 7$, $C_{i,j}$ is unstable minimal cone(Simons [33]). For $i+j\geq 8$, and $(i,j)\neq(2,6)$, $C_{i,j}$ are area minimizing, and $C_{2,6}$ is not area minimizing but it is one-sided minimizer. (See [2], [10], [24], [27] and [28]).

There are analogous objects as the cone $C_{m,m}$ in the theory of Allen-Cahn equation. They are the so-called saddle-shaped solutions, which are solutions in $\mathbb{R}^{2m}$ of (1) vanishing exactly on the cone $C_{m,m}$ (See [8, 20, 31] for discussion on the dimension 2 case, and Cabré-Terra [5, 6] and Cabré [7] for higher dimension case). We denote them by U_m. In this paper, we will simply call it saddle solution. It has been proved in [7] that these solutions are unique in the class of symmetric functions. Furthermore in [5, 6] it is proved that for $2\leq m\leq 3$, the saddle solution is unstable, while for $m\geq 7$, they are stable [7]. It is conjectured in [7] that for $m\geq 4$, U_m should be stable. In this paper, we confirm this conjecture and prove the following

Theorem 1. *The saddle solution U_m is stable for $m=4,5,6$.*

As a corollary, Theorem 1 together with the result of Cabré tells us that U_m is stable for $m\geq 4$ and unstable for $m\leq 3$.

We remark that actually U_m is conjectured to be a minimizer of the corresponding energy functional for all $m\geq 4$. But this seems to be difficult to prove at this moment.

Let us now briefly explain the main idea of the proof. We focus on the case of $m=4$. That is, saddle solution in $\mathbb{R}^8$.

Suppose u depends only on the variables $s:=\sqrt{x_1^2+...+x_m^2}$ and $t:=\sqrt{x_{m+1}^2+...+x_{2m}^2}$. Then (1) reduces to

$$-\partial_s^2u-\partial_t^2u-\frac{m-1}{s}\partial_su-\frac{m-1}{t}\partial_tu=u-u^3. \tag{2}$$

Throughout the paper, we use the notation

$$\Omega:=\{(s,t):s>t>0\},\ \Omega^*:=\{(s,t):s>|t|>0\}.$$

Then the saddle solution satisfies $U(s,t)=-U(t,s)$ and $U>0$ in Ω. We will use L to denote the linearized Allen-Cahn operator around U :

$$L\eta:=\Delta\eta-\left(3U^2-1\right)\eta.$$

By definition, U is stable if and only if:

$$\int_{\mathbb{R}^8}(\eta\cdot L\eta)\leq 0,\ \text{for any }\eta\in C_0^\infty\left(\mathbb{R}^8\right).$$

To prove Theorem 1, we would like to construct a positive function Φ satisfying

$$L\Phi \leq 0 \text{ in } \mathbb{R}^8. \tag{3}$$

It is known that the existence of such a supersolution implies the stability of U. We define

$$f := \left(\tanh\left(\frac{s}{t}\right)\frac{\sqrt{2}s}{\sqrt{s^2+t^2}} + \frac{1}{4.2}\left(1-e^{-\frac{s}{2t}}\right)\right)(s+t)^{-2.5},$$

$$h := -\left(\tanh\left(\frac{t}{s}\right)\frac{\sqrt{2}t}{\sqrt{s^2+t^2}} + \frac{1}{4.2}\left(1-e^{-\frac{t}{2s}}\right)\right)(s+t)^{-2.5}.$$

Then we set

$$\Phi := fU_s + hU_t + 0.00007\left(s^{-1.8}e^{-\frac{t}{3}} + t^{-1.8}e^{-\frac{s}{3}}\right).$$

Here U_s, U_t are the derivatives of U with respect to s and t. We will prove in Section 3 that Φ satisfies (3). The choice of f is governed by the Jacobi fields of the Simons cone, which are of the form $c_1(s+t)^{-2} + c_2(s+t)^{-3}$. Note that $2.5 \in (2,3)$. We also point out that the admissable constants choosen here are not unique.

The key ingredients of our proof are some estimates of the first and second derivatives of U, obtained in the next section. These estimates are partly inspired by the explicit saddle solution in the plane of the elliptic sine-Gordon equation

$$-\Delta u = \sin u. \tag{4}$$

The double well potential of this equation is $1+\cos u$. It can be checked(see [25]) that the function

$$4\arctan\left(\frac{\cosh\left(\frac{y}{\sqrt{2}}\right)}{\cosh\left(\frac{x}{\sqrt{2}}\right)}\right) - \pi$$

is a saddle solution to (4). However, in dimension $2m$ with $m > 1$, (we believe)the saddle solution of (4) does not have explicit formula. More generally, one may conjecture that the saddle solution in dimension 8 is stable for general Allen-Cahn type equations of the form

$$\Delta u = F'(u),$$

where F is a double well potential. However, as we shall see later on in this paper, it seems that the stability of the saddle solution will also depend on the nonlinearity F.(At least, our computations have used the explicitly formula of the one dimensional heteroclinic solution)

This paper is organized as follows. In Section 2, we obtain some point-wise estimates for the derivatives of U in dimension 8. The key will be the estimate of $u_s + u_t$. In Section 3, we use these estimates to show that Φ is a supersolution of the linearized operator in dimension 8. In Section 4, we briefly discuss the case of dimensions 10 and 12.

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2. Estimates for the saddle solution and its derivatives in $\mathbb{R}^8$.

In this section, we analyze the saddle solution in dimension 8. That of dimension 10 and 12 follows from straightforward modifications.

One of the main difficulties in the proof of the stability stems from the fact that we don't have an explicit formula for the saddle solution. Hence we need to estimate U_4 and its derivatives. This will be the main aim of this section.

To begin with, we would like to control U_4 from below and above. For this purpose, we shall construct suitable sub and super solutions.

Recall that U_1 is the saddle solution in dimension 2. Let

$$s = \sqrt{x_1^2 + ... + x_4^2}, t = \sqrt{x_5^2 + ... + x_8^2}.$$

Then the function $U_1(s,t)$ satisfies

$$-\partial_s^2 U_1 - \partial_t^2 U_1 = U_1 - U_1^3. \tag{5}$$

In the rest of the paper, Δ will represent the Laplacian operator in dimension 8. That is, in the (s,t) coordinate,

$$\Delta := \partial_s^2 + \partial_t^2 + \frac{3}{s}\partial_s + \frac{3}{t}\partial_t.$$

Let $H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$ be the one dimensional heteroclinic solution:

$$-H'' = H - H^3.$$

It has the following expansion:

$$H(x) = 1 - 2e^{-\sqrt{2}x} + O\left(e^{-2\sqrt{2}x}\right), \text{ for } x \text{ large}.$$

Moreover, $\sqrt{2}H' = 1 - H^2$.

For simplicity, in the rest of this section, we will also write U_4 as u. Recall that bounded solutions of Allen-Cahn equation satisfy the Modica estimate:

$$\frac{1}{2}|\nabla u|^2 \leq F(u) := \frac{(1-u^2)^2}{4}. \tag{6}$$

This inequality will be used frequently later on in our analysis. Note that it provides an upper bound for the gradient. The lower bound of the gradient turns out to be much more delicate. Nevertheless, we will prove in this section that

$$u_s u + u_{ss} \geq 0 \text{ in } \Omega. \tag{7}$$

Let $d(s,t) := \frac{1}{2}u^2 + u_s$. Then inequality (7) implies $\partial_s d \geq 0$ and hence

$$d(s,t) \geq d(t,t), \text{ in } \Omega.$$

This inequality will give us a lower bound of u_s, provided we have some information of $d(t,t)$. The proof of (7) is quite nontrivial and requires many delicate estimates. It will be one of the main contents in this section.

Following Cabré [7], we introduce the new variables

$$y=\frac{s+t}{\sqrt{2}},z=\frac{s-t}{\sqrt{2}}.$$

Then the Allen-Cahn equation has the form:

$$-\partial_y^2 u-\partial_z^2 u-\frac{6}{y^2-z^2}\left(y\partial_y u-z\partial_z u\right)=u-u^3. \tag{8}$$

The estimates obtained in this paper rely crucially on the following maximum principle, due to Cabré [7].

Theorem 2 (Proposition 2.2 of [7])**.** *Suppose $c\geq 0$ in Ω. Then the maximum principle holds for the operator $L-c$.*

It is known that $u_s>0$ and $u_t<0$ in Ω. Moreover, based on this maximum principle, it is proved in [7] that $u_{st}\geq 0, u_{tt}\leq 0$ in Ω. But u_{ss} will change sign in Ω. Indeed, u_{ss} is positive near the origin and y axis. But we don't know the precise region where u_{ss} is positive. Here we point out that the estimate of the upper bound of $|u_{tt}|$ near the s axis is the most difficult one.

Differentiating equation (2) with respect to s and t, we obtain

$$Lu_s=\frac{3}{s^2}u_s,\ Lu_t=\frac{3}{t^2}u_t. \tag{9}$$

Lemma 3. *In Ω, U_1 satisfies*

$$t\partial_s U_1+s\partial_t U_1\leq 0.$$

Proof. Consider the linearized operator around U_1 :

$$P\phi:=\partial_s^2\phi+\partial_t^2\phi+\left(1-3U_1^2\right)\phi.$$

We compute $P\left(\partial_s U_1\right)=0, P\left(\partial_t U_1\right)=0$. Moreover,

$$P\left(t\partial_s U_1\right)=2\partial_s\partial_t U_1,\ \text{and } P\left(s\partial_t U_1\right)=2\partial_s\partial_t U_1.$$

Hence, using the fact that $\partial_s\partial_t U_1\geq 0$ in Ω, we get

$$P\left(t\partial_s U_1+s\partial_t U_1\right)\geq 0.$$

By the maximum principle(Theorem 2), and the boundary condition:

$$t\partial_s U_1+s\partial_t U_1=0,\ \text{on } \partial\Omega,$$

we get

$$t\partial_s U_1+s\partial_t U_1\leq 0,\ \text{in } \Omega.$$

This completes the proof. □

The next result is more or less standard.

Lemma 4. *The functions $H(y)H(z)$ and $U_1(s,t)$ are super solutions of U_4 in $\mathbb{R}^8$. Consequently, $U_4(s,t)\leq U_1(s,t)\leq H(y)H(z)$ in Ω.*

Proof. We first compute

$$\begin{aligned}
&-\Delta\left(H\left(y\right)H\left(z\right)\right)+\left(H\left(y\right)H\left(z\right)\right)^{3}-H\left(y\right)H\left(z\right)\\
&=-H''\left(y\right)H\left(z\right)-H\left(y\right)H''\left(z\right)+\left(H\left(y\right)H\left(z\right)\right)^{3}-H\left(y\right)H\left(z\right)\\
&-\frac{6yH'\left(y\right)H\left(z\right)-6zH\left(y\right)H'\left(z\right)}{y^{2}-z^{2}}\\
&=H\left(y\right)H\left(z\right)\left(1-H\left(y\right)^{2}\right)\left(1-H\left(z\right)^{2}\right)\\
&-\frac{6yH'\left(y\right)H\left(z\right)-6zH\left(y\right)H'\left(z\right)}{y^{2}-z^{2}}.
\end{aligned}$$

Note that when $y>0$, the function $\frac{yH'(y)}{H(y)}$ is monotone decreasing. It follows that

$$yH'\left(y\right)H\left(z\right)-zH\left(y\right)H'\left(z\right)\leq 0,\ \text{if } y\geq z.$$

This in turn implies that

$$-\Delta\left(H\left(y\right)H\left(z\right)\right)+\left(H\left(y\right)H\left(z\right)\right)^{3}-H\left(y\right)H\left(z\right)\geq 0.$$

Therefore $H\left(y\right)H\left(z\right)$ is a supersolution.

Next, by Lemma 3, we have

$$\begin{aligned}
&-\Delta\left(U_{1}\left(s,t\right)\right)-U_{1}\left(s,t\right)+U_{1}^{3}\left(s,t\right)\\
&=-\frac{3}{s}\partial_{s}U_{1}-\frac{3}{t}\partial_{t}U_{1}\geq 0.
\end{aligned}$$

Hence U_1 is also a supersolution of U_4, in Ω. Indeed, the fact that $U_1\left(y,z\right)\leq H\left(y\right)H\left(z\right)$ has already been proved in [31]. □

Note that although U_1 is a supersolution, we still don't have explicit formula for U_1. On the other hand, using $H\left(y\right)H\left(z\right)$, the upper bound near the origin can be improved by iterating the solution once. Indeed, after some tedious computation, we can show that u is bounded from above near the origin by $0.434yz$. Note that for y,z small, the supersolution $H\left(y\right)H\left(z\right)\sim 0.5yz$.

We remark that in Ω, the saddle solution is not concave. However, we conjecture that its level lines should be convex. But we don't know how to prove it at this moment.

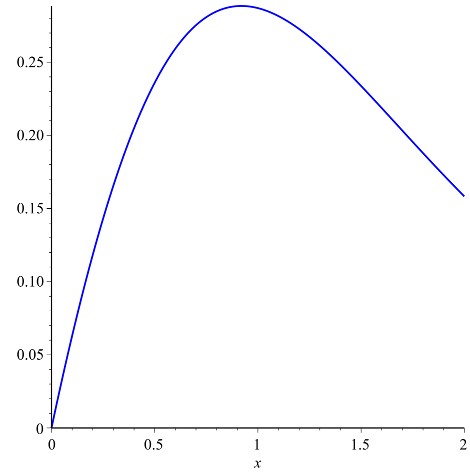
Next, we want to find (explicit) subsolutions of u. In $\mathbb{R}^2$, it is known[31] that the function $H\left(\frac{y}{\sqrt{2}}\right)H\left(\frac{z}{\sqrt{2}}\right)$ is a subsolution of U_1. In higher dimensions, the construction of (explicit) subsolutions are more delicate. We have the following

Lemma 5. *For $a\in(0,0.45)$, the function $H\left(ay\right)H\left(az\right)$ is a subsolution of u.*

Proof. Let us denote $H\left(ay\right)H\left(az\right)$ by η and write $\tilde{y}=ay,\tilde{z}=az$. Then $-\Delta\eta-\eta+\eta^{3}$ is equal to

$$\begin{aligned}
&H\left(\tilde{y}\right)H\left(\tilde{z}\right)\left(2a^{2}-1-a^{2}H^{2}\left(\tilde{y}\right)-a^{2}H^{2}\left(\tilde{z}\right)+H^{2}\left(\tilde{y}\right)H^{2}\left(\tilde{z}\right)\right)\\
&-\frac{3a^{2}\sqrt{2}}{\tilde{y}^{2}-\tilde{z}^{2}}\left(\tilde{y}H\left(\tilde{z}\right)-\tilde{z}\tilde{H}\left(\tilde{y}\right)+H\left(\tilde{y}\right)H\left(\tilde{z}\right)\left(\tilde{z}H\left(\tilde{z}\right)-\tilde{y}H\left(\tilde{y}\right)\right)\right).
\end{aligned}$$

Figure 1. ρ



This is an explicit function of the variables $\tilde{y}$ and $\tilde{z}$. One can verify directly that it is negative when $a \in (0, 0.45)$. □

We remark that this subsolution is not optimal, especially regarding the decaying rate away from the Simons cone. In the sequel, we shall write the supersolution $H(y)H(z)$ as u^*. We also set $\phi = u^* - u \geq 0$. To estimate ϕ, we introduce the function

$$\rho(z) = H'(z) \int_0^z \left(H'^{-2} \int_s^{+\infty} H'^2 \right) ds.$$

Note that the function ρ can be explicitly written down and it satisfies

$$\rho'' - \left(3H^2 - 1\right)\rho = -H'.$$

Lemma 6. *In* Ω, *we have:*

$$\phi \leq \frac{4H(y)(H(z) + zH'(z))}{y^2 - z^2}.$$

Moreover,

$$\phi \leq \frac{5}{4}\left(\frac{1}{t} - \frac{1}{s}\right) H(y)\rho(z), \text{ for } z > 1. \tag{10}$$

Proof. The function ϕ satisfies

$$-\Delta\phi + \left(3u^2 - 1\right)\phi = -3u\phi^2 - \phi^3 - \Delta u^* + u^{*3} - u^*.$$

Recall that u^* is a supersolution and we have

$$\begin{aligned} -\Delta u^* + u^{*3} - u^* &= H(y)H(z)\left(1 - H(y)^2\right)\left(1 - H(z)^2\right) \\ &\quad - \frac{6yH'(y)H(z)}{y^2 - z^2} + \frac{6zH(y)H'(z)}{y^2 - z^2}. \end{aligned} \tag{11}$$

Let $g(z) = \frac{1}{2}(H(z) + zH'(z))$. We compute

$$\Delta\left(\frac{g(z)}{y^2 - z^2}\right) = \left(\frac{6y^2 + 2z^2}{(y^2 - z^2)^3} - \frac{6y}{y^2 - z^2}\frac{2y}{(y^2 - z^2)^2}\right)g$$
$$+ \left(\frac{6z^2 + 2y^2}{(y^2 - z^2)^3} - \frac{6z}{y^2 - z^2}\frac{2z}{y^2 - z^2}\right)g$$
$$+ 2\frac{2z}{y^2 - z^2}\frac{1}{y^2 - z^2}g' - \frac{6z}{y^2 - z^2}\frac{1}{y^2 - z^2}g' + \frac{g''}{y^2 - z^2}.$$

The left hand side is equal to

$$\frac{-4y^2 - 4z^2}{(y^2 - z^2)^3}g - \frac{2z}{(y^2 - z^2)^2}g' + \frac{g''}{y^2 - z^2}.$$

It follows that

$$\Delta\left(\frac{H(y)}{y^2 - z^2}g(z)\right) = \frac{-4y^2 - 4z^2}{(y^2 - z^2)^3}H(y)g - \frac{2z}{(y^2 - z^2)^2}H(y)g' - \frac{H(y)H(z)}{y^2 - z^2}$$
$$+ \left(H''(y) + \frac{6yH'(y)}{y^2 - z^2}\right)\frac{g}{y^2 - z^2} - \frac{4y}{(y^2 - z^2)^2}H'(y)g$$
$$= \left(\frac{-4y^2 - 4z^2}{(y^2 - z^2)^3}H(y) - \frac{\sqrt{2}H(y)H'(y)}{y^2 - z^2} + \frac{2yH'(y)}{(y^2 - z^2)^2}\right)g$$
$$- \frac{2z}{y^2 - z^2}H(y)g' + \frac{H(y)g''}{y^2 - z^2}.$$

Let a be a constant to be determined later on, we have

$$L\left(\frac{aH(y)g}{y^2 - z^2} - \phi\right) \leq -\frac{aH(y)H(z)}{y^2 - z^2} - \Delta u^* - u^* + u^{*3}$$
$$+ \frac{3a(u^{*2} - u^2)}{y^2 - z^2}H(y)g$$
$$+ \left(\frac{-4y^2 - 4z^2}{(y^2 - z^2)^3}H(y) - \frac{\sqrt{2}H(y)H'(y)}{y^2 - z^2} + \frac{2yH'(y)}{(y^2 - z^2)^2}\right)ag$$
$$- \frac{2azH(y)}{(y^2 - z^2)^2}g' - 3u\phi^2 - \phi^3.$$

Let us denote $\frac{aH(y)g}{y^2 - z^2} - \phi$ by η. Note that for z close to 0,

$$H(y)H(z)\left(1 - H(y)^2\right)$$
$$\left(1 - H(z)^2\right) \leq \frac{6yH'(y)H(z)}{y^2 - z^2}.$$

If we choose $a = 8$, then $L\eta \leq 0$, in the region where $\eta \leq 0$. Hence by maximum principle,

$$\phi \leq \frac{4H(y)(H(z) + zH'(z))}{y^2 - z^2} \text{ in } \Omega. \tag{12}$$

It remains to prove (10). We first observe that due to (12), the inequality (10) is true for $z = 1$. Let $\tilde{\eta} = \left(\frac{1}{t} - \frac{1}{s}\right) H(y)\rho(z)$. We compute

$$\begin{aligned}
&L\tilde{\eta} + 3\left(u^2 - u^{*2}\right)\tilde{\eta} \\
&= \left(\frac{1}{s^3} - \frac{1}{t^3}\right) H(y)\rho + \left(\frac{\sqrt{2}}{s^2} - \frac{\sqrt{2}}{t^2}\right) H'(y)\rho \\
&+ \left(\frac{\sqrt{2}}{s^2} + \frac{\sqrt{2}}{t^2}\right) H(y)\rho' + \left(\frac{1}{t} - \frac{1}{s}\right)\left(-2H^3(y) + \frac{3\sqrt{2}}{2}\left(\frac{1}{s} + \frac{1}{t}\right) H'(y)\right)\rho \\
&+ \left(\frac{1}{t} - \frac{1}{s}\right)\left(\frac{3\sqrt{2}}{2}\left(\frac{1}{s} - \frac{1}{t}\right)\rho' - H'(z)\right) H(y) \\
&= \left(\frac{1}{s^3} - \frac{1}{t^3}\right) H(y)\rho + \frac{\sqrt{2}}{2}\left(\frac{1}{t^2} - \frac{1}{s^2}\right) H'(y)\rho \\
&+ \sqrt{2}\left(\frac{1}{t^2} + \frac{1}{s^2} - \frac{3}{2}\left(\frac{1}{t} - \frac{1}{s}\right)^2\right) H(y)\rho' \\
&+ \left(\frac{1}{t} - \frac{1}{s}\right)\left(-2H^3(y)\rho - H'(z) H(y)\right).
\end{aligned}$$

We can verify $L\left(\frac{5}{4}\tilde{\eta} - \phi\right)$ is negative in the region where $\frac{5}{4}\tilde{\eta} < \phi$. Hence by maximum principle, $\phi \leq \frac{5}{4}\tilde{\eta}$. This finishes the proof. □

With the estimate of ϕ at hand, we see from Modica estimate that

$$\begin{aligned}
u_y &\leq \frac{|\nabla u|}{\sqrt{2}} \leq \frac{1}{2}\left(1 - u^2\right) \\
&\leq \frac{1}{2}\left(1 + H(y) H(z)\right)\left(1 - H(y) H(z) + \phi\right) \\
&\leq \frac{1}{2}\left(1 - H(y)^2 H(z)^2\right) + \frac{3}{2}\frac{H(y)\left(1 + H(y) H(z)\right)\left(H(z) + zH'(z)\right)}{y^2 - z^2}.
\end{aligned}$$

In particular, as $y \to +\infty$, u_y decays at least like $O\left(y^{-2}\right)$. (Note that u_y decays exponentially fast away from the Simons cone). However, we expect that u_y decays like $O\left(y^{-3}\right)$. Let us consider

$$\rho_1(z) := H'(z)\int_0^z \left(H'^{-2}\int_s^{+\infty} tH'^2(t)\,dt\right) ds.$$

It satisfies

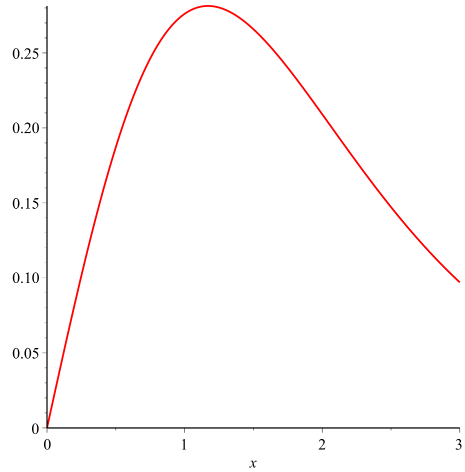
$$-\rho_1'' + \left(3H^2 - 1\right)\rho_1 = zH'(z).$$

Then intuitively, near the Simons cone, for y large,

$$u_y \sim \frac{12}{y^3}\rho_1(z).$$

However, it turns out to be quite delicate and difficult to get an explicit global bound of u_y with this decay rate. The estimate of u_y will be one of our main aim in this section.

FIGURE 2. ρ_1



Lemma 7. *In* Ω, $tu_s + su_t \leq 0$.

Proof. By monotonicity, we know that $u_s + u_t \geq 0$. Let us define

$$\eta := t^k u_s + s^k u_t.$$

We have

$$L\eta = u_s t^k \left(\frac{3}{s^2} + \frac{k^2 + 2k}{t^2} \right) + u_t s^k \left(\frac{3}{t^2} + \frac{k^2 + 2k}{s^2} \right) + 2k u_{st} \left(s^{k-1} + t^{k-1} \right).$$

We write this equation as

$$L\eta - \left(\frac{3}{t^2} + \frac{k^2 + 2k}{s^2} \right) g = u_s t^k \left(\frac{3}{s^2} + \frac{k^2 + 2k}{t^2} - \left(\frac{3}{t^2} + \frac{k^2 + 2k}{s^2} \right) \right) + 2k u_{st} \left(s^{k-1} + t^{k-1} \right).$$

Suppose $k \geq 1$. Using the fact that $u_s \geq 0$ and $u_{st} \geq 0$, we obtain

$$L\eta - \left(\frac{3}{t^2} + \frac{k^2 + 2k}{s^2} \right) \eta \geq 0.$$

Hence from the maximum principle,

$$t^k u_s + s^k u_t \leq 0 \text{ in } \Omega, \text{ if } k \geq 1.$$

□

The above lemma in particular gives us a lower bound of $|u_t|$ in terms of u_s. That is,

$$|u_t| \geq \frac{t}{s} u_s \text{ in } \Omega.$$

We also note that this inequality implies that $u_s + u_t$ has the following decaying property:

(13) $$u_s + u_t \leq \frac{z}{y+z} u_s.$$

Lemma 8. *In Ω, we have*

$$u_s \leq 2\left(e^{0.85t} + \frac{4.9}{\sqrt{t}}\right) e^{-0.85s}.$$

Proof. u_s satisfies

$$Lu_s - \frac{3u_s}{s^2} = 0.$$

Let a, b be parameters to be determined and $\eta = \left(e^{at} + \frac{b}{\sqrt{t}}\right) e^{-as}$. We have

$$\begin{aligned} L\eta - \frac{3\eta}{s^2} = &\left(2a^2 + \frac{3a}{t} - \frac{3a}{s} + 1 - 3u^2 - \frac{3}{s^2}\right) e^{a(t-s)} \\ &- \frac{3b}{4t^{\frac{5}{2}}} e^{-as} + \frac{b}{\sqrt{t}}\left(a^2 - \frac{3a}{s}\right) e^{-as} \\ &+ \left(1 - 3u^2 - \frac{3}{s^2}\right) \frac{b}{\sqrt{t}} e^{-as}. \end{aligned}$$

Note that in the region $z > 2$, u is close to 1. More precisely,

$$u \geq \max\left\{H(0.45y) H(0.45z), u^* - \phi\right\}.$$

Let $\Gamma := \Omega \cap \{z > 2\}$. Choose $a = \frac{3.4}{4}$, $b = 4.9$. We then have

$$L\eta - \frac{3\eta}{s^2} \leq 0, \text{ in } \Gamma.$$

Note that on $\partial\Gamma$, $u_s \leq 2\eta$. The desired estimate then follows from the maximum principle. □

Lemma 9. $\frac{u_s}{s} - u_{ss} \geq 0$ *in* Ω^*.

Proof. Let $\eta = u_s s^{-1} - u_{ss}$. Then by (22), $\eta \geq 0$ on $\partial\Omega^*$. We compute

$$L\eta = \frac{8}{s^3} u_s - \frac{8u_{ss}}{s^2} - 6u_s^2 u.$$

Hence

$$L\eta - \frac{8}{s^2}\eta \leq 0.$$

By maximum principle, $\eta \geq 0$. □

Lemma 10. *In Ω^*, we have*

$$\frac{u_s}{s} + \frac{u_t}{t} - u_{ss} - u_{tt} \geq 0.$$

Proof. Let $\eta = \frac{u_s}{s} + \frac{u_t}{t} - u_{ss} - u_{tt}$. Then $\eta = 0$ on $\partial\Omega^*$. We have

$$\begin{aligned} L\eta &= \frac{8u_s}{s^3} + \frac{8u_t}{t^3} - \frac{8u_{ss}}{s^2} - \frac{8u_{tt}}{t^2} - 6u\left(u_s^2 + u_t^2\right) \\ &= \frac{8}{t^2}\eta + \left(\frac{8}{t^2} - \frac{8}{s^2}\right)\left(u_{ss} - \frac{u_s}{s}\right) - 6u\left(u_s^2 + u_t^2\right). \end{aligned}$$

FIGURE 3. $z/H(z)$

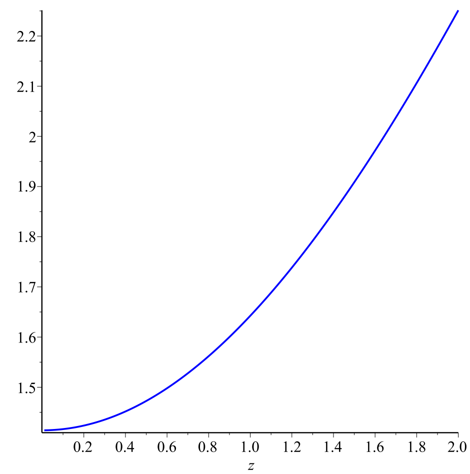
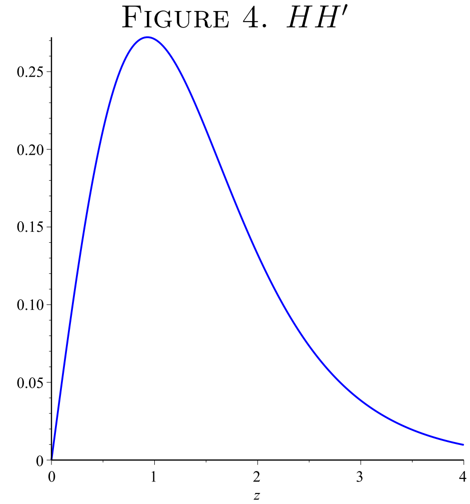


FIGURE 4. HH'



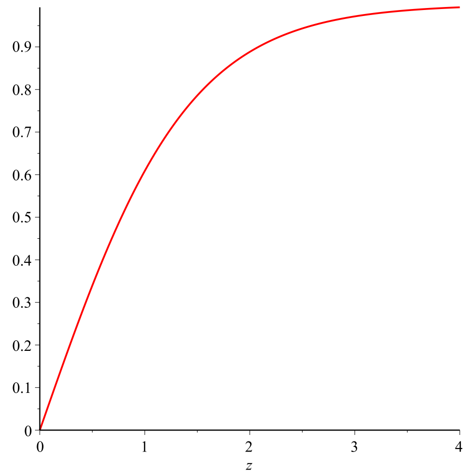
Using the fact that $u_{ss} - \frac{u_s}{s} \leq 0$, we find that $L\eta - \frac{8}{t^2}\eta \leq 0$. Then by maximum principle, $\eta \geq 0$. □

We know that for y large, u_y decays like $O\left(y^{-3}\right)$. However, to estimate the second derivatives of u, we need to have some explicit global estimate of $u_s + u_t$ in Ω. We shall prove that $u_s + u_t$ can be bounded by functions of the form

$$\left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left(au_z + bu_z^{\alpha}\right),$$

with suitable constants a, b, α. We would like to prove the following(non-optimal)

FIGURE 5. H



Proposition 11. *In Ω, we have*

$$u_s + u_t \le \left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left(2\left(u_s - u_t\right) + \sqrt{u_s - u_t}\right). \tag{14}$$

Proof. $u_s + u_t$ satisfies

$$L\left(u_s + u_t\right) = \frac{3u_s}{s^2} + \frac{3u_t}{t^2}.$$

We write it in the form:

$$L\left(u_s + u_t\right) - \frac{3}{2}\left(\frac{1}{s^2} + \frac{1}{t^2}\right)\left(u_s + u_t\right) = \frac{3}{2}\left(\frac{1}{s^2} - \frac{1}{t^2}\right)\left(u_s - u_t\right).$$

Let $\alpha \in [0, 1]$ be a parameter. Define the function

$$\eta_\alpha = \left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left(u_s - u_t\right)^\alpha.$$

Using the fact that $\Delta\left(t^{-2}\right) = \Delta\left(s^{-2}\right) = 0$, we get

$$\begin{aligned} L\eta_\alpha = {} & \alpha\left(u_s - u_t\right)^{\alpha-1}\left(\frac{4}{s^3}\left(u_{ss} - u_{st}\right) - \frac{4}{t^3}\left(u_{st} - u_{tt}\right)\right) \\ & + \left(\frac{1}{t^2} - \frac{1}{s^2}\right) L\left(\left(u_s - u_t\right)^\alpha\right). \end{aligned} \tag{15}$$

We compute,

$$\begin{aligned} L\left(\left(u_s - u_t\right)^\alpha\right) = {} & \left(u_s - u_t\right)^{\alpha-1}\left(\frac{3\alpha u_s}{s^2} - \frac{3\alpha u_t}{t^2} + (1-\alpha)\left(1 - 3u^2\right)\left(u_s - u_t\right)\right) \\ & + \alpha(\alpha - 1)\left(u_s - u_t\right)^{\alpha-2}\left|\nabla\left(u_s - u_t\right)\right|^2. \end{aligned}$$

Note that the term $\frac{3\alpha u_s}{s^2}-\frac{3\alpha u_t}{t^2}$ is positive. However, for $\alpha\in[0,1]$, we also have,

$$\begin{aligned}
&\frac{3\alpha u_s}{s^2}-\frac{3\alpha u_t}{t^2}-\frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)(u_s-u_t)\\
&=\frac{3(\alpha-1)}{2}\left(\frac{1}{t^2}+\frac{1}{s^2}\right)(u_s-u_t)-\frac{3\alpha}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s+u_t)\\
;&\leq-\frac{3\alpha}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s+u_t)\leq 0.
\end{aligned}$$

¿From this, we know that intuitively, for $\alpha=1$, $L\eta_\alpha-\frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)\eta_\alpha$ is negative near the Simons cone; while for $0<\alpha<1$, it is negative away from the Simons cone. In view of this, we consider a combination of η_1 and η_α, where $\alpha=\frac{1}{2}$(this choice may not be optimal). That is,

$$\begin{aligned}
h&:=\eta_1+\frac{1}{2}\eta_{\frac{1}{2}}\\
&=\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(u_s-u_t+\frac{1}{2}(u_s-u_t)^\alpha\right).
\end{aligned}$$

We then set $h^*:=h-(u_s+u_t)$. Observe that $h^*\geq 0$ on $\partial\Omega$. We would like to use the maximum principle to show $h^*\geq 0$ in Ω.

Using (15), we compute

$$\begin{aligned}
&Lh^*-\frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)h^*\\
&=\left(1+\frac{1}{2}\alpha(u_s-u_t)^{\alpha-1}\right)\left(\frac{4}{s^3}(u_{ss}-u_{st})-\frac{4}{t^3}(u_{st}-u_{tt})\right)\\
&\quad+\left(\frac{1}{t^2}-\frac{1}{s^2}\right)L\left(u_s-u_t+\frac{1}{2}(u_s-u_t)^\alpha\right)\\
&\quad-\frac{3}{2}\left(\frac{1}{t^2}+\frac{1}{s^2}\right)h+\frac{3}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s-u_t).
\end{aligned}$$

We first observe that

$$\begin{aligned}
&\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s-u_t)^{\alpha-1}\left(\frac{3\alpha u_s}{s^2}-\frac{3\alpha u_t}{t^2}\right)-\frac{3}{2}\left(\frac{1}{t^2}+\frac{1}{s^2}\right)\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s-u_t)^\alpha\\
&=3\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s-u_t)^{\alpha-1}\left(\frac{\alpha u_s}{s^2}-\frac{\alpha u_t}{t^2}-\frac{1}{2}\left(\frac{1}{t^2}+\frac{1}{s^2}\right)(u_s-u_t)\right)\\
&=3\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_s-u_t)^{\alpha-1}\left((\alpha-1)\left(\frac{u_s}{s^2}-\frac{u_t}{t^2}\right)+\frac{1}{2}\left(\frac{1}{s^2}-\frac{1}{t^2}\right)(u_s+u_t)\right).
\end{aligned}$$

Let us denote the right hand side by I. We know that $I \leq 0$. It follows that

$$
\begin{aligned}
& Lh^* - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)h^* \\
& \leq \left(1+\alpha\left(u_s-u_t\right)^{\alpha-1}\right)\left(\frac{4}{s^3}\left(u_{ss}-u_{st}\right)-\frac{4}{t^3}\left(u_{st}-u_{tt}\right)\right) \\
& +(1-\alpha)\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(\left(u_s-u_t\right)^{\alpha}\left(1-3u^2\right)-\alpha\left(u_s-u_t\right)^{\alpha-2}\left|\nabla\left(u_s-u_t\right)\right|^2\right) \\
& +\frac{3}{2}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(u_s-u_t\right)+\frac{1}{2}I. \qquad (16)
\end{aligned}
$$

Suppose to the contrary that the inequality (14) was not true. Consider the region Γ where $u_s + u_t > h$. Now we would like to show that in Γ, $Lh^* - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)h^* \leq 0$. We have

$$
\begin{aligned}
& \frac{1}{s^3}\left(u_{ss}-u_{st}\right)-\frac{1}{t^3}\left(u_{st}-u_{tt}\right) \\
& = \frac{u_{ss}}{s^3}+\frac{u_{tt}}{t^3}-\left(\frac{1}{s^3}+\frac{1}{t^3}\right)u_{st} \\
& = \frac{u^3-u}{s^3}-\frac{3}{s^3}\left(\frac{u_s}{s}+\frac{u_t}{t}\right)+\left(\frac{1}{t^3}-\frac{1}{s^3}\right)u_{tt}-\left(\frac{1}{s^3}+\frac{1}{t^3}\right)u_{st}.
\end{aligned}
$$

In particular, since $u_{tt} < 0$ and $u_{st} > 0$,

$$
\frac{1}{s^3}\left(u_{ss}-u_{st}\right)-\frac{1}{t^3}\left(u_{st}-u_{tt}\right) \leq \frac{u^3-u}{s^3}-\frac{3}{s^3}\left(\frac{u_s}{s}+\frac{u_t}{t}\right). \qquad (17)
$$

On the other hand,

$$
\begin{aligned}
\left|\nabla\left(u_s-u_t\right)\right|^2 &= \left(u_{ss}-u_{st}\right)^2+\left(u_{st}-u_{tt}\right)^2 \\
&\geq \frac{1}{2}\left(u_{ss}+u_{tt}-2u_{st}\right)^2 \\
&\geq \frac{1}{2}\left(u^3-u-3\left(\frac{u_s}{s}+\frac{u_t}{t}\right)\right)^2. \qquad (18)
\end{aligned}
$$

Inserting estimates (17), (18) into (16), and using the fact that

$$
1-u^2 \geq u_s-u_t,
$$

we conclude that

$$
Lh^* - \frac{3}{2}\left(\frac{1}{s^2}+\frac{1}{t^2}\right)h^* \leq 0, \text{ in } \Gamma.
$$

By maximum principle, $h^* \geq 0$. The proof is then completed. □

We remark that the estimates in the previous Proposition is not optimal, since we have not used the information of u_{st}, which is, intuitively, of the order $-\frac{1}{2}\left(u_{ss}+u_{tt}\right)$.

Lemma 12. *u satisfies*

$$
u - u^3 + u_{ss} \geq 0 \text{ in } \Omega. \qquad (19)
$$

Proof. This inequality follows directly from the equation

$$u_{ss} + u_{tt} + \frac{3}{s} u_s + \frac{3}{t} u_t + u - u^3 = 0,$$

and the fact that in Ω,

$$\frac{3}{s} u_s + \frac{3}{t} u_t \leq 0, \ u_{tt} \leq 0.$$

Here we give another proof using the maximum principle. We have

$$\Delta u^\sigma = \sigma u^{\sigma-1} \Delta u + \sigma (\sigma - 1) u^{\sigma-2} |\nabla u|^2,$$

which implies

$$\begin{aligned} L u^\sigma &= \sigma u^{\sigma-1} \left(u^3 - u\right) + \left(1 - 3u^2\right) u^\sigma + \sigma (\sigma - 1) u^{\sigma-2} |\nabla u|^2 \\ &= (1 - \sigma) u^\sigma + (\sigma - 3) u^{\sigma+2} + \sigma (\sigma - 1) u^{\sigma-2} |\nabla u|^2 . \end{aligned}$$

In particular,

$$Lu = -2u^3, Lu^3 = -2u^3 + 6u |\nabla u|^2 .$$

Hence

$$L \left(u - u^3\right) = -6u |\nabla u|^2 . \tag{20}$$

Define $\eta = u - u^3 + u_{ss}$. Applying (20) and (24), we get

$$L\eta = -6u |\nabla u|^2 + \frac{6}{s^2} u_{ss} - \frac{6}{s^3} u_s + 6u_s^2 u.$$

Therefore, in Ω^*,

$$L\eta - \frac{6}{s^2}\eta = -\frac{6}{s^2} \left(u - u^3\right) - \frac{6}{s^3} u_s - 6u_t^2 u \leq 0.$$

By the maximum principle, $\eta \geq 0$. This finishes the proof. □

It turns out that the estimate of Lemma 12 is also not optimal. Indeed, u_{ss} can be estimated by u_s. This is the following

Lemma 13. *u satisfies*

$$\sqrt{2} u_s u + u_{ss} \geq 0, \ \textit{in } \Omega.$$

Proof. We compute

$$L (u_s u) = \frac{3}{s^2} u_s u + 2u_{ss} u_s + 2u_{st} u_t + u_s \left(u^3 - u\right).$$

Let $\eta = a u_s u + u_{ss}$, where $a > 0$ is a constant to be chosen later on. We have

$$\begin{aligned} L\eta = &\frac{3a}{s^2} u_s u + 2a u_{ss} u_s + 2a u_{st} u_t + a u_s \left(u^3 - u\right) \\ &+ \frac{6}{s^2} u_{ss} - \frac{6}{s^3} u_s + 6u_s^2 u. \end{aligned}$$

We can write it in the form

$$L\eta = \left(2au_s + \frac{6}{s^2}\right)\eta + \frac{3a}{s^2}u_s u + 2au_{st}u_t + au_s\left(u^3 - u\right) - \frac{6}{s^3}u_s + 6u_s^2 u - \left(2au_s + \frac{6}{s^2}\right)(au_s u).$$

That is,

$$L\eta - \left(2au_s + \frac{6}{s^2}\right)\eta = -\frac{3a}{s^2}u_s u + 2au_{st}u_t - \frac{6}{s^3}u_s + \left(6 - 2a^2\right)u_s^2 u + au_s\left(u^3 - u\right).$$

By Modica estimate, $1 - u^2 \geq \sqrt{2}\,|\nabla u| \geq \sqrt{2}u_s$. Hence

$$L\eta - \left(2au_s + \frac{6}{s^2}\right)\eta \leq \left(6 - 2a^2\right)u_s^2 u - \sqrt{2}au_s^2 u = \left(6 - 2a^2 - \sqrt{2}a\right)u_s^2 u.$$

In particular, if we choose a to be $\sqrt{2}$, then $L\eta - \left(2au_s + \frac{6}{s^2}\right)\eta \leq 0$. In this case, by the maximum principle, $\eta \geq 0$. Hence

$$\sqrt{2}u_s u + u_{ss} \geq 0.$$

□

We remark that this estimate together with Modica estimate implies (19).

Lemma 14. *We have*

$$\sqrt{2}u_t u + u_{st} \leq 0 \text{ in } \Omega.$$

Proof. Let $\eta = \sqrt{2}u_t u + u_{st}$. Then

$$L\eta = \frac{3\sqrt{2}}{t^2}u_t u + 2\sqrt{2}u_{st}u_s + 2\sqrt{2}u_{tt}u_t + \sqrt{2}u_t\left(u^3 - u\right) + \left(\frac{3}{t^2} + \frac{3}{s^2}\right)u_{st} + 6u_s u_t u.$$

This can be written as

$$L\eta - \left(2\sqrt{2}u_s + \frac{3}{t^2} + \frac{3}{s^2}\right)\eta = \frac{3\sqrt{2}}{t^2}u_t u + 2\sqrt{2}u_{tt}u_t + \sqrt{2}u_t\left(u^3 - u\right) + 6u_s u_t u - \sqrt{2}u_t u\left(2\sqrt{2}u_s + \frac{3}{t^2} + \frac{3}{s^2}\right).$$

Since $u_{tt} \leq 0$ and $u_t \leq 0$ in Ω, we get

$$L\eta - \left(2\sqrt{2}u_s + \frac{3}{t^2} + \frac{3}{s^2}\right)\eta \geq 0, \text{ in } \Omega.$$

By the maximum principle, $\sqrt{2}u_t u + u_{st} \leq 0$. □

Next we prove the following non-optimal estimate:

Lemma 15. *In* Ω, *we have*

$$2\left(u_s + u_t\right) + u_{st} + u_{ss} \geq 0.$$

Proof. Let $\eta = 2\left(u_s + u_t\right) + u_{st} + u_{ss}$. First of all, $\eta \geq 0$ on $\partial\Omega$. We compute

$$\begin{aligned} L\eta = &\frac{6}{s^2}u_s + \frac{6}{t^2}u_t + \left(\frac{3}{s^2} + \frac{3}{t^2}\right)u_{st} + 6u_s u_t u \\ &+ \frac{6}{s^2}u_{ss} - \frac{6}{s^3}u_s + 6u_s^2 u \end{aligned}$$

$$\begin{aligned} L\eta - \frac{6}{s^2}\eta = &-\frac{6}{s^2}\left(u_s + u_t\right) + \left(\frac{3}{t^2} - \frac{3}{s^2}\right)\left(2u_t + u_{st}\right) - \frac{6}{s^3}u_s \\ &+ 6u_s u\left(u_s + u_t\right). \end{aligned}$$

Using Lemma 14, we obtain

$$2u_t + u_{st} \leq \left(2 - \sqrt{2}u\right)u_t.$$

Hence

$$\begin{aligned} L\eta - \frac{6}{s^2}\eta \leq &\left(6u_s u - \frac{6}{s^2}\right)\left(u_s + u_t\right) - \frac{6}{s^3}u_s \\ &+ \left(2 - \sqrt{2}u\right)\left(\frac{3}{t^2} - \frac{3}{s^2}\right)u_t. \end{aligned}$$

Applying the estimate of $u_s + u_t$(Proposition 11), we see that $L\eta - \frac{6}{s^2}\eta \leq 0$. It then follows from the maximum principle that $\eta \geq 0$. □

Lemma 16. *In* Ω, *we have*

$$2\left(u_s + u_t\right) - u_{st} - u_{tt} \geq 0.$$

Proof. Let $\eta = 2\left(u_s + u_t\right) - u_{st} - u_{tt}$. First of all, since $u_{tt} \leq 0$ in Ω, we have $\eta \geq 0$ on $\partial\Omega$. We compute

$$\begin{aligned} L\eta = &\frac{6u_s}{s^2} + \frac{6u_t}{t^2} - \left(\frac{3}{s^2} + \frac{3}{t^2}\right)u_{st} - 6u_s u_t u \\ &- \frac{6}{t^2}u_{tt} + \frac{6}{t^3}u_t - 6u_t^2 u. \end{aligned}$$

Therefore,

$$\begin{aligned}
L\eta-\frac{6}{t^2}\eta &= \left(\frac{6}{s^2}-\frac{12}{t^2}\right)u_s-\frac{6u_t}{t^2}-\left(\frac{3}{s^2}-\frac{3}{t^2}\right)u_{st}+\frac{6}{t^3}u_t \\
&-6u_tu\left(u_s+u_t\right) \\
&= \left(\frac{6}{s^2}-\frac{12}{t^2}\right)\left(u_s+u_t\right)+\left(\frac{3}{t^2}-\frac{3}{s^2}\right)\left(2u_t+u_{st}\right) \\
&+\frac{6}{t^3}u_t-6u_tu\left(u_s+u_t\right) \\
&\leq \left(\frac{6}{s^2}-\frac{12}{t^2}-6u_tu\right)\left(u_s+u_t\right) \\
&+\frac{6}{t^3}u_t+\left(\frac{3}{t^2}-\frac{3}{s^2}\right)\left(2-\sqrt{2}u\right)u_t \\
&\leq 0.
\end{aligned}$$

It then follows from the maximum principle that $\eta\geq 0$. □

It follows immediately from these lemmas that

$$4\left(u_s+u_t\right)+u_{ss}-u_{tt}\geq 0.$$

We conjecture that

$$\begin{aligned}
u\left(u_s+u_t\right)+u_{ss}+u_{st}&\geq 0,\\
u\left(u_s+u_t\right)-u_{st}-u_{tt}&\geq 0.
\end{aligned}$$

However, we are not able to prove them in this paper. The main difficulty here is the following: In $L\left(u\left(u_s+u_t\right)\right)$, we have u_{tt} term. Hence one needs to handle terms like $u_{st}+u_{tt}$(In particular in the region $t<1$). The lower bound of this term is not easy to derive, because $L\left(u_{tt}\right)=\frac{6}{t^2}u_{tt}-\frac{6}{t^3}u_t+6u_t^2u$ and $\frac{u_t}{t^3}$ blows up as $t\to 0$.

Lemma 17.

$$\frac{u}{y}+\frac{u}{z}-u_y-u_z\geq 0,\ \ in\ \Omega^*.$$

Proof. Let $\eta=\frac{u}{y}+\frac{u}{z}-u_y-u_z$. We first observe that $\eta=0$ on $\partial\Omega^*$. We have

$$\begin{aligned}
L\eta &= -\frac{2u^3}{y}-2\frac{u_y}{y^2}+u\left(\frac{2}{y^3}-\frac{6}{y\left(y^2-z^2\right)}\right) \\
&-\frac{2u^3}{z}-2\frac{u_z}{z^2}+u\left(\frac{2}{z^3}+\frac{6}{z\left(y^2-z^2\right)}\right) \\
&-\frac{6y^2+6z^2}{\left(y^2-z^2\right)^2}u_y+\frac{12yz}{\left(y^2-z^2\right)^2}u_z \\
&+\frac{12yz}{\left(y^2-z^2\right)^2}u_y-\frac{6y^2+6z^2}{\left(y^2-z^2\right)^2}u_z.
\end{aligned}$$

That is,

$$\begin{aligned} L\eta - \frac{6}{(y+z)^2}\eta &= -\frac{2u^3}{y} - 2\frac{u_y}{y^2} + u\left(\frac{2}{y^3} - \frac{6}{y(y^2-z^2)}\right) \\ &\quad - \frac{2u^3}{z} - 2\frac{u_z}{z^2} + u\left(\frac{2}{z^3} + \frac{6}{z(y^2-z^2)}\right) \\ &\quad - \frac{6}{(y+z)^2}\left(\frac{u}{y} + \frac{u}{z}\right) \\ &= -2u^3\left(\frac{1}{y} + \frac{1}{z}\right) - \frac{2u_y}{y^2} - \frac{2u_z}{z^2} + \frac{2u}{y^3} + \frac{2u}{z^3}. \end{aligned}$$

On the other hand, we know that in Ω,

$$\begin{aligned} yu_y - zu_z &= \frac{1}{2}(s+t)(u_s+u_t) - \frac{1}{2}(s-t)(u_s-u_t) \\ &= su_t + tu_s < 0, \end{aligned}$$

while in Ω_r,

$$yu_y - zu_z > 0.$$

Suppose at some point p, $\frac{u}{y} + \frac{u}{z} - u_y - u_z < 0$. Then due to symmetry, we can assume $p \in \Omega$. Since $u - yu_y > u - zu_z$, we have, at p,

$$\frac{u}{z} - u_z < 0.$$

We then write

$$L\eta - \frac{6}{(y+z)^2}\eta - \frac{2}{y^2}\eta \leq \left(\frac{2}{z^2} - \frac{2}{y^2}\right)\left(\frac{u}{z} - u_z\right) \leq 0.$$

This contradicts with the maximum principle. Hence

$$\frac{u}{y} + \frac{u}{z} - u_y - u_z \geq 0.$$

□

The above lemma in particular implies that

$$u \geq yu_y \text{ in } \Omega. \tag{21}$$

On the other hand, we conjecture that $u_{yy} < 0$ in Ω^*. If this is true, then we will also have (21) .

Taking the z derivative in (21), we find that

$$u_z - yu_{yz} \geq 0, \text{ if } z = 0. \tag{22}$$

With this estimates at hand, we want to prove the following

Lemma 18.

$$-\frac{u_t}{t} + u_{st} + u_{tt} \geq 0, \ \ in\ \Omega.$$

Proof. Let $a > 0$ be a parameter and $\eta = -\frac{au_t}{t} + u_{st} + u_{tt}$. We compute

$$
\begin{aligned}
L\eta &= -\frac{2a}{t^3}u_t + \frac{2a}{t^2}u_{tt} + \left(\frac{3}{s^2} + \frac{3}{t^2}\right) u_{st} + 6u_s u_t u \\
&\quad + \frac{6}{t^2}u_{tt} - \frac{6}{t^3}u_t + 6u_t^2 u \\
&= \frac{6+2a}{t^2}\eta + \left(\frac{3}{s^2} + \frac{-3-2a}{t^2}\right) u_{st} + 6u_t u \left(u_s + u_t\right).
\end{aligned}
$$

It follows that for $a \geq 0$, $L\eta - \frac{6+2a}{t^2}\eta \leq 0$.

Let us choose $a = 1$. It remains to verify that $\eta \geq 0$ on $\partial\Omega$. If $z = 0$, then $u_{st} = 0$ and $u_{tt} = -u_{yz}$. Then

$$
-u_t + tu_{tt} = \frac{u_z}{\sqrt{2}} + \frac{y}{\sqrt{2}}\left(-u_{yz}\right) = \frac{1}{\sqrt{2}}\left(u_z - yu_{yz}\right).
$$

By (22), we know that $-u_t + tu_{tt} \geq 0$ on the Simons cone. This finishes the proof. □

Lemma 19. *We have the following estimate(not optimal):*

$$
3\left(\frac{1}{t} - \frac{1}{s}\right) u_s + u_{ss} + 2u_{st} \geq 0, \text{ in } \Omega.
$$

Proof. Let a, d be two parameters. Define

$$
\eta = a\left(\frac{1}{t} - \frac{1}{s}\right) u_s + u_{ss} + du_{st}.
$$

We compute

$$
\begin{aligned}
L\eta &= \frac{3a}{s^2}\left(\frac{1}{t} - \frac{1}{s}\right) u_s - \frac{2a}{t^2}u_{st} + \frac{2a}{s^2}u_{ss} + a\left(\frac{1}{s^3} - \frac{1}{t^3}\right) u_s \\
&\quad + \frac{6}{s^2}u_{ss} - \frac{6}{s^3}u_s + 6u_s^2 u + d\left(\frac{3}{s^2} + \frac{3}{t^2}\right) u_{st} + 6du_s u_t u.
\end{aligned}
$$

Then

$$
\begin{aligned}
&L\eta - \frac{6+2a}{s^2}\eta \\
&= \left(-\frac{6+2a}{s^2}a\left(\frac{1}{t} - \frac{1}{s}\right) + \frac{3a}{s^2}\left(\frac{1}{t} - \frac{1}{s}\right) + a\left(\frac{1}{s^3} - \frac{1}{t^3}\right) - \frac{6}{s^3}\right) u_s \\
&+ \left(-\frac{6+2a}{s^2}d - \frac{2a}{t^2} + \left(\frac{3}{s^2} + \frac{3}{t^2}\right) d\right) u_{st} + 6u_s\left(u_s + du_t\right) u.
\end{aligned}
$$

The right hand side is equal to

$$
\begin{aligned}
&\left(\left(\frac{1}{t} - \frac{1}{s}\right)\frac{-3a - 2a^2}{s^2} + a\left(\frac{1}{s^3} - \frac{1}{t^3}\right) - \frac{6}{s^3}\right) u_s \\
&+ \left(-\frac{3+2a}{s^2}d - \frac{2a-3d}{t^2}\right) u_{st} + 6u_s\left(u_s + du_t\right) u.
\end{aligned}
$$

Let us take $d = 2, a = 3$. Since $|u_t| \geq \frac{t}{s} u_s$, we have

$$u_s + 2u_t \leq 0, \text{ if } s \leq 2t.$$

Then applying Proposition 11 in the region $\{s > 2t\}$, we get

$$L\eta - \frac{9}{s^2}\eta \leq 0.$$

By maximum principle, $\eta \geq 0$. The proof is finished. □

Next we want to estimate u_{st} in terms of $u_{ss} + u_{tt}$. This is the content of the following

Lemma 20.

$$3\left(\frac{1}{t} - \frac{1}{s}\right)(u_s - u_t) + 2u_{st} + u_{ss} + u_{tt} \geq 0, \text{ in } \Omega.$$

Proof. Let $\eta = a\left(\frac{1}{t} - \frac{1}{s}\right)(u_s - u_t) + 2u_{st} + u_{ss} + u_{tt}$. Then

$$\begin{aligned} L\eta = & -a\left(\frac{1}{t^3} - \frac{1}{s^3}\right)(u_s - u_t) + \frac{2a}{t^2}(u_{tt} - u_{st}) \\ & + \frac{2a}{s^2}(u_{ss} - u_{st}) + a\left(\frac{1}{t} - \frac{1}{s}\right)\left(\frac{3u_s}{s^2} - \frac{3u_t}{t^2}\right) \\ & + \frac{6u_{ss}}{s^2} - \frac{6u_s}{s^3} + \frac{6u_{tt}}{t^2} - \frac{6u_t}{t^3} + \left(\frac{6}{s^2} + \frac{6}{t^2}\right)u_{st} + 6(u_s + u_t)^2 u. \end{aligned}$$

Let us denote $h_1 = u_{ss} + u_{st}, h_2 = u_{st} + u_{tt}$, and $h = h_1 + h_2$. Then the terms in $L\eta$ involving second order derivatives can be written as

$$\begin{aligned} & \frac{6+2a}{s^2}h_1 + \frac{6+2a}{t^2}h_2 - \left(\frac{4a}{s^2} + \frac{4a}{t^2}\right)u_{st} \\ & = (3+a)\left(\frac{1}{t^2} + \frac{1}{s^2}\right)h + (3+a)\left(\frac{1}{t^2} - \frac{1}{s^2}\right)(u_{tt} - u_{ss}) \\ & - \left(\frac{4a}{t^2} + \frac{4a}{s^2}\right)u_{st}. \end{aligned}$$

On the other hand, the terms in $L\eta$ involving u_s, u_t can be written as

$$\begin{aligned} & \left(-a\left(\frac{1}{t^3} - \frac{1}{s^3}\right) + a\left(\frac{1}{t} - \frac{1}{s}\right)\left(\frac{3}{2s^2} + \frac{3}{2t^2}\right) - \frac{3}{s^3} + \frac{3}{t^3}\right)(u_s - u_t) \\ & + \left(a\left(\frac{1}{t} - \frac{1}{s}\right)\left(\frac{3}{2s^2} - \frac{3}{2t^2}\right) - \frac{3}{s^3} - \frac{3}{t^3}\right)(u_s + u_t) + 6(u_s + u_t)^2 u \end{aligned}$$

Hence $L\eta - (3+a)\left(\frac{1}{t^2}+\frac{1}{s^2}\right)\eta$ is equal to

$$\left(-a\left(\frac{1}{t^3}-\frac{1}{s^3}\right)+a\left(\frac{1}{t}-\frac{1}{s}\right)\left(\frac{3}{2s^2}+\frac{3}{2t^2}\right)-\frac{3}{s^3}+\frac{3}{t^3}-a\left(3+a\right)\left(\frac{1}{t}-\frac{1}{s}\right)\left(\frac{1}{t^2}+\frac{1}{s^2}\right)\right)(u_s-u_t)$$
$$+\left(3+a\right)\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_{tt}-u_{ss})-\left(\frac{4a}{t^2}+\frac{4a}{s^2}\right)u_{st}$$
$$+\left(a\left(\frac{1}{t}-\frac{1}{s}\right)\left(\frac{3}{2s^2}-\frac{3}{2t^2}\right)-\frac{3}{s^3}-\frac{3}{t^3}\right)(u_s+u_t)+6\left(u_s+u_t\right)^2 u.$$

The coefficient before u_s-u_t, divided by $\frac{1}{t}-\frac{1}{s}$ is

$$\begin{aligned}&(3-a)\left(\frac{1}{t^2}+\frac{1}{st}+\frac{1}{s^2}\right)+a\left(\frac{3}{2s^2}+\frac{3}{2t^2}\right)-a\left(3+a\right)\left(\frac{1}{t^2}+\frac{1}{s^2}\right)\\&=\left(3-a+\frac{3a}{2}-a\left(3+a\right)\right)\left(\frac{1}{s^2}+\frac{1}{t^2}\right)+\frac{3-a}{st}\\&=\left(3-\frac{5}{2}a-a^2\right)\left(\frac{1}{s^2}+\frac{1}{t^2}\right)+\frac{3-a}{st}.\end{aligned}$$

Let us choose $a=3$. Using Lemma 19, we obtain

$$\begin{aligned}&\left(3+a\right)\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(u_{tt}-u_{ss})-\left(\frac{4a}{t^2}+\frac{4a}{s^2}\right)u_{st}\\&\leq 6\left(\frac{1}{t^2}-\frac{1}{s^2}\right)(-u_{ss}-2u_{st})\\&\leq 18\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(\frac{1}{t}-\frac{1}{s}\right)u_s.\end{aligned}$$

Then Applying the estimate of u_s+u_t, we get

$$L\eta-6\left(\frac{1}{t^2}+\frac{1}{s^2}\right)\eta\leq 0.$$

By maximum principle, $\eta\geq 0$. This completes the proof. ∎

Proposition 21. *In* Ω,

$$u_{st}+u_{ss}+\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\left(2\left(u_s-u_t\right)+\sqrt{u_s-u_t}\right)\geq 0. \tag{23}$$

Proof. The functions u_{ss} and u_{st} satisfy

$$Lu_{ss}=\frac{6}{s^2}u_{ss}-\frac{6}{s^3}u_s+6u_s^2 u, \tag{24}$$

$$Lu_{st}=\left(\frac{3}{t^2}+\frac{3}{s^2}\right)u_{st}+6u_s u_t u. \tag{25}$$

Hence

$$\begin{aligned} &L\left(u_{ss}+u_{st}\right)-\frac{6}{s^{2}}\left(u_{ss}+u_{st}\right) \\ &=\left(\frac{3}{t^{2}}-\frac{3}{s^{2}}\right)u_{st}-\frac{6u_{s}}{s^{3}}+6u_{s}u\left(u_{s}+u_{t}\right). \end{aligned}$$

Let $\eta=\left(\frac{1}{t^{2}}-\frac{1}{s^{2}}\right)\left(u_{s}-u_{t}+\left(u_{s}-u_{t}\right)^{\alpha}\right)$. Then same computation as before yields

$$\begin{aligned} &L\left(\eta+u_{st}+u_{ss}\right)-\frac{6}{s^{2}}\left(\eta+u_{ss}+u_{st}\right) \\ &=\left(1+\frac{1}{2}\alpha\left(u_{s}-u_{t}\right)^{\alpha-1}\right)\left(\frac{4}{s^{3}}\left(u_{ss}-u_{st}\right)-\frac{4}{t^{3}}\left(u_{st}-u_{tt}\right)\right) \\ &+\left(\frac{1}{t^{2}}-\frac{1}{s^{2}}\right)L\left(u_{s}-u_{t}+\frac{1}{2}\left(u_{s}-u_{t}\right)^{\alpha}\right) \\ &+\left(\frac{3}{t^{2}}-\frac{3}{s^{2}}\right)u_{st}-\frac{6u_{s}}{s^{3}}+6u_{s}u\left(u_{s}+u_{t}\right)-\frac{6}{s^{2}}\eta. \end{aligned}$$

In the region Γ where (23) is not true, we have

$$\begin{aligned} &\frac{1}{s^{3}}\left(u_{ss}-u_{st}\right)-\frac{1}{t^{3}}\left(u_{st}-u_{tt}\right) \\ &\leq-\left(\frac{2}{s^{3}}+\frac{1}{t^{3}}\right)u_{st}+\frac{1}{t^{3}}u_{tt} \\ &-\frac{1}{s^{3}}\left(\frac{1}{t^{2}}-\frac{1}{s^{2}}\right)\left(u_{s}-u_{t}+\sqrt{u_{s}-u_{t}}\right). \end{aligned}$$

Denote

$$J:=\frac{1}{s^{3}}\left(\frac{1}{t^{2}}-\frac{1}{s^{2}}\right)\left(u_{s}-u_{t}+\frac{1}{2}\sqrt{u_{s}-u_{t}}\right).$$

Using the fact that $u_{tt}\leq 0$, we also have

$$\begin{aligned} \left|\nabla\left(u_{s}-u_{t}\right)\right|^{2}&=\left(u_{ss}-u_{st}\right)^{2}+\left(u_{st}-u_{tt}\right)^{2} \\ &\geq\left(2u_{st}+J\right)^{2}+u_{st}^{2} \\ &=5u_{st}^{2}+4u_{st}J+J^{2}. \end{aligned}$$

Then applying the decay of u_s, we get

$$L\left(\eta+u_{st}+u_{ss}\right)-\frac{6}{s^{2}}\left(\eta+u_{ss}+u_{st}\right)\leq 0.$$

This implies that $\eta+u_{st}+u_{ss}\geq 0$. □

Lemma 22. $u_{s}u+u_{ss}\geq 0$, *in* Ω.

Proof. Let $\eta = u_s u + u_{ss}$. The computation in Lemma 13 tells us that

$$\begin{aligned} & L\eta - \left(2u_s + \frac{6}{s^2}\right)\eta \\ & = -\frac{3}{s^2}u_s u + 2u_{st}u_t - \frac{6}{s^3}u_s \\ & + 4u_s^2 u + u_s\left(u^3 - u\right). \end{aligned}$$

This can be written as

$$\begin{aligned} & L\eta - \left(2u_s - 2u_t + \frac{6}{s^2}\right)\eta \\ & = -\frac{3}{s^2}u_s u + 2\left(u_{st} + u_{ss}\right)u_t + 2\left(u_s + u_t\right)u_s u - \frac{6}{s^3}u_s \\ & + 2u_s^2 u + u_s\left(u^3 - u\right). \end{aligned}$$

Note that by Modica estimate,

$$\begin{aligned} 1 - u^2 \geq & \sqrt{2\left(u_s^2 + u_t^2\right)}. \\ & = 2u_s + \sqrt{2\left(u_s^2 + u_t^2\right)} - 2u_s \\ & = 2u_s + \frac{2\left(u_t^2 - u_s^2\right)}{\sqrt{2\left(u_s^2 + u_t^2\right)} + 2u_s}. \end{aligned}$$

We then deduce

$$\begin{aligned} L\eta \leq & -\frac{3}{s^2}u_s u + 2\left(u_{st} + u_{ss}\right)u_t + 2\left(u_s + u_t\right)u_s u - \frac{6}{s^3}u_s \\ & + u_s u\frac{2\left(u_s^2 - u_t^2\right)}{\sqrt{2\left(u_s^2 + u_t^2\right)} + 2u_s}. \end{aligned}$$

Now inserting the estimate of u_{st} obtained in Lemma 20 into the proof of Proposition 11, we can bootstrap the estimate of $u_s + u_t$ to

$$u_s + u_t \leq \left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left(u_s - u_t + \frac{1}{2}\sqrt{u_s - u_t}\right). \tag{26}$$

Applying these estimates, we obtain $L\eta \leq 0$. Hence $\eta \geq 0$. □

Similarly, we have the following

Lemma 23. $-u_t u - u_{st} \geq 0$, *in* Ω.

The proof of this lemma is almost identical to that of Lemma 22. We omit the details. Next we would like to prove the following

Lemma 24. *In* Ω, *we have*

$$\frac{3}{2}\left(\frac{1}{t} - \frac{1}{s}\right)u_s + u_{ss} + u_{st} \geq 0.$$

Proof. Let

$$\eta = a\left(\frac{1}{t} - \frac{1}{s}\right) u_s + u_{ss} + u_{st}.$$

We compute

$$\begin{aligned} L\eta = &\frac{3a}{s^2}\left(\frac{1}{t} - \frac{1}{s}\right) u_s - \frac{2a}{t^2} u_{st} + \frac{2a}{s^2} u_{ss} + a\left(\frac{1}{s^3} - \frac{1}{t^3}\right) u_s \\ &+ \frac{6}{s^2} u_{ss} - \frac{6}{s^3} u_s + 6u_s^2 u + \left(\frac{3}{s^2} + \frac{3}{t^2}\right) u_{st} + 6u_s u_t u. \end{aligned}$$

Then

$$\begin{aligned} &L\eta - \frac{6+2a}{s^2}\eta \\ &= \left(-\frac{6+2a}{s^2} a\left(\frac{1}{t} - \frac{1}{s}\right) + \frac{3a}{s^2}\left(\frac{1}{t} - \frac{1}{s}\right) + a\left(\frac{1}{s^3} - \frac{1}{t^3}\right) - \frac{6}{s^3}\right) u_s \\ &+ \left(-\frac{6+2a}{s^2} - \frac{2a}{t^2} + \frac{3}{s^2} + \frac{3}{t^2}\right) u_{st} + 6u_s\left(u_s + u_t\right) u. \end{aligned}$$

The right hand side is equal to

$$\begin{aligned} &\left(\left(\frac{1}{t} - \frac{1}{s}\right)\frac{-3a - 2a^2}{s^2} + a\left(\frac{1}{s^3} - \frac{1}{t^3}\right) - \frac{6}{s^3}\right) u_s \\ &+ \left(-\frac{6+2a}{s^2} - \frac{2a}{t^2} + \frac{3}{s^2} + \frac{3}{t^2}\right) u_{st} + 6u_s\left(u_s + u_t\right) u \\ &= \left(\left(\frac{1}{t} - \frac{1}{s}\right)\frac{-3a - 2a^2}{s^2} + a\left(\frac{1}{s^3} - \frac{1}{t^3}\right) - \frac{6}{s^3}\right) u_s \\ &+ \left(\frac{-3 - 2a}{s^2} + \frac{3 - 2a}{t^2}\right) u_{st} + 6u_s\left(u_s + u_t\right) u. \end{aligned}$$

Hence applying Lemma 20 and (26) , we get

$$L\eta - \frac{9}{s^2}\eta \leq 0.$$

By maximum principle, $\eta \geq 0$. The proof is finished. □

Similarly, we have

Lemma 25.

$$\frac{3}{2}\left(\frac{1}{s} - \frac{1}{t}\right) u_t - u_{st} - u_{tt} \geq 0.$$

Proof. Let

$$\eta = a\left(\frac{1}{s} - \frac{1}{t}\right) u_t - u_{st} - u_{tt}.$$

We compute

$$\begin{aligned} L\eta &= \frac{3a}{t^2}\left(\frac{1}{s}-\frac{1}{t}\right)u_t + \frac{2a}{t^2}u_{tt} - \frac{2a}{s^2}u_{st} - a\left(\frac{1}{s^3}-\frac{1}{t^3}\right)u_t \\ &\quad - \left(\frac{3}{s^2}+\frac{3}{t^2}\right)u_{st} - 6u_s u_t u - \frac{6}{t^2}u_{tt} + \frac{6}{t^3}u_t - 6u_t^2 u. \end{aligned}$$

Then

$$\begin{aligned} & L\eta - \frac{6-2a}{t^2}\eta \\ &= \left(-\frac{6-2a}{t^2}a\left(\frac{1}{s}-\frac{1}{t}\right) + \frac{3a}{t^2}\left(\frac{1}{s}-\frac{1}{t}\right) - a\left(\frac{1}{s^3}-\frac{1}{t^3}\right) + \frac{6}{t^3}\right)u_t \\ &+ \left(\frac{6-2a}{t^2} - \frac{2a}{s^2} - \frac{3}{s^2} - \frac{3}{t^2}\right)u_{st} - 6u_t\left(u_s+u_t\right)u. \end{aligned}$$

The left hand side is equal to

$$\begin{aligned} &\left(\frac{a\left(6-2a\right)-2a+6}{t^3} + \frac{-a\left(6-2a\right)+3a}{st^2} - \frac{a}{s^3}\right)u_t \\ &+ \left(\frac{-3-2a}{s^2} + \frac{3-2a}{t^2}\right)u_{st} - 6u_t\left(u_s+u_t\right)u. \end{aligned}$$

If we choose $a = \frac{3}{2}$, then this is equal to

$$\begin{aligned} &\left(\frac{a\left(6-2a\right)-2a+6}{t^3} + \frac{-a\left(6-2a\right)+3a}{st^2} - \frac{a}{s^3}\right)u_t \\ &- \frac{6}{s^2}u_{st} - 6u_t\left(u_s+u_t\right)u \\ &= \left(\frac{15}{2t^3} - \frac{3}{2s^3}\right)u_t - \frac{6}{s^2}u_{st} - 6u_t\left(u_s+u_t\right)u. \end{aligned}$$

This will be negative. Hence $\eta \geq 0$. □

Proposition 26. *In* Ω,

$$u_s + u_t - u_{st} - u_{tt} \geq 0.$$

Proof. Let $\eta = u_s + u_t - u_{st} - u_{tt}$. Then

$$\begin{aligned} L\eta &= \frac{3u_s}{s^2} + \frac{3u_t}{t^2} - \left(\frac{3}{s^2}+\frac{3}{t^2}\right)u_{st} - 6u_s u_t u - \frac{6}{t^2}u_{tt} + \frac{6}{t^3}u_t - 6u_t^2 u \\ &= \frac{6}{t^2}\eta - \frac{6}{t^2}\left(u_s+u_t-u_{st}\right) + \frac{3u_s}{s^2} + \frac{3u_t}{t^2} \\ &- \left(\frac{3}{s^2}+\frac{3}{t^2}\right)u_{st} - 6u_s u_t u - \frac{6}{t^2}u_{tt} + \frac{6}{t^3}u_t - 6u_t^2 u \\ &= \frac{6}{t^2}\eta + \left(-\frac{6}{t^2}+\frac{3}{s^2}\right)u_s + \left(-\frac{3}{t^2}\right)u_t + \left(\frac{3}{t^2}-\frac{3}{s^2}\right)u_{st} + \frac{6}{t^3}u_t - 6u_t u\left(u_s+u_t\right). \end{aligned}$$

We write it as

$$L\eta - \frac{6}{t^2}\eta = \left(\frac{3}{t^2} - \frac{3}{s^2}\right)(u_{st} + u_t) + \left(-\frac{6}{t^2} + \frac{3}{s^2}\right)(u_s + u_t)$$
$$+ \frac{6}{t^3}u_t - 6u_t u (u_s + u_t).$$

It then follows from the estimate of $u_s + u_t$ that $L\eta - \frac{6}{t^2}\eta \le 0$, which implies $\eta \ge 0$. □

Similarly, we have

Proposition 27. *In Ω, we have*

$$u_s + u_t + u_{st} + u_{ss} \ge 0.$$

The proof of Proposition 27 is similar to the proof of Proposition 26. That is, denoting

$$\eta = u_s + u_t + u_{st} + u_{ss}.$$

We can show that $L\eta - \frac{6}{s^2}\eta \le 0$, which then implies that $\eta \ge 0$. We sketch the proof below.

Proof. Let $\eta = u_s + u_t + u_{st} + u_{ss}$. First of all, since $u_s u + u_{ss} \ge 0$, we obtain $\eta \ge 0$ on $\partial\Omega$. We compute

$$L\eta = \frac{3}{s^2}u_s + \frac{3}{t^2}u_t + \left(\frac{3}{s^2} + \frac{3}{t^2}\right)u_{st} + 6u_s u_t u$$
$$+ \frac{6}{s^2}u_{ss} - \frac{6}{s^3}u_s + 6u_s^2 u.$$

Then we get

$$L\eta - \frac{6}{s^2}\eta = -\frac{3}{s^2}(u_s + u_t) + \left(\frac{3}{t^2} - \frac{3}{s^2}\right)(u_{st} + u_t)$$
$$- \frac{6}{s^3}u_s + 6u_s u (u_s + u_t).$$

Using the fact that $u_t u + u_{st} \le 0$ and applying the estimate of $u_s + u_t$, we see that $L\eta - \frac{6}{s^2}\eta \le 0$. It then follows from the maximum principle that $\eta \ge 0$. □

An immediate consequence of Proposition 27 and estimate (26) is the following

Corollary 28. *In Ω, we have*

$$\left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left(u_s - u_t + \frac{1}{2}\sqrt{u_s - u_t}\right) + u_{st} + u_{ss} \ge 0.$$

Now we would like to establish a lower bound on u_s. Let us define the function $E := u^2 + 2u_s$. By Lemma 22 and Lemma 23, we have

$$u_s u - u_t u + u_{ss} - u_{st} \ge 0. \tag{27}$$

This implies $\partial_z E \geq 0$. We will slightly abuse the notation and still write the function u, u_s in (y,z) variables as $u(y,z), u_s(y,z)$. Recall that $u = H(y)H(z) - \phi$. From (27) and the fact that $u \leq H(y)H(z)$, we get the following estimate in Ω :

$$\begin{aligned} 2u_s(y,z) &\geq 2u_s(y,0) + u^2(y,0) - u^2(y,z) \\ &\geq 2u_s(y,0) - (H(y)H(z))^2 \\ &= 2\partial_s (H(y)H(z) - \phi)|_{(z=0)} - (H(y)H(z))^2 . \end{aligned} \tag{28}$$

We have mentioned that the estimate of u_{tt} is most delicate. To conclude this section, let us derive certain upper bound on $|u_{tt}|$. Note that so far we have good control on $|u_{ss}|$ and $|u_{st}|$, in terms of u_s and u_t respectively. We first recall that

$$u_{ss} + u_{tt} + \frac{3}{s}u_s + \frac{3}{t}u_t = -u\left(1-u^2\right).$$

Since

$$2u_s(y,z) \geq 2u_s(y,0) - u^2(y,z),$$

there holds

$$1 - u^2(y,z) \leq 2u_s(y,z) + 1 - 2u_s(y,0).$$

Hence

$$u_{ss} + u_{tt} + \frac{3}{s}u_s + \frac{3}{t}u_t \geq -u\left(2u_s(y,z) + 1 - 2u_s(y,0)\right). \tag{29}$$

This together with the fact that u_{tt} is negative, clearly gives us an upper bound on $|u_{tt}|$.

3. Construction of supersolution in dimension 8

In this section, we will prove Theorem 1. Consider function ϕ of the form $fu_s + hu_t$. We have

$$\begin{aligned} L\phi &= u_s\Delta f + 2u_{ss}f_s + 2u_{st}f_t + \frac{3}{s^2}u_s f \\ &\quad + u_t\Delta h + 2u_{st}h_s + 2u_{tt}h_t + \frac{3}{t^2}u_t h. \end{aligned}$$

Hence

$$\begin{aligned} L\phi &= \left(\Delta f + \frac{3}{s^2}f\right)u_s + \left(\Delta h + \frac{3}{t^2}h\right)u_t \\ &\quad + 2u_{ss}f_s + (2f_t + 2h_s)u_{st} + 2u_{tt}h_t. \end{aligned} \tag{30}$$

Let us write it as

$$L\phi = C_s u_s + C_{st}u_{st} + C_{ss}u_{ss} + C_{tt}u_{tt} + C_t u_t,$$

where

$$C_s := \Delta f + \frac{3}{s^2}f,\ C_{st} := 2f_t + 2h_s,$$

$$C_{ss} := 2f_s,\ C_{tt} := 2h_t,\ C_t := \Delta h + \frac{3}{t^2}h.$$

Recall that in dimension $n \geq 14$, Cabré [7] made the choice $f = t^{-\alpha}, h = -s^{\alpha}$, for suitable constant $\alpha > 0$. In our case, to construct a supersolution, we choose

$$f(s,t) := \left(\tanh\left(\frac{s}{t}\right) \frac{\sqrt{2}s}{\sqrt{s^2+t^2}} + \frac{1}{4.2}\left(1 - e^{-\frac{s}{2t}}\right) \right) (s+t)^{-2.5},$$

$$h(s,t) := -\left(\tanh\left(\frac{t}{s}\right) \frac{\sqrt{2}t}{\sqrt{s^2+t^2}} + \frac{1}{4.2}\left(1 - e^{-\frac{t}{2s}}\right) \right) (s+t)^{-2.5}.$$

We now define $\Phi_0 := 0.00007\left(s^{-1.8}e^{-\frac{t}{3}} + t^{-1.8}e^{-\frac{s}{3}}\right)$ and $\Phi_1 = fu_s + hu_t$. Then we set

$$\Phi := \Phi_0 + \Phi_1. \tag{31}$$

Note that $\Phi > 0$ and $\Phi(s,t) = \Phi(t,s)$. The reason that we choose this specific f, h, instead of the more natural choice of $\mu := (s+t)^{-2.5}(u_s - u_t)$, is the following: Although near the Simons cone, the function μ is well behaved, it does not satisfies $L\mu \leq 0$ away from the Simon cone(for instance, when $(s,t) = (3,2)$). To deal with this issue, we have multiplied the term $\tanh\left(\frac{s}{t}\right)\frac{s}{\sqrt{s^2+t^2}}$. Next, we add a small perturbation term Φ_0, because the function $fu_s + hu_t$ essentially decays as e^{t-s} and is not a good supersolution when $s - t$ is very large, where the linearized operator looks like $-\Delta + 2$. Finally, the term $\frac{1}{4.2}\left(1 - e^{-\frac{s}{2t}}\right)$ is used to control the sign of $L\Phi$ near the point $(5, 2.5)$.

We would like to show that Φ is a supersolution of the linearized operator L. Due to symmetry, in the sequel, we only need to consider the problem in Ω.

Our first observation is the following fact:

$$C_s < 0, C_{st} < 0, C_{ss} < 0, \text{ in } \Omega.$$

We emphasize that C_t and C_{tt} may change sign. As a matter of fact, C_t changes sign near the Simons cone, and in most part of Ω, C_t is negative. When $t > \frac{1}{2}$, C_{tt} is negative in the region(approximately desribed by) $2s/5 < t < 3s/5$. Moreover, in this region, $|C_{tt}|$ is small compared to $|C_{ss}|$.(See Figure 11). It follows that $C_s u_s$ and $C_{st}u_{st}$ are negative, which can be regarded as "good" terms. The "bad" terms are $C_{ss}u_{ss}$ and $C_t u_t$.

The main idea of our proof is to control the other positive terms using $C_s u_s + C_{st}u_{st}$ and $C_{ss}u_{ss}$, based on the estimates obtained in the previous section. We have the following

Proposition 29. *Let Φ be the function defined by (31). For all (s,t), we have $L\Phi \leq 0$.*

Proof. Since Φ is even with respect to the Simons cone, it will be suffice to prove this inequality in Ω.

Lemma 24 tells us that

$$\frac{3}{2}\left(\frac{1}{t} - \frac{1}{s}\right) u_s + u_{st} + u_{ss} \geq 0. \tag{32}$$

For notational convenience, we will also write the coefficient as λ. That is

$$\lambda := \frac{3}{2}\left(\frac{1}{t} - \frac{1}{s}\right).$$

It plays an important role in our analysis, since it measures how close is u_{st} to u_{ss}.

FIGURE 6. C_t/C_s

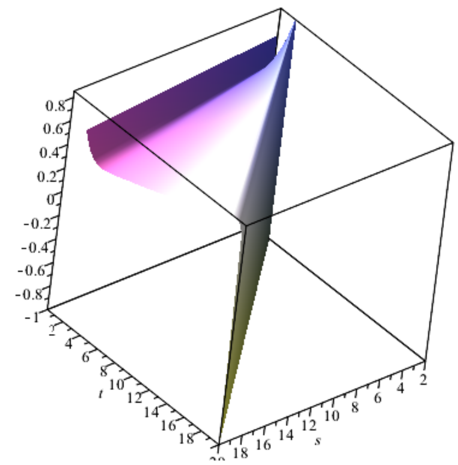
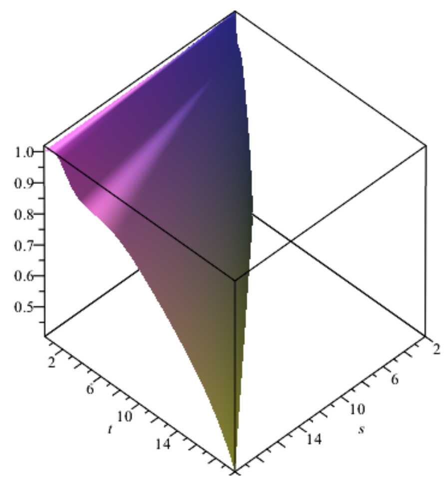


FIGURE 7. $C_{ss}/(C_{st}-C_{tt})$



Note that by Proposition 27 and the fact that $|u_t| \geq \frac{t}{s}u_s$, we have

$$\frac{s-t}{s}u_s + u_{st} + u_{ss} \geq u_s + u_t + u_{st} + u_{ss} \geq 0. \tag{33}$$

Hence when $t < \frac{3}{2}$, the inequality (32) is weaker than (33) . Moreover, (33) has the following simple consequence: If $|u_t| = au_s$ for some constant $a < 1$, then

$$u_{ss} + u_{st} + (1-a)\,u_s \geq 0. \tag{34}$$

This estimate is useful, because a priori, we don't know the precise value of $|u_t/u_s|$, although it is always bounded from below by t/s. At this stage, it will be crucial to have some information on the ratio C_t/C_s. We have

$$C_t/C_s < 0.9, \text{ if } s/10 < t < s.$$

FIGURE 8. $T(1-\lambda)$

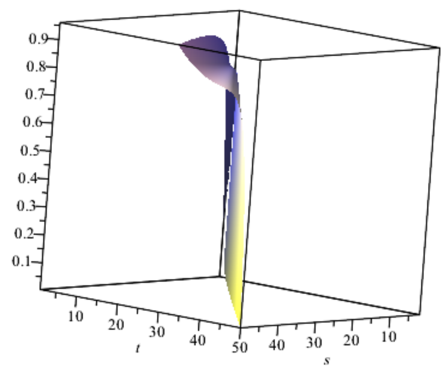
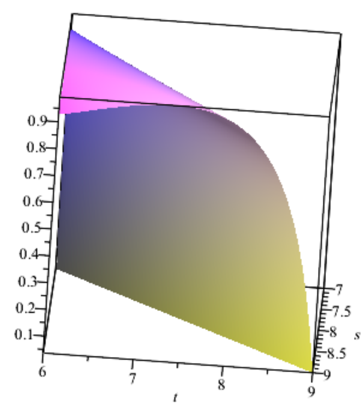


FIGURE 9. $L\Phi/(C_t - C_s)$



See Figure 6 for detailed information on the function C_t/C_s. This tells us that $C_s u_s + C_t u_t < 0$, for $s/10 < t < s$. Moreover, it turns out that $C_s u_s + C_t u_t + C_{st} u_{st} + C_{tt} u_{tt}$ can be used to control the term $C_{ss} u_{ss}$. Indeed, first of all, we have(See Figure 7), in the region $s/10 < t < s$,

$$\frac{C_{ss}}{C_{st} - C_{tt}} < 1.$$

Then in this region, we can use (34) to estimate $u_{st} + u_{ss}$, and applying Proposition 26 to deduce

$$2\left(u_s + u_t\right) + u_{ss} - u_{tt} \geq 0. \tag{35}$$

FIGURE 10. $(C_s - C_t)t/L\Phi_0$

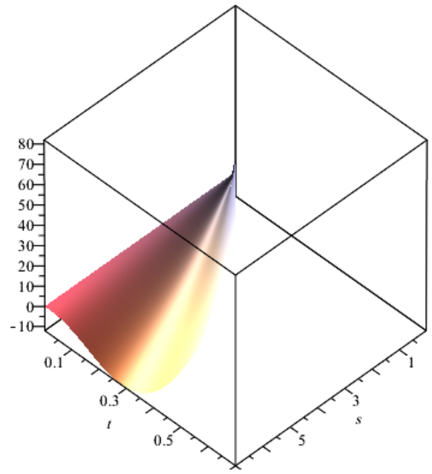
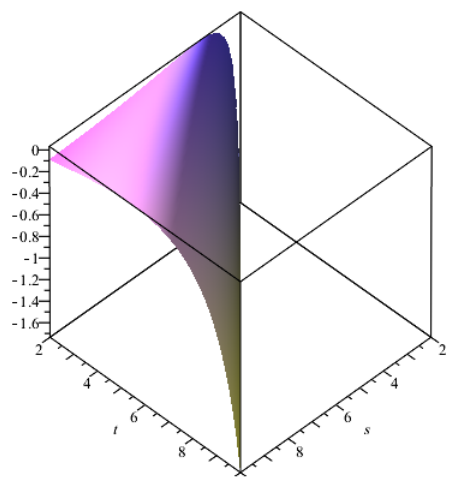


FIGURE 11. C_{tt}/C_{ss}



It follows that if $C_{tt} > 0$, then

$$\begin{aligned} & C_s u_s + C_{st} u_{st} + C_{ss} u_{ss} + C_{tt} u_t + C_t u_t \\ & \leq \left(C_s - (1-a) C_{ss} - a C_t + (1-a) \max \{ C_{st} - C_{ss}, 0 \} \right) u_s, \end{aligned} \tag{36}$$

where a can be choosen to be $|u_t| / u_s$ or λ.

With the help of these estimates, let us consider the subregion

$$E_1 := \left\{ (s,t) : 0.65 s < t < s \text{ and } t > \frac{1}{2} \right\}.$$

Consider the function $T(r) := \frac{(1-r)C_{ss}}{C_s+(1-r)\max\{C_{st}-C_{ss},0\}-rC_t}$. It turns out that

$$0 < \frac{(1-r)\,C_{ss}}{C_s+(1-r)\max\{C_{st}-C_{ss},0\}-rC_t} < 1 \text{ in } E_1, \text{ for all } r \in [1-\lambda, 1).$$

See Figure 8 on the picture of this function in the case of $r = 1-\lambda$. We conclude that

$$L\Phi_1 < 0, \text{ in } E_1.$$

We also observe

$$L\left(s^{-1.8}e^{-\frac{t}{3}}\right) = -0.36s^{-3.8}e^{-\frac{t}{3}} + s^{-1.8}e^{-\frac{t}{3}}\left(-\frac{1}{t}+\frac{10}{9}-3u^2\right). \tag{37}$$

Hence $L\Phi_0$ will be positive only in the region close to the Simons cone, where $-\frac{1}{t}+\frac{10}{9}-3u^2 > 0$. Note that we always have $u \geq H(0.45y)\,H(0.45z)$. In E_1, we then verify that $L(\Phi_1+\Phi_0) \leq 0$, see Figure 9. Hence we conclude that

$$L\Phi \leq 0 \text{ in } E_1.$$

Next we consider the region $E_2 := \{(s,t) : 1/2 < t < 0.65s\}$. In this region, $L\Phi_1$ may be positive. However, we already know that, if $C_{tt} > 0$, then (36) holds. Moreover, if $C_{tt} < 0$, then by(Lemma 18)

$$-\frac{1}{t}u_t + u_{st} + u_{tt} \geq 0, \tag{38}$$

we get

$$\begin{aligned} &C_s u_s + C_{st}u_{st} + C_{ss}u_{ss} + C_{tt}u_t + C_t u_t \\ &\leq \left(C_s - (1-a)\,C_{ss} - aC_t + \frac{1}{t}\left|C_{tt}\right|\right)u_s, \end{aligned} \tag{39}$$

We emphasize that in the region where $C_{tt} < 0$, actually $|C_{tt}/C_{ss}|$ is small. On the other hand, by Lemma 8, u_s has the following decay

$$u_s \leq 2\left(e^{0.85t}+\frac{4.9}{\sqrt{t}}\right)e^{-0.85s}. \tag{40}$$

Note that here we can also use the Modica estimate $u_s \leq \frac{1-u^2}{\sqrt{2}}$ to estimate u_s. In particular, (40) implies that u_s decays at least like $e^{-0.5t}$ along the line $t = 0.65s$. We also should keep in mind that (40) does not mean that u_s blows up as $t \to 0$. Indeed, by $u_{st} > 0$, we know that $u_s(s,0) < u_s(s,t)$ in Ω. Now we recall that Φ_0 decays like $s^{-1.8}e^{-\frac{t}{3}}$. In particular, along the line $s = 0.65t$, u_s decays faster than $L\Phi_0$. From (36), (39), (40) and (37), we can indeed verify that

$$L\Phi_0 + L\Phi_1 \leq 0, \text{ in } E_2.$$

Now we would like to consider $E_3 := \{(s,t) : t < 1/2\}$. Here $|C_t|$ could be large compared to $|C_s|$. Hence the above arguments does not work. However, as $t \to 0$, we know that $u_t/t \to u_{tt}$. This implies that for t small, u_t is of the order tu_{tt}. More precisely, we have

$$u_t(s,t) = \int_0^t u_{tt}(s,r)\,dr.$$

On the other hand, we can estimate u_{tt} using the fact that(Lemma 9) $u_s/s \geq u_{ss}$ and the Allen-Cahn equation

$$u_{ss} + u_{tt} + \frac{3}{s}u_s + \frac{3}{t}u_t + u - u^3 = 0.$$

Indeed, since $u_s/s + t/u_t < 0$, we have

$$|u_{ss} + u_{tt}| < u - u^3.$$

Let $u_{tt}^* = \min_{r\in(0,t)} u_{tt}(s,r)$. It turns out that $|2(C_s - C_t)\, t u_{tt}^*| < L\Phi_0$. Hence $L\Phi \leq 0$ in E_3. See Figure 10.

Combing the above analysis in E_1, E_2, E_3, we get the desired inequality. The proof is thus completed. □

Proposition 29 tells us that Φ is a supersolution and it follows from standard arguments(see, for instance, [7, 29]) that the saddle solution u is stable in dimension 8. This finishes the proof of Theorem 1.

4. Stability in dimension 10 and 12

In this section, we indicate the necessary changes needed in order to prove the stability of the saddle solution in dimension $n = 10$ and $n = 12$.

We construct supersolution in the form $\Phi_0 + \Phi_1$, where $\Phi_1 = f u_s + h u_t$, with

$$f(s,t) := \tanh\left(\frac{s}{t}\right)\frac{s}{\sqrt{s^2+t^2}}(s+t)^{-\frac{n-3}{2}},$$

$$h(s,t) := -\tanh\left(\frac{t}{s}\right)\frac{t}{\sqrt{s^2+t^2}}(s+t)^{-\frac{n-3}{2}}.$$

In principle, the cases $n = 10$ and 12 are easier than the dimension 8 case. Observe that in the definiton of f, we don't need the term $1 - \frac{1}{4}e^{-\frac{s}{2t}}$. In the previous section, this term is used to control the behaviour of the supersolution near the point $(s,t) = (5, 2.5)$. The reason that we choose $(s+t)^{-\frac{n-3}{2}}$ is as follows. Let us consider the function $\phi := (s+t)^{\alpha}(u_s - u_t)$. Then $L\phi$ is equal to

$$\begin{aligned}
&(s+t)^{\alpha-2}\left(2\alpha(\alpha-1) + \left(\frac{n}{2}-1\right)\alpha\left(s^{-1}+t^{-1}\right)(s+t) + \left(\frac{n}{2}-1\right)s^{-2}(s+t)^2\right)u_s \\
&-(s+t)^{\alpha-2}\left(2\alpha(\alpha-1) + \left(\frac{n}{2}-1\right)\alpha\left(s^{-1}+t^{-1}\right)(s+t) + \left(\frac{n}{2}-1\right)t^{-2}(s+t)^2\right)u_t \\
&+2\alpha(u_{ss}-u_{tt})(s+t)^{\alpha-1}.
\end{aligned}$$

When $s = t$, as $t \to +\infty$, $L\phi$ asymptotically looks like

$$\begin{aligned}
&2(\alpha(\alpha-1) + (n-2)\alpha + (n-2))(s+t)^{\alpha-2}(u_s - u_t) \\
&= 2\left(\alpha^2 + (n-3)\alpha + n - 2\right)(u_s - u_t)
\end{aligned}$$

The roots of the equation $\alpha^2 + (n-3)\alpha + n - 2 = 0$ are given by

$$\frac{3-n\pm\sqrt{(n-3)^2 - 4(n-2)}}{2}. \tag{41}$$

These roots are real only when $n \geq 8$ or $n = 2$. Furthermore, when $n \geq 8$, $L\phi$ will be negative at the Simons cone only if the exponent α is between these two roots. At this stage, it is worth mentioning that the Jacobi operator of the corresponding Simons cone has kernels of the form $c_1 r^{\beta_1} + c_2 r^{\beta_2}$, where $\beta_{1,2}$ are also given by (41).

Now for the Φ_0 part, we choose

$$\Phi_0 = c\left(s^{-\frac{n-4}{2}} e^{-\frac{t}{3}} + t^{-\frac{n-4}{2}} e^{-\frac{s}{3}}\right),$$

where $c = 0.001$. We can prove similar estimates of the derivatives of u. More precisely, we have

$$\begin{aligned} u_s u + u_{ss} &\geq 0, \\ -u_t u - u_{st} &\geq 0, \\ \frac{n-2}{4}\left(\frac{1}{t} - \frac{1}{s}\right) u_s + u_{st} + u_{ss} &\geq 0, \\ u_s + u_t + u_{st} + u_{ss} &\geq 0, \\ \left(\frac{1}{t^2} - \frac{1}{s^2}\right)\left(u_s - u_t + \frac{1}{2}\sqrt{u_s - u_t}\right) + u_{st} + u_{ss} &\geq 0, \\ -\frac{1}{t} u_t + u_{st} + u_{tt} &\geq 0. \end{aligned}$$

Then direct computation shows that $L\Phi \leq 0$. Hence the saddle solution is also stable in dimension 10 and 12.

References

[1] G. Alberti; L. Ambrosio; X. Cabré, On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property. Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday. *Acta Appl. Math.* 65 (2001), no. 1-3, 9–33.

[2] H. Alencar; A. Barros; O. Palmas; J. G. Reyes; W. Santos, $O(m) \times O(n)$-invariant minimal hypersurfaces in $\mathbb{R}^{m+n}$. *Ann. Global Anal. Geom.* 27 (2005), no. 2, 179–199.

[3] L. Ambrosio; X. Cabré, Entire solutions of semilinear elliptic equations in R3 and a conjecture of De Giorgi. *J. Amer. Math. Soc.* 13 (2000), no. 4, 725–739.

[4] E. Bombieri; De Giorgi; Giusti, Minimal cones and the Bernstein problem. *Invent. Math.* 7 (1969), 243–268.

[5] X. Cabré; J. Terra, Saddle-shaped solutions of bistable diffusion equations in all of $\mathbb{R}^{2m}$. *J. Eur. Math. Soc. (JEMS)* 11 (2009), no. 4, 819–843.

[6] X. Cabré; J. Terra, Qualitative properties of saddle-shaped solutions to bistable diffusion equations. *Comm. Partial Differential Equations* 35 (2010), no. 11, 1923–1957.

[7] X. Cabré, Uniqueness and stability of saddle-shaped solutions to the Allen-Cahn equation. J. Math. Pures Appl. (9) 98 (2012), no. 3, 239–256.

[8] H. Dang; P. C. Fife; L. A. Peletier, Saddle solutions of the bistable diffusion equation. *Z. Angew. Math. Phys.* 43 (1992), no. 6, 984–998.

[9] E. De Giorgi, Convergence problems for functionals and operators, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131188, Pitagora, Bologna (1979).

[10] G. De Philippis; E. Paolini, A short proof of the minimality of the Simons cone, *Rend. Sem. Mat. Univ. Padova*, 121(2009), 233–241.

[11] M. Del Pino; M. Kowalczyk; J. Wei, Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$. *J. Differential Geom.* 93 (2013), no. 1, 67–131.
[12] M. Del Pino; M. Kowalczyk; J. Wei, On De Giorgi's conjecture in dimension $N \geq 9$. *Ann. of Math.* (2) 174 (2011), no. 3, 1485–1569.
[13] M. Del Pino; M. Kowalczyk; J. Wei, Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$. *J. Differential Geom.* 93 (2013), no. 1, 67–131.
[14] L. Dupaigne; A. Farina, Stable solutions of $-\Delta u = f(u)$. *J. Eur. Math. Soc. (JEMS)* 12 (2010), no. 4, 855–882.
[15] L. Dupaigne; A. Farina, Classification and Liouville-type theorems for semilinear elliptic equations in unbounded domains, arXiv:1912.11639.
[16] A. Farina; E. Valdinoci, The state of the art for a conjecture of De Giorgi and related problems, in: Recent Progress on Reaction–Diffusion Systems and Viscosity Solutions, World Sci. Publ., Hackensack, NJ, 2009, pp. 74–96.
[17] A. Farina; E. Valdinoci, 1D symmetry for semilinear PDEs from the limit interface of the solution.*Comm. Partial Differential Equations* 41 (2016), no. 4, 665–682.
[18] A. Farina; E. Valdinoci, 1D symmetry for solutions of semilinear and quasilinear elliptic equations. *Trans. Amer. Math. Soc.* 363 (2011), no. 2, 579–609.
[19] N. Ghoussoub; C. Gui, On a conjecture of De Giorgi and some related problems. *Math. Ann.* 311 (1998), no. 3, 481–491.
[20] C. Gui, Hamiltonian identities for elliptic partial differential equations. *J. Funct. Anal.* 254 (2008), no. 4, 904–933.
[21] R. Hardt; L. Simon, Area minimizing hypersurfaces with isolated singularities. *J. Reine Angew. Math.* 362 (1985), 102–129.
[22] D. Jerison; R. Monneau, Towards a counter-example to a conjecture of De Giorgi in high dimensions. *Ann. Mat. Pura Appl.* (4) 183 (2004), no. 4, 439–467.
[23] M. Kowalczyk; Y. Liu; F. Pacard, The classification of four-end solutions to the Allen-Cahn equation on the plane. *Anal. PDE* 6 (2013), no. 7, 1675–1718.
[24] F. H. Lin, Minimality and stability of minimal hypersurfaces in $\mathbb{R}^n$, *Bull. Austral. Math. Soc.* 36(1987), 209–214.
[25] Y. Liu; J. Wei, A complete classification of finite Morse index solutions to elliptic sine-Gordon equation in the plane, arxiv 1806.06921.
[26] Y. Liu; K. Wang; J. Wei, Global minimizers of the Allen-Cahn equation in dimension $n \geq 8$. *J. Math. Pures Appl.* (9) 108 (2017), no. 6, 818–840.
[27] G. Lawlor, A sufficient criterion for a cone to be area-minimizing. *Mem. Amer. Math. Soc.* 91 (1991), no. 446, vi+111 pp.
[28] B. Lawson, The equivariant Plateau problem and interior regularity, *Trans. Amer. Math. Soc.* 173 (1972).
[29] F. Pacard; J. Wei, Stable solutions of the Allen-Cahn equation in dimension 8 and minimal cones. *J. Funct. Anal.* 264 (2013), no. 5, 1131–1167.
[30] O. Savin, Regularity of flat level sets in phase transitions. *Ann. of Math.* (2) 169 (2009), no. 1, 41–78.
[31] M. Schatzman, On the stability of the saddle solution of Allen-Cahn's equation. *Proc. Roy. Soc. Edinburgh Sect. A* 125 (1995), no. 6, 1241–1275.
[32] L. Simon, Entire solutions of the minimal surface equation, *J. Differential Geometry* 30 (1989), 643–688.
[33] J. Simons, Minimal varieties in riemannian manifolds. *Ann. of Math.* (2) 88(1968), 62–105.

Y. Liu: Department of Mathematics, University of Science and Technology of China, Hefei, China
E-mail address: `yliumath@ustc.edu.cn`

K. Wang: School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei, China
E-mail address: `wangkelei@whu.edu.cn`

J. Wei: Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: `jcwei@math.ubc.ca`