EXISTENCE OF POSITIVE WEAK SOLUTIONS FOR FRACTIONAL LANE–EMDEN EQUATIONS WITH PRESCRIBED SINGULAR SETS

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Abstract. In this paper, we consider the problem of the existence of positive weak solutions of

$$\begin{cases} (-\Delta)^s u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega \end{cases}$$

having prescribed isolated interior singularities.

We prove that if $\frac{n}{n-2s} < p < p_1$ for some critical exponent p_1 defined in the introduction which is related to the stability of the singular solution u_s, and if S is a closed subset of Ω, then there are infinitely many positive weak solutions with S as its singular set.

We also show the existence of solutions to the fractional Yamabe problem with singular set to be the whole space $\mathbb{R}^n$. These results are the extension of Chen and Lin's result [9] to the fractional case.

1. Introduction

In this paper we are concerned with the existence of positive weak solutions with a prescribed singular set of the fractional version of the Lane-Emden equation

$$\begin{cases} (-\Delta)^s u = u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1$, Ω is a smooth open set in $\mathbb{R}^n$ with $n \geq 2$ and

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy$$

is the fractional Laplacian. Here $c_{n,s}$ is a normalization constant.

First, let us recall some results for the classical case when $s = 1$. Consider the Lane-Emden equation

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The corresponding equation to (1.2) in $\mathbb{R}^n$ for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ when $n \geq 3$ with an isolated singularity at the origin,

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \mathbb{R}^n \backslash \{0\}, \\ u > 0 & \text{in } \mathbb{R}^n \backslash \{0\}, \\ \lim_{|x| \to 0} u(x) = +\infty, \end{cases} \tag{1.3}$$

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is studied in [5] and [9]. The following result classifies all solutions of (1.3):

Proposition 1.1 ([9]). *Suppose that* $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ *and* u *is a solution of* (1.3). *Then either*

$$u(x) = c_{p,}|x|^{-\frac{2}{p-1}},$$

where $c_{p,n} = [\frac{1}{p-1}(n-2-\frac{2}{p-1})]^{\frac{1}{p-1}}$, *or there exists a constant* $\alpha > 0$ *such that*

$$\lim_{|x|\to\infty} u(x)|x|^{n-2} = \alpha.$$

Conversely, for any $\alpha > 0$, *there exists a unique solution* $u_\alpha(x)$ *of (1.3) such that*

$$\lim_{|x|\to\infty} u(x)|x|^{n-2} = \alpha.$$

Using this proposition, Chen and Lin [9] constructed positive weak solutions of (1.2) with prescribed interior singular set. The basic cells in the construction of Chen and Lin are the radially symmetric solutions u_α of (1.3) which have the following asymptotic behaviour at 0 and ∞:

$$u_\alpha(x) = \begin{cases} c_{p,n}|x|^{-\frac{2}{p-1}} & \text{as } |x| \to 0, \\ \alpha|x|^{-(n-2)} & \text{as } |x| \to \infty. \end{cases} \tag{1.4}$$

These solutions have the same behaviour near the origin but they converge uniformly to 0 on any compact subset of $\mathbb{R}^n \setminus \{0\}$ as $\alpha \to 0$. Thus given k different points $x_1, \cdots, x_k \in \Omega$, the function

$$u_*(x) = \sum_{i=1}^{k} u_\alpha(x - x_i) \tag{1.5}$$

constitutes for α small, a good approximate solution to (1.2) with isolated singularities at the points $x_1, \ldots x_k$. Then a variational argument establishes the existence of actual singular solution to (1.2) near $u_*(x)$. So the key point in the proof is the existence of a solution to (1.3) which satisfies (1.4).

A related question is to look at the Yamabe problem on $\mathbb{R}^n$ for the scalar curvature

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u > 0, \tag{1.6}$$

and consider the measure $d\mu = (1 + |x|^2)^{-\frac{n-2}{2}}\, dx$. Through stereographic projection, this problem is equivalent to the Yamabe problem on the sphere $\mathbb{S}^n$ with its canonical metric. Schoen and Yau conjectured in [28] that weak solutions for (1.6) that satisfy $u \in L^{\frac{n+2}{n-2}}(\mathbb{R}^n, d\mu)$ must have a singular set of Hausdorff dimension less or equal than $\frac{n-2}{2}$. A counterexample was provided by [26] in dimensions 4 and 6, while [9] gave examples of weak solutions that are singular in the whole $\mathbb{R}^n$. However, in all these examples the conformal metric $g_u = u^{\frac{n+2}{n-2}}|dx|^2$ is not complete.

On the other hand, complete metrics that are solutions to (1.6) have been constructed by Mazzeo and Pacard [25] when the singular set is a smooth submanifold of dimension less than $\frac{n-2}{2}$. By the work of Schoen and Yau [28] for complete metrics, this dimension is sharp. The construction in [25] also relies on the existence of ODE solutions with the asymptotic behavior (1.4) which allow to construct a suitable approximate solution that is singular along the prescribed submanifold.

Finally, the Yamabe problem on $\mathbb{R}^n$ with isolated singularities has been considered in [27, 24].

Let us go back to the non-local equation (1.1). In this paper, we would like to obtain similar results as in Chen-Lin [9] for the fractional case $0 < s < 1$. As we have mentioned before, the key point is the existence of fast decaying singular entire solutions to

$$\begin{cases} (-\Delta)^s u = u^p & \text{in } \mathbb{R}^n \backslash \{0\}, \\ u > 0 & \text{in } \mathbb{R}^n \backslash \{0\}, \\ \lim\limits_{|x|\to 0} u(x) = +\infty. \end{cases} \tag{1.7}$$

For this equation, it is known in [17] that

$$u_s(x) = A_{p,n}|x|^{-\frac{2s}{p-1}}, \tag{1.8}$$

where

$$A_{p,n}^{p-1} = \lambda\big(\tfrac{n-2s}{2} - \tfrac{2s}{p-1}\big) \tag{1.9}$$

and

$$\lambda(\alpha) = 2^{2s}\frac{\Gamma(\frac{n+2s+2\alpha}{4})\Gamma(\frac{n+2s-2\alpha}{4})}{\Gamma(\frac{n-2s-2\alpha}{4})\Gamma(\frac{n-2s+2\alpha}{4})},$$

is a singular solution to (1.7). By virtue of the following Hardy's inequality [20, 16]

$$\Lambda_{n,s}\int_{\mathbb{R}^n}\frac{\phi^2}{|x|^{2s}}\,dx \le \|\phi\|^2_{H^s(\mathbb{R}^n)} \tag{1.10}$$

with the optimal constant given by

$$\Lambda_{n,s} = 2^{2s}\frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

this solution u_s is stable if

$$\Lambda_{n,s} > pA_{p,n}^{p-1}, \tag{1.11}$$

which is equivalent to

$$H(n,s) > pK(n,p,s),$$

where

$$H(n,s) := \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}, \quad K(n,p,s) := \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})}.$$

Concerning the roots of

$$H(n,s) = pK(n,p,s), \tag{1.12}$$

recently, Luo, Wei and Zou [23] have proved the following classification result: there exists $n_0(s) \in \mathbb{N}^+$, such that for $n \le n_0(s)$, there exists only one root p_1 of (1.12) and it satisfies $\frac{n}{n-2s} < p_1 < \frac{n+2s}{n-2s}$. On the contrary, for $n > n_0(s)$, there exist exactly two roots p_1, p_2 and they satisfy $\frac{n}{n-2s} < p_1 < \frac{n+2s}{n-2s} < p_2 < +\infty$ (for the explicit expression of p_1 and p_2, see Theorem 1.1 in [23]). In summary there exists a unique $p_1 \in (\frac{n}{n-2s}, \frac{n+2s}{n-2s})$ such that for $\frac{n}{n-2s} < p < \frac{n+2s}{n-2s}$,

$$u_s \text{ is stable if and only if } \frac{n}{n-2s} < p < p_1.$$

In order to find a good approximate solution for our problem, it is crucial that there exists an entire solution to (1.7) which has the same asymptotic behaviour as $u_s(x)$ in (1.8) near 0, but which has a faster decay at ∞.

Our first result shows that:

Theorem 1.2. *Suppose* $\frac{n}{n-2s} < p < p_1$. *Then for every* $\alpha \in (0, \infty)$, *there exists a solution* u_α *of* (1.7) *with the following asymptotic behaviour:*

$$u_\alpha(x) = \begin{cases} A_{p,n}|x|^{-\frac{2s}{p-1}} & \text{as } |x| \to 0, \\ \alpha|x|^{-(n-2s)} & \text{as } |x| \to \infty. \end{cases}$$

Moreover, we have $0 < u_\alpha(x) < u_s(x)$ *for* $0 < \alpha < \infty$, *and*

$$\lim_{\alpha \to 0} u_\alpha(x) = 0, \ \lim_{\alpha \to \infty} u_\alpha(x) = u_s(x)$$

uniformly in any compact set in $\mathbb{R}^n \backslash \{0\}$.

Unlike the corresponding result when $s = 1$ which can be proved by a simple phase-plane analysis, the proof of Theorem 1.2 is quite involved: first we use Kelvin transformation to a subcritical problem with a weight. Then we use a blow up argument to establish the existence of entire solutions to this subcritical problem.

We conjecture that outside the stability regime, i.e., for $p_1 \le p < \frac{n+2s}{n-2s}$, this theorem is also true. We hope to return to this problem elsewhere.

Using Theorem 1.2, we follow the arguments in Chen-Lin [9] to prove the following existence results. While the general scheme of the proofs in [9] is also valid in the fractional case, there are extra difficulties and subtleties that come from the non-locality of the operator $(-\Delta)^s$, for instance when handling cutoff functions.

Theorem 1.3. *Suppose that* $0 < s < 1$, $\frac{n}{n-2s} < p < p_1$, Ω *is a bounded smooth domain in* $\mathbb{R}^n$ *and* S *is a closed subset of* Ω.

Then there exist two distinct sequences of solutions of (1.1) *having* S *as their singular set such that one sequence converges to* 0 *in* $L^q(\Omega)$, *and the other sequence converges to a smooth positive solution of* (1.1) *in* $L^q(\Omega)$ *for* $1 < q < p^* := \frac{n(p-1)}{2s}$.

The next result concerns the existence of weak solutions to the fractional Yamabe problem

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}} \text{ in } \mathbb{R}^n, \quad u > 0, \tag{1.13}$$

with prescribed singular set. For the regular case, see the references [19, 21, 22].

In the case of an isolated singularity for (1.13), the authors have established in [6] the asymptotic behavior of solutions in the spirit of [5], while radially symmetric solutions have been constructed in [13, 14]. In addition, Ao, Dela Torre, González and Wei [1] have recently proved the existence of solutions to 1.13 that are singular at a prescribed set of isolated points through a gluing method together with a Lyapunov-Schimidt reduction.

In this paper we are interested in the existence of solutions to (1.13) with non-isolated singular sets, in fact, when the Hausdorff dimension $dim_{\mathcal{H}}$ of the singular set is greater than 0. In the case of complete metrics that are singular along a smooth submanifold S, a dimension estimate was shown in [18], and it includes, in particular, the case $dim_{\mathcal{H}}(S) < \frac{n-2\gamma}{2}$. On the contrary, if one removes the completeness assumption, one may have very general singular sets as in the classical Yamabe problem. This is our next result:

Let $d\mu = (1 + |x|^2)^{-\frac{n-2s}{2}} dx$, and assume that $s \in \left(0, \min\{1, \frac{n}{4}\}\right)$ satisfies the following:

H. There exists an integer $m > \frac{n+2s}{2}$ such that

$$\frac{\Gamma(\frac{m+2s}{4})^2}{\Gamma(\frac{m-2s}{4})^2} > p\frac{\Gamma(\frac{m}{2}-\frac{s}{p-1})\Gamma(s+\frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{m-2s}{2}-\frac{s}{p-1})}$$

for $p = \frac{n+2s}{n-2s}$.

Theorem 1.4. *Suppose* $0 < s < \min\left\{1, \frac{n}{4}\right\}$ *satisfies assumption* **H.** *above. Then there exist positive weak solutions of* (1.13) *in* $L^{\frac{n+2s}{n-2s}}(\mathbb{R}^n, d\mu)$ *whose singular set is the whole* $\mathbb{R}^n$.

Remark 1.5. *For* $s = 1$*, it is shown in* [9] *that for* $n \geq 9$*, assumption* **H.** *is satisfied for* $m = \frac{n+3}{2}$ *for* n *odd and* $m = \frac{n+4}{2}$ *for* n *even. So at least for* s *close to* 1 *and* $n \geq 9$*, we can find some* m *such that* **H.** *is satisfied.*

The paper is organized as follows. In Section 2, we will deal with fast decaying entire solutions of (1.7) and give the proof of Theorem 1.2. Instead of ODE methods, we use a blow up argument which has been used in [12] to prove this result. Once the existence of singular entire solutions is obtained, we then follow the idea in Chen and Lin's paper [9]. We use these entire solutions to construct approximate solutions for (1.1) with finitely many isolated singular points in Section 3. In Section 4, we give complete proofs of Theorems 1.3 and 1.4.

2. Proof of Theorem 1.2: Construction of entire radial singular solutions of (1.7)

In this section, we consider the entire solutions for (1.7). We will use the Kelvin transform and a blow up argument in [12] to prove Theorem 1.2.

It is known [17] that

$$(-\Delta)^s u = u^p \text{ in } \mathbb{R}^n \tag{2.1}$$

has a singular solution of the form

$$u_s(r) = A_{p,n} r^{-\frac{2s}{p-1}},$$

where $r = |x|$ and the constant $A_{p,n}$ is defined in (1.9). We would like to construct another singular solution to (2.1) with fast decay at infinity. More precisely,

$$u_\alpha(r) = \begin{cases} A_{p,n} r^{-\frac{2s}{p-1}} & \text{as } r \to 0, \\ \alpha r^{-(n-2s)} & \text{as } r \to \infty. \end{cases} \tag{2.2}$$

First of all, the Kelvin transform $v(x) = |x|^{-(n-2s)} u\big(\frac{x}{|x|^2}\big)$ of $u(x)$ satisfies

$$(-\Delta)^s v(x) = |x|^\beta v^p(x) \quad \text{in } \mathbb{R}^n, \tag{2.3}$$

where $\beta = p(n-2s) - (n+2s)$. The corresponding singular solution is

$$v_s(x) = A_{p,n} |x|^{\frac{2s}{p-1} - (n-2s)},$$

since $p > \frac{n}{n-2s}$ implies $\frac{2s}{p-1} - (n-2s) < 0$.

We show that under the conditions $\frac{n}{n-2s} < p < \frac{n+2s}{n-2s}$ and

$$\frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2} > p\frac{\Gamma(\frac{n}{2}-\frac{s}{p-1})\Gamma(s+\frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2}-\frac{s}{p-1})}, \tag{2.4}$$

v_s is a stable solution to (2.3). Indeed, the corresponding quadratic form is given by

$$\begin{aligned} Q(v_s)(\phi) &= \int_{\mathbb{R}^n} (-\Delta)^s \phi \cdot \phi - p|x|^\beta v_s^{p-1} \phi^2 \\ &= \|\phi\|_{H^s}^2 - p \int_{\mathbb{R}^n} |x|^\beta v_s^{p-1} \phi^2 \\ &= \|\phi\|_{H^s}^2 - p A_{p,n}^{p-1} \int_{\mathbb{R}^n} |x|^{-2s} \phi^2 \, dx. \end{aligned}$$

By Hardy's inequality

$$\Lambda_{n,s} \int_{\mathbb{R}^n} \frac{\phi^2}{|x|^{2s}} \leq \|\phi\|_{H^s}^2,$$

we have

$$Q(v_s)(\phi) \geq \left(1 - \frac{p A_{p,n}^{p-1}}{\Lambda_{n,s}}\right) \|\phi\|_{H^s}^2 > 0,$$

provided that $\Lambda_{n,s} > p A_{p,n}^{p-1}$, which is equivalent to (2.4) by direct calculation. This implies that v_s is stable.

So we have proven that under (2.4), v_s is a stable singular solution of (2.3). We will use this singular solution to construct an entire stable radial solution to (2.3) following the idea in [12].

We will work with the localized extension problem which is due to Caffarelli-Silvestre [7]. Write $X = (x,t) \in \mathbb{R}_+^{n+1}$. For $0 < s < \sigma < 1$, and $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1+|y|)^{-(n+2s)} \, dy)$, the s-harmonic extension $\bar{v}(X)$ of $u(x)$ is given by

$$\bar{v}(X) = \int_{\mathbb{R}^n} P(X,y) u(y) \, dy,$$

where

$$P(X,y) = p_{n,s} t^{2s} \left| X - (y,0) \right|^{-(n+2s)}$$

and $p_{n,s}$ is chosen so that $\int_{\mathbb{R}^n} P(X,y) \, dy = 1$. Then $\bar{v} \in C^2(\mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$, $t^{1-2s} \partial_t \bar{v} \in C(\overline{\mathbb{R}_+^{n+1}})$ and $\bar{v}$ satisfies

$$\begin{cases} \nabla \cdot (t^{1-2s} \nabla \bar{v}) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ \bar{v} = v & \text{on } \partial \mathbb{R}_+^{n+1} \\ -\lim\limits_{t \to 0} t^{1-2s} \partial_t \bar{v} = \kappa_s (-\Delta)^s v & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases}$$

where

$$\kappa_s = \frac{2^{1-2s} \Gamma(1-s)}{\Gamma(s)}.$$

In our case, let $\lambda \in (0,1)$ and denote by B_r^{n+1} the ball in $\mathbb{R}^{n+1}$ of radius r centered at the origin. Writing $\bar{v}_s$ as the s-harmonic extension of v_s, we consider the extension problem in the unit ball

$$\begin{cases} \nabla \cdot (t^{1-2s} \nabla \bar{v}) = 0 & \text{in } B_1^{n+1} \cap \mathbb{R}_+^{n+1} \\ \bar{v} = \lambda \bar{v}_s & \text{on } \partial B_1^{n+1} \cap \mathbb{R}_+^{n+1} \\ -\lim\limits_{t \to 0} t^{1-2s} \partial_t \bar{v} = \kappa_s \left| x \right|^\beta v^p & \text{on } B_1^{n+1} \cap \partial \mathbb{R}_+^{n+1}. \end{cases} \tag{2.5}$$

Since $\bar{v}_s$ is a positive super-solution of (2.5), we get a minimal solution $v = v_\lambda$. The family (v_λ) is non-decreasing in λ. Moreover, by the truncation method (see [2], [11]), v_λ is bounded for each fixed $\lambda \in (0,1)$. The moving plane method (see, for instance, [4]) shows that v_λ is axially symmetric in x with respect to the t-axis and is decreasing in $|x|$. Since v_s is stable, we have $v_\lambda \uparrow \bar{v}_s$ as $\lambda \uparrow 1$ by the classical convexity argument. The readers are referred to [3] or Section 3.2.2 in the monograph [15]. Note also that having $|x|^\beta$ in the equation does not modify the arguments.

Now we perform a blow-up argument to obtain a radial entire stable solution. Let $\lambda_j \uparrow 1$ and

$$m_j = \left\|v_{\lambda_j}\right\|_{L^\infty} = v_{\lambda_j}(0), \quad R_j = m_j^{\frac{p-1}{2s}},$$

so that m_j, $R_j \to \infty$ as $j \to \infty$. Define

$$\bar{V}_j(x) = m_j^{-1} v_{\lambda_j}\left(\frac{x}{R_j}\right).$$

Then $0 \le \bar{V}_j \le \min\{1, \bar{v}_s\}$ solves

$$\begin{cases} \nabla\cdot(t^{1-2s}\nabla \bar{V}_j) = 0 & \text{in } B_{R_j}^{n+1} \cap \mathbb{R}_+^{n+1} \\ \bar{V}_j = \lambda_j \bar{v}_s & \text{on } \partial B_{R_j}^{n+1} \cap \mathbb{R}_+^{n+1} \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t \bar{V}_j = \kappa_s |x|^\beta \bar{V}_j^p & \text{on } B_{R_j}^{n+1} \cap \partial\mathbb{R}_+^{n+1} \end{cases}$$

By elliptic estimates, we can extract a subsequence of $(\bar{V}_j)$ converging in $C_{loc}(\overline{\mathbb{R}_+^{n+1}})$ to some axially symmetric function $\bar{V}$ which satisfies $0 \le \bar{V} \le \min\{1, \bar{v}_s\}$, $\bar{V}(0) = 1$ and solves

$$\begin{cases} \nabla\cdot(t^{1-2s}\nabla \bar{V}) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t \bar{V} = \kappa_s |x|^\beta \bar{V}^p & \text{on } \partial\mathbb{R}_+^{n+1}. \end{cases}$$

Therefore, it trace $v = \bar{V}(\cdot, 0)$ is a bounded, smooth and radially decreasing solution to

$$(-\Delta)^s v = |x|^\beta v^p \quad \text{in } \mathbb{R}^n.$$

Moreover, from $v \le v_s$ one also deduces that v is stable.

It has the following asymptotic behaviour

$$v(x) = \begin{cases} 1 + O(|x|^2) & \text{as } |x| \to 0, \\ A_{p,n}|x|^{\frac{2s}{p-1}-(n-2s)} & \text{as } |x| \to \infty. \end{cases}$$

If we denote $v_\alpha(x) = \alpha v\left(\alpha^{\frac{p-1}{p(n-2s)-n}} x\right)$, then $v_\alpha(x)$ is also solution to (2.3) and the corresponding solution $u_\alpha(x) = |x|^{-(n-2s)} v_\alpha(\frac{x}{|x|^2})$ is a solution of (2.1) which satisfies

$$u_\alpha(x) = \begin{cases} A_{p,n}|x|^{-\frac{2s}{p-1}} & \text{as } |x| \to 0, \\ \alpha|x|^{-(n-2s)} & \text{as } |x| \to \infty. \end{cases} \tag{2.6}$$

Moreover, by the above argument and the maximum principle, one can check that $0 < u_\alpha(x) < u_s(x)$ for $0 < \alpha < \infty$, and

$$\lim_{\alpha\to 0} u_\alpha(x) = 0, \quad \lim_{\alpha\to\infty} u_\alpha(x) = u_s(x)$$

uniformly in any compact set in $\mathbb{R}^n \backslash \{0\}$.

Thus we finish the proof of Theorem 1.2.

3. Construction of approximate solutions

We construct approximate solutions of (1.1) in this section. The ideas here follow closely [9]; our new difficulty is to estimate the non-local terms that come from $(-\Delta)^s$ and $\|\cdot\|_{H^s}$.

A pair of functions $(\bar{u}, \bar{f})$ is called *quasi-solution* of (1.1) if

$$(-\Delta)^s \bar{u} = \bar{u}^p + \bar{f} \quad \text{in } \Omega.$$

Given the family of solutions $\{u_\alpha\}$ that we have just constructed, we have

$$\lim_{\alpha\to 0} \int_{\mathbb{R}^n} u_\alpha^q \, dx = 0 \tag{3.1}$$

for $0 < q < p^*$, where p^* is defined by

$$p^* = \frac{n(p-1)}{2s}.$$

Under our hypothesis $\frac{n}{n-2s} < p < p_1$, which yields the stability condition (1.11), let ε_0 be

$$\varepsilon_0 = 1 - \frac{pA_{p,n}^{p-1}}{\Lambda_{n,s}} > 0.$$

The reason for this choice will become clear in the proof.

Lemma 3.1. *Fix $p_0, q_0 > 0$ such that $p < p_0 < p^*$, $\frac{2n}{n+2s} < q_0 < \frac{n}{2s}$. Let $\eta > 0$ and $\{\bar{x}_1, \cdots, \bar{x}_k\} \subset \Omega$.*

Then a quasi-solution $(\bar{u}, \bar{f})$ of (1.1) can be constructed to satisfy the following:

(i) $\bar{u} := \bar{u}_k$ is smooth except at $\bar{x}_j$, $1 \le j \le k$. At $\bar{x}_j$, $\bar{u}_k$ has the asymptotic behavior

$$\lim_{x\to\bar{x}_j} \bar{u}_k(x)|x-\bar{x}_j|^{\frac{2s}{p-1}} = A_{p,n}. \tag{3.2}$$

(ii)

$$\int_\Omega \bar{u}_k^{p_0} \, dx < \eta, \quad \int_\Omega \bar{f}_k^{q_0} \, dx < \eta. \tag{3.3}$$

(iii) Set

$$Q_k(\phi) = \left(1 + \sum_{j=1}^k 3^{-j}\varepsilon_0 - \varepsilon_0\right) \|\phi\|_{H^s}^2 - p\int_\Omega \bar{u}_k^{p-1}\phi^2 \, dx, \tag{3.4}$$

for $\phi \in H_0^s(\Omega)$. Then Q_k is positive definite and equivalent to the H_0^s-norm in $H_0^s(\Omega)$.

Proof. We will construct $(\bar{u}_k, \bar{f}_k)$ by induction on k. First we note that for $\alpha \in (0,\infty)$, we have

$$\begin{aligned} p\int_{\mathbb{R}^n} u_\alpha^{p-1}(x)\phi^2(x)\,dx &\le p\int_{\mathbb{R}^n} u_s^{p-1}(x)\phi^2(x)\,dx \\ &= pA_{p,n}\int_{\mathbb{R}^n} |x|^{-2s}\phi^2(x)\,dx \\ &= (1-\varepsilon_0)\|\phi\|_{H^s}^2, \end{aligned} \tag{3.5}$$

for any $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Here we have used Hardy's inequality (1.10) in the last step.

Now let $\chi(x) = \chi(|x|)$ be a smooth cut-off function such that $\chi(t) = 1$ for $0 \le t \le \frac{1}{2}$ and $\chi(t) = 0$ for $t \ge 1$. For $r > 0$, we denote $\chi_r(x) = \chi(\frac{x}{r})$.

For $k = 0$, the pair $(0, 0)$ is a (trivial) quasi-solution and satisfies *(i)-(iii)* of Lemma 3.1.

Next, suppose that the conclusions of Lemma 3.1 hold true for $(k-1)$ points $(\bar{x}_1, \cdots, \bar{x}_{k-1})$ and $(\bar{u}_{k-1}, \bar{f}_{k-1})$ is a quasi-solution satisfying *(i)-(iii)*. Let

$$0 < r_k < \frac{1}{2} \min\{\mathrm{dist}(\bar{x}_k, \partial\Omega), |\bar{x}_k - \bar{x}_j|, 1 \le j \le k-1\}$$

and define

$$\bar{u}_k = \bar{u}_{k-1} + \chi_{r_k}(x - \bar{x}_k) u_\alpha(x - \bar{x}_k) \tag{3.6}$$

where r_k, α are to be chosen later. Then

$$\|\bar{u}_k - \bar{u}_{k-1}\|_{L^{p_0}} \le \|u_\alpha\|_{L^{p_0}}.$$

If α is small enough, we have

$$\|u_\alpha\|_{L^{p_0}} < \frac{1}{2}(\eta - \|\bar{u}_{k-1}\|_{L^{p_0}}),$$

which yields

$$\|\bar{u}_k\|_{L^{p_0}} < \frac{1}{2}(\eta + \|\bar{u}_{k-1}\|_{L^{p_0}}) < \eta.$$

For simplicity, denote $\chi_k := \chi_{r_k}$, we may also take $\bar{x}_k = 0$. Set

$$\begin{aligned}
\bar{f}_k &= (-\Delta)^s \bar{u}_k - \bar{u}_k^p \\
&= \bar{f}_{k-1} - (\bar{u}_k^p - \bar{u}_{k-1}^p - \chi_k^p u_\alpha^p) \\
&\quad + \Big[(\chi_k - \chi_k^p) u_\alpha^p + c_{n,s} PV \int_\Omega \frac{u_\alpha(y)(\chi_k(x) - \chi_k(y))}{|x-y|^{n+2s}}\, dy\Big] \\
&=: f_{k-1} - g_1 + g_2.
\end{aligned}$$

We note that $\bar{u}_{k-1}$ is smooth in $B(\bar{x}_k, r_k)$. Hence,

$$\int_\Omega g_1^{q_0}\, dx = \int_{B(\bar{x}_k, r_k)} |g_1|^{q_0}\, dx \le C \int_{B(\bar{x}_k, r_k)} (1 + u_\alpha^{(p-1)q_0})\, dx, \tag{3.7}$$

which can be small if both r_k, α are small and $\frac{2n}{n+2s} \le q_0 < \frac{n}{2s}$. Moreover,

$$\begin{aligned}
\|g_2\|_{L^\infty} &\le \|(\chi_k - \chi_k^p) u_\alpha^p\|_{L^\infty} + \Big\| c_{n,s} PV \int_\Omega \frac{u_\alpha(y)(\chi_k(x) - \chi_k(y))}{|x-y|^{n+2s}}\, dy \Big\|_{L^\infty} \\
&\le C(\|u_\alpha\|_{L^\infty(D_k)} + \|u_\alpha\|_{L^q}),
\end{aligned} \tag{3.8}$$

where $D_k = \{x \in \mathbb{R}^n, \frac{r_k}{2} < |x| < r_k\}$ for some $1 < q < p^*$, which is small if α is small enough by (3.1). We conclude from (3.7) and (3.8) that

$$\|f_k\|_{L^{q_0}} \le \|f_{k-1}\|_{L^{q_0}} + \|g_1\|_{L^{q_0}} + \|g_2\|_{L^{q_0}} < \eta.$$

This proves 3.3 for $\bar{u}_k$ and $\bar{f}_k$.

The proof of *(iii)* is divided into two steps:

Step 1. There exists a constant $C_0 > 0$ independent of α such that

$$p \int_\Omega \bar{u}_k^{p-1} \phi^2\, dx \le (1 + \sum_{j=1}^{k-1} 3^{-j} \varepsilon_0 - \varepsilon_0) \|\phi\|_{H^s}^2 + C_0 \int_\Omega \phi^2\, dx \tag{3.9}$$

holds true for any $\phi \in H_0^s(\Omega)$.

Let $\chi_i \in \mathcal{C}^\infty(\Omega), i = 1,2$, such that $\chi_1^2 + \chi_2^2 = 1$ and the support of χ_1 is disjoint from $B(\bar{x}_k, r_k)$ and the support of χ_2 is disjoint from $\{\bar{x}_1, \cdots, \bar{x}_{k-1}\}$. Then, by our induction hypothesis and the initial claim (3.5),

$$\begin{aligned} p\int_\Omega \bar{u}_k^{p-1}\phi^2\,dx &= p\int_\Omega \bar{u}_k^{p-1}\chi_1^2\phi^2\,dx + p\int_\Omega \bar{u}_k^{p-1}\chi_2^2\phi^2\,dx \\ &\leq (1 + \sum_{j=1}^{k-1} 3^{-j}\varepsilon_0 - \varepsilon_0)\|\chi_1\phi\|_{H^s}^2 + (1-\varepsilon_0)\|\chi_2\phi\|_{H^s}^2 + C\int_\Omega \phi^2\,dx \\ &\leq (1 + \sum_{j=1}^{k-1} 3^{-j}\varepsilon_0 - \varepsilon_0)(\|\chi_1\phi\|_{H^s}^2 + \|\chi_2\phi\|_{H^s}^2) + C\int_\Omega \phi^2\,dx \\ &\leq (1 + \sum_{j=1}^{k} 3^{-j}\varepsilon_0 - \varepsilon_0)\|\phi\|_{H^s}^2 + C\int_\Omega \phi^2\,dx. \end{aligned}$$

Here we have used that

$$\begin{aligned} &\|\chi_1\phi\|_{H^s}^2 + \|\chi_2\phi\|_{H^s}^2 \\ &= \|\phi\|_{H^s}^2 + \frac{c_{n,s}}{2}\int_\Omega\int_\Omega \frac{\phi(x)\phi(y)[(\chi_1(x)-\chi_1(y))^2 + (\chi_2(x)-\chi_2(y))^2]}{|x-y|^{n+2s}}\,dxdy \\ &\leq \|\phi\|_{H^s}^2 + C\int_\Omega \phi^2\,dx. \end{aligned}$$

Step 2. Denote $\delta_0 = 1 + \sum_{j=1}^{k-1} 3^{-j}\varepsilon_0 - \varepsilon_0$. We can find a finite dimensional subspace $\mathcal{N}$ of $H_0^s(\Omega)$ such that

$$C_0\int_\Omega \phi^2\,dx \leq 3^{-(k+1)}\delta_0^{-1}\varepsilon_0 Q_{k-1}(\phi) \tag{3.10}$$

for all $\phi \in H_0^s(\Omega)$ which is orthogonal to $\mathcal{N}$ with respect to the quadratic form Q_{k-1} and C_0 is the constant stated in (3.9).

Consider, for instance, the k-th eigenspace $\bar{\mathcal{N}}$ for $(-\Delta)^s$ with Dirichlet conditions for $k \geq l$ with l large. Since the eigenvalues of $(-\Delta)^s$ increases to $+\infty$, we can take $\mathcal{N} = \bar{\mathcal{N}}^\perp$ with respect to Q_{k-1} in $H_0^s(\Omega)$.

For any $\phi \in H_0^s(\Omega)$, decompose $\phi = \phi_1 + \phi_2$ where $\phi_1 \in \mathcal{N}$ and $\phi_2 \in \mathcal{N}^\perp$ (with respect to Q_{k-1}). Let $\bar{u}_k = \bar{u}_{k-1} + v_k$ and $B_k = B(\bar{x}_k, r_k)$. Then,

$$\begin{aligned} p\int_\Omega \bar{u}_k^{p-1}\phi^2\,dx &= p\int_\Omega \bar{u}_k^{p-1}(\phi_1+\phi_2)^2\,dx \\ &= p\int_\Omega \bar{u}_k^{p-1}\phi_1^2\,dx + p\int_\Omega \bar{u}_k^{p-1}\phi_2^2\,dx + 2p\int_\Omega \bar{u}_k^{p-1}\phi_1\phi_2\,dx \\ &\leq p\int_\Omega \bar{u}_k^{p-1}\phi_2^2\,dx + p\int_\Omega \bar{u}_{k-1}^{p-1}\phi_1^2\,dx + 2p\int_\Omega \bar{u}_{k-1}^{p-1}\phi_1\phi_2\,dx \\ &\quad + C\left(\int_\Omega v_k^{p-1}\phi_1^2\,dx + \int_\Omega v_k^{p-1}\phi_1\phi_2\,dx + \int_{B_k}(\phi_1^2 + \phi_1\phi_2)\,dx\right), \end{aligned} \tag{3.11}$$

where we have used that $\bar{u}_k = \bar{u}_{k-1}$ outside B_k, and that in B_k,

$$\bar{u}_k^{p-1} = (\bar{u}_{k-1} + v_k)^{p-1} \leq C(1 + v_k^{p-1}).$$

To estimate the sum of the first three terms in the last step of (3.11), we utilize Step 1 and the induction hypothesis to obtain

$$p\int_\Omega \bar{u}_k^{p-1}\phi_2^2\,dx + p\int_\Omega \bar{u}_{k-1}^{p-1}\phi_1^2\,dx + 2p\int_\Omega \bar{u}_{k-1}^{p-1}\phi_1\phi_2\,dx \\ \leq \delta_0(\|\phi_1\|_{H^s}^2 + \|\phi_2\|_{H^s}^2) + 2p\int_\Omega \bar{u}_{k-1}^{p-1}\phi_1\phi_2\,dx + C_0\int_\Omega \phi_2^2\,dx. \tag{3.12}$$

Since $\phi_1 \perp \phi_2$, by the definition of the quadratic form Q_{k-1} and its associated bilinear form we have

$$p\int_\Omega \bar{u}_{k-1}^{p-1}\phi_1\phi_2 = \delta_0\int_\Omega\int_\Omega \frac{(\phi_1(x)-\phi_1(y))(\phi_2(x)-\phi_2(y))}{|x-y|^{n+2s}}\,dxdy. \tag{3.13}$$

Therefore, recalling the definition of the H^s norm, we have from (3.12) and (3.10),

$$\begin{aligned} &p\int_\Omega \bar{u}_k^{p-1}\phi_2^2\,dx + p\int_\Omega \bar{u}_{k-1}^{p-1}\phi_1^2\,dx + 2p\int_\Omega \bar{u}_{k-1}^{p-1}\phi_1\phi_2\,dx \\ &\leq \delta_0\|\phi\|_{H^s}^2 + C_0\int_\Omega \phi_2^2\,dx \\ &\leq \delta_0\|\phi\|_{H^s}^2 + 3^{-(k+1)}\delta_0^{-1}\varepsilon_0 Q_{k-1}(\phi) \\ &\leq (\delta_0 + 3^{-(k+1)}\varepsilon_0)\|\phi\|_{H^s}^2. \end{aligned}$$

Now we estimate the last line in (3.11). For any $\tilde{\varepsilon} > 0$, using Hölder's inequality and (3.5) we have

$$\begin{aligned} \int_\Omega v_k^{p-1}(|\phi_1\phi_2| + \phi_1^2)\,dx &\leq \tilde{\varepsilon}\int_\Omega v_k^{p-1}\phi^2\,dx + C_{\tilde{\varepsilon}}\int_\Omega v_k^{p-1}\phi_1^2\,dx \\ &\leq C\tilde{\varepsilon}(1-\varepsilon_0)\|\phi\|_{H^s}^2 + C_{\tilde{\varepsilon}}\int_\Omega v_k^{p-1}\phi_1^2\,dx \\ &\leq \frac{3^{-(k+1)}}{2}\varepsilon_0\|\phi\|_{H^s}^2 + C_{\tilde{\varepsilon}}\int_{B_k} v_k^{p-1}\phi_1^2\,dx, \end{aligned}$$

provided that $\tilde{\varepsilon}$ is small enough. Next, for the last two terms of (3.11),

$$\begin{aligned} \int_{B_k}(\phi_1^2 + |\phi_1\phi_2|)\,dx &\leq C_{\tilde{\varepsilon}}\int_{B_k}\phi_1^2\,dx + \tilde{\varepsilon}\int_\Omega \phi^2\,dx \leq C_{\tilde{\varepsilon}}\int_{B_k}\phi_1^2\,dx + C_1\varepsilon\|\phi\|_{H^s}^2 \\ &\leq \frac{3^{-(k+1)}}{2}\varepsilon_0\|\phi\|_{H^s}^2 + C_{\tilde{\varepsilon}}\int_{B_k}\phi_1^2\,dx. \end{aligned}$$

In conclusion, we have that (3.11) can be estimated by

$$p\int_\Omega \bar{u}_k^{p-1}\phi^2\,dx \leq \left(\delta_0 + 2\cdot 3^{-(k+1)}\varepsilon_0\right)\|\phi\|_{H^s}^2 + C_1\int_{B_k}(v_k^{p-1}\phi_1^2 + \phi_1^2)\,dx,$$

where C is a constant independent of α and r_k. Since $\mathcal{N}$ has finite dimension, r_k can be chosen so small such that

$$C_1\int_{B_k}\phi_1^2\,dx \leq 2^{-1}3^{-(k+1)}\varepsilon_0 Q_{k-1}(\phi_1) \leq 2^{-1}3^{-(k+1)}\varepsilon_0\|\phi\|_{H^s}^2.$$

After fixing r_k, we may choose α small, then by Theorem 1.2 and (3.5), we have

$$\begin{aligned} C_1 \int v_k^{p-1}\phi_1^2\,dx &\le 2^{-1}3^{-(k+1)}\varepsilon_0 Q_{k-1}(\phi_1)\,dx \\ &\le 2^{-1}3^{-(k+1)}\varepsilon_0 Q_{k-1}(\phi)\,dx \\ &\le 2^{-1}3^{-(k+1)}\varepsilon_0 \|\phi\|_{H^s}^2. \end{aligned}$$

This completes the proof of the Lemma. □

Let $(\bar u, \bar f)$ be a quasi-solution. If $u = \bar u + v$ is a solution, then v satisfies

$$\begin{cases} (-\Delta)^s v + \bar u^p - (\bar u + v)^p + \bar f = 0 & \text{in } \Omega, \\ \bar u + v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

Define

$$E(\phi) = \frac{1}{2}\|\phi\|_{H^s}^2 - \int_\Omega F(\bar u, \phi)\,dx + \int_\Omega \bar f \phi\,dx \tag{3.15}$$

for $\phi \in H_0^s(\Omega)$, where

$$F(s,t) = \frac{1}{p+1}\left(|s+t|^p(s+t) - s^{p+1} - (p+1)s^p t\right). \tag{3.16}$$

Lemma 3.2. *$E \in \mathcal{C}^1(H_0^s;\mathbb{R})$ and any critical point v of E satisfies*

$$\begin{cases} (-\Delta)^s v - |\bar u + v|^p - \bar u^p + \bar f = 0 & \text{in } \Omega, \\ \bar v + v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

Moreover, E satisfies (P.S.) condition.

Proof. The first part of this lemma is standard. To prove the Palais-Smale condition, suppose that there is a sequence $\{v_j\} \subset H_0^s(\Omega)$ such that

$$E(v_j) \to C \quad \text{and} \quad E'(v_j) \to 0 \quad \text{in } H_0^s(\Omega) \tag{3.18}$$

as $j \to \infty$. We would like to show that there exists a strongly convergent sequence of $\{v_j\}$. It is straightforward to calculate

$$\begin{aligned} \langle E'(v_j), \phi\rangle = \frac{c_{n,s}}{2}\int_\Omega\int_\Omega \frac{(\phi(x)-\phi(y))(v_j(x)-v_j(y))}{|x-y|^{n+2s}}\,dxdy \\ - \int_\Omega (|\bar u + v_j|^p - \bar u^p)\phi\,dx + \int_\Omega \bar f\phi\,dx. \end{aligned} \tag{3.19}$$

Case 1. Let $p \le 2$. A direct computation shows that

$$\begin{aligned} &E(v_j) - \frac{1}{p+1}\langle E'(v_j), v_j\rangle \\ &\quad = \frac{p-1}{2(p+1)}\|v_j\|_{H^s}^2 - \frac{1}{p+1}\int_\Omega \Big[|\bar u + v_j|^p \bar u - \bar u^{p+1} - p\bar u^p v_j\Big]dx \\ &\quad + \frac{p}{p+1}\int_\Omega \bar f v_j\,dx. \end{aligned}$$

Since $p \le 2$, there holds the inequality

$$\Big||\bar u + v_j|^p\bar u - \bar u^{p+1} - p\bar u^p v_j\Big| \le \frac{p(p-1)}{2}\bar u^{p-1}v_j^2.$$

So we have by (3.18) that,

$$\frac{p-1}{2(p+1)}\left[\|v_j\|_{H^s}^2 - p\int_\Omega \bar{u}^{p-1}v_j^2\right] \le C(1+\|v_j\|_{H^s}). \tag{3.20}$$

By Lemma 3.1,

$$\|v_j\|_{H^s}^2 - p\int_\Omega \bar{u}^{p-1}v_j^2\,dx \ge C\|v_j\|_{H^s}^2. \tag{3.21}$$

Combining the above two estimates, we have the boundedness of $\|v_j\|_{H^s}$. Furthermore, by (3.19), we have

$$\begin{aligned}\langle E'(v_i)-E'(v_j), v_i-v_j\rangle &= \|v_i-v_j\|_{H^s}^2 \\ &\quad - \int_\Omega (|v_i+\bar{u}|^p - |v_j+\bar{u}|^p)(v_i-v_j)\,dx.\end{aligned} \tag{3.22}$$

Since

$$\begin{aligned}\big||1+x|^p-|1+y|^p\big| &\le p\max\{(1+|x|)^{p-1},(1+|y|)^{p-1}\}|x-y| \\ &\le p|x-y| + C(|x|^{p-1}+|y|^{p-1})|x-y|,\end{aligned}$$

we have from (3.22)

$$\begin{aligned}\|v_i-v_j\|_{H^s}^2 - p\int_\Omega \bar{u}^{p-1}|v_i-v_j|^2\,dx &\le C\int_\Omega (|v_i|^{p-1}+|v_j|^{p-1})(v_i-v_j)^2\,dx \\ &\quad + o(1)\|v_i-v_j\|_{H^s}.\end{aligned} \tag{3.23}$$

By Hölder's inequality, the first term of the right hand side can be estimated by

$$\begin{aligned}&\int_\Omega (|v_i|^{p-1}+|v_j|^{p-1})|v_i-v_j|^2\,dx \\ &\quad\le \left(\int_\Omega (|v_i|^{p-1}+|v_j|^{p-1})^{q'}\,dx\right)^{\frac{1}{q'}}\left(\int_\Omega |v_i-v_j|^{2q}\,dx\right)^{\frac{1}{q}}\,dx \\ &\quad\le C\left(\int_\Omega |v_i|^{q'(p-1)}+|v_j|^{q'(p-1)}\,dx\right)^{\frac{1}{q'}}\|v_i-v_j\|_{L^{2q}}^2,\end{aligned}$$

where $\frac{1}{q'}+\frac{1}{q}=1$, $\frac{1}{q}=1-\frac{(n-2s)(p-1)}{2n}>1-\frac{2s}{n}$, so that $2q<\frac{2n}{n-2s}$ and $q'(p-1)=\frac{2n}{n-2s}$.

By construction, there exists a subsequence of $\{v_j\}$, still denoted by v_j, which converges in $L^{2q}(\Omega)$. Then by the above inequality, from (3.23) we have that $\{v_j\}$ strongly converges in $H_0^s(\Omega)$.

Case 2. Let $p>2$. We first use (3.19) with $\phi=v_j^-$ to get

$$\langle E'(v_j), v_j^-\rangle = -\|v_j^-\|_{H^s}^2 + \int_\Omega (\bar{u}^p - \left|\bar{u}-v_j^-\right|^p)v_j^-\,dx + \int_\Omega \bar{f}v_j^-\,dx.$$

Here we have used the fact that $\langle v_j^+, v_j^-\rangle = 0$. Since $1-|1-x|^p\le px$ for all $x\ge 0$, we have

$$(1-o(1))\left\|v_j^-\right\|_{H^s}^2 - p\int_\Omega \bar{u}^{p-1}(v_j^-)^2\,dx \le C\left\|v_j^-\right\|_{H^s},$$

as $j \to \infty$. Now the boundedness of $\left\|v_j^-\right\|_{H^s}$ follows from Lemma 3.1(iii). Once we have that, it is easy to see

$$\begin{aligned}\left|\int_\Omega F(\bar u, -v_j^-)\right| &\le C\int_\Omega \left|v_j^-\right|^{p+1} + \bar u^{p-1}(v_j^-)^2\,dx\\ &\le C\int_\Omega \left(\left\|v_j^-\right\|_{H^s}^{p+1} + \left\|v_j^-\right\|_{H^s}^2\right)\\ &\le C\end{aligned}$$

so that

$$\begin{aligned}&E(v_j)\\ &= \frac{1}{2}\left\|v_j^+\right\|_{H^s}^2 + \frac{1}{2}\left\|v_j^-\right\|_{H^s}^2 - \int_\Omega F(\bar u, v_j^+)\,dx - \int_\Omega F(\bar u, v_j^-)\,dx + \int_\Omega \bar f(v_j^+ - v_j^-)\,dx\\ &= E(v_j^+) + O(1)\end{aligned}$$

as $j \to \infty$.

Next we look for a bound of $\left\|v_j^+\right\|_{H^s}$ by putting $\phi = v_j^+$ in (3.19), which yields

$$\langle E'(v_j), v_j^+\rangle = \|v_j^+\|_{H^s}^2 - \int_\Omega (|\bar u + v_j^+|^p - \bar u^p)v_j^+\,dx + \int_\Omega \bar f v_j^+\,dx. \tag{3.24}$$

From the elementary inequality

$$(1+x)^p x - x - \frac{2}{p+1}((1+x)^{p+1} - 1 - (p+1)x) \ge \frac{p-1}{p+1}x^{p+1},$$

which holds for $x \ge 0$, we have

$$\frac{p-1}{p+1}\int_\Omega (v_j^+)^{p+1} \le 2E(v_j) - \langle E'(v_j), v_j^+\rangle - \int_\Omega \bar f v_j^+\,dx + O(1). \tag{3.25}$$

For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$((\bar u + v_j^+)^p - \bar u^p)v_j^+ \le (p+\varepsilon)\bar u^{p-1}(v_j^+)^2 + C_\varepsilon (v_j^+)^{p+1}.$$

By (3.24),

$$\begin{aligned}&\|v_j^+\|_{H^s}^2 - (p+\varepsilon)\int_\Omega \bar u^{p-1}(v_j^+)^2\,dx\\ &\qquad\le \langle E'(v_j), v_j^+\rangle - \int_\Omega \bar f v_j^+\,dx + C_\varepsilon\int_\Omega (v_j^+)^{p+1}\,dx + O(1).\end{aligned}$$

Together with Lemma 3.1 and (3.25) we see for any fixed $\varepsilon > 0$ small,

$$\|v_j^+\|_{H^s}^2 \le C(1 + \|v_j^+\|_{H^s})$$

for all sufficiently large j. Hence we have established the boundedness of $\|v_j\|_{H^s} = \sqrt{\left\|v_j^+\right\|_{H^s}^2 + \left\|v_j^-\right\|_{H^s}^2}$. We can use the argument in Case 1 to obtain strongly convergence in $H_0^s(\Omega)$. Therefore, the (P.S.) condition is satisfied and the proof is complete.

□

Lemma 3.3. *Let $\{\bar x_1, \cdots, \bar x_k\}$ be any set of finite points in Ω. Then there exists at least two distinct solutions having $\{\bar x_1, \cdots, \bar x_k\}$ as its singular set.*

Proof. We claim that there exist positive numbers $\eta_0, \rho, \theta > 0$ such that if $(\bar{u}, \bar{f})$ is a quasi-solution of (1.1) with $0 < \eta < \eta_0$, then

$$E(u) \geq \theta > 0$$

for $u \in H_0^s(\Omega)$ such that $\rho \leq \|u\|_{H_0^s} \leq 2\rho$. After the claim, the existence part follows immediately. Indeed, one solution can be obtained from minimizing

$$\min_{\|u\|_{H^s} \leq \rho} E(u) \leq E(0) = 0 < \theta,$$

and the other solution obtained from the mountain pass lemma.

First, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F_+(s,t)| \leq \frac{p}{2}(1+\varepsilon)s^{p-1}t^2 + C_\varepsilon t^{p+1}.$$

Thus

$$2E(v) \geq \|v\|_{H^s}^2 - p(1+\varepsilon)\int_\Omega \bar{u}^{p-1}v^2\,dx - C_\varepsilon \int_\Omega v^{p+1}\,dx - 2\eta_0\Big(\int_\Omega v^{\frac{2n}{n-2s}}\,dx\Big)^{\frac{n-2s}{2n}},$$

using Lemma 3.1 to estimate $\bar{f}$. By Sobolev's embedding and Lemma 3.1 again, for $\varepsilon > 0$ small

$$2E(v) \geq C_1\|v\|_{H^s}^2 - C_2\left(\|v\|_{H^s}^{p+1} + \eta_0\|v\|_{H^s}\right).$$

Then the claim follows easily.

Suppose that $u = \bar{u} + v$ is any solution of (1.1) with $v \in H_0^s(\Omega)$. Since $p < \frac{n+2s}{n-2s}$ is subcritical, a standard bootstrap argument shows that $v \in \mathcal{C}^\infty(\Omega\backslash\{\bar{x}_1, \cdots, \bar{x}_k\})$ (see, for instance, [14] for the regularity of equation (1.1)).

If we assume that $\bar{x}_j$ is a removable singularity for u, by *(i)* of Lemma 3.1, $-\bar{u} \leq v \leq c - \bar{u}$ in a neighborhood of $\bar{x}_j$ which implies that $v \notin L^{p+1}(\Omega)$ (note that $\frac{2s(p+1)}{p-1} > n$). This contradicts the fact that $v \in H_0^s(\Omega)$ since $p + 1 < \frac{2n}{n-2s}$, and completes the proof of the Lemma.

□

Lemma 3.4. *Let v be a solution of* (3.14) *and $v \in H_0^s(\Omega)$. Then $|v|^a \in H_0^s(\Omega)$ for some $a > 1$. The constant a depends on p, q_0 and n.*

Proof. By Lemma 3.3, we know $v \in \mathcal{C}^\infty(\Omega\backslash\{\bar{x}_1, \cdots, \bar{x}_k\})$. We claim that there exists $\delta = \delta(p, q_0, n) > 0$ such that $|x - \bar{x}_i|^{-\delta}v \in L^{\frac{2n}{n-2s}}(\Omega)$ for $1 \leq i \leq k$.

Let $\eta(x) = (|x - \bar{x}_i|^2 + \sigma^2)^{-\frac{\delta}{2}}$, $1 \leq i \leq k$, for $\sigma > 0$ small. Using the weak formulation of (3.14) with the test function $\eta^2 v$ we have

$$\begin{aligned}
\|\eta v\|_{H^s}^2 &- \frac{c_{n,s}}{2}\int_\Omega\int_\Omega \frac{v(x)v(y)(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dxdy \\
&= \int_\Omega [(\bar{u}+v)^p - \bar{u}^p]\eta^2 v\,dx - \int_\Omega \bar{f}\eta^2 v\,dx \\
&\leq (p+\varepsilon)\int_\Omega \bar{u}^{p-1}\eta^2 v^2\,dx + C_\varepsilon\int_\Omega \eta^2 v^{p+1}\,dx - \int_\Omega \bar{f}\eta^2 v\,dx.
\end{aligned}$$

By the elementary inequality $2xy \leq x^2 + y^2$ and similar argument in the proof of Lemma 2.2 of [12], we have

$$\begin{aligned}
&\frac{c_{n,s}}{2}\int_\Omega\int_\Omega \frac{v(x)v(y)(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dxdy \\
&\leq C\int_\Omega\int_\Omega \left(v(x)^2\frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}} + v(y)^2\frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\right) dxdy \\
&\leq C\int_\Omega \frac{v(x)^2}{(\sigma^2+|x-\bar{x}_i|^2)^{\delta+s}}\,dx \\
&\leq C.
\end{aligned}$$

Thus, using Lemma 3.1, we deduce that for ε small enough,

$$\begin{aligned}
\|\eta v\|_{H^s}^2 \leq C + C&\left(\int_\Omega \eta^{2q}\,dx\right)^{\frac{1}{q}}\left(\int_\Omega v^{\frac{2n}{n-2s}}\,dx\right)^{\frac{(n-2s)(p+1)}{2n}} \\
&+ C\left(\int_\Omega (\bar{f}\eta^2)^{\frac{2n}{n+2s}}\,dx\right)^{\frac{n+2s}{2n}}\left(\int_\Omega v^{\frac{2n}{n-2s}}\,dx\right)^{\frac{n-2s}{2n}},
\end{aligned} \tag{3.26}$$

where we have applied Hölder's inequality with exponents $\frac{1}{q'} = \frac{(n-2s)(p+1)}{2n}$, $\frac{1}{q} = 1 - \frac{(n-2s)(p+1)}{2n}$ for the first integral, and with exponents $q' = \frac{2n}{n-2s}$, $q = \frac{2n}{n+2s}$ for the second one.

Since $\bar{f} \in L^{q_0}$ with $q_0 > \frac{2n}{n+2s}$, δ can be chosen so small that the integrands involving η are integrable around $\bar{x}_i$ even if $\sigma = 0$, so that the right hand side of the above (3.26) has an upper bounded independent of σ. In other words, if δ is small then we have

$$\|\eta v\|_{H^s}^2 \leq C.$$

Letting $\sigma \to 0$, we have $|x - x_i|^{-\delta} v \in H_0^s(\Omega)$, and the claim is proved.

Recalling (3.2), we know that $|v| \leq C|x - \bar{x}_i|^{-\tau}$ for $\tau \geq \frac{2s}{p-1}$. Then the above claim implies that $v \in L^{\frac{2n}{n-2s}a_0}$ for some $a_0 > 1$ which depends only on p, q_0, n, s. To estimate $\||v|^{a_1}\|_{H^s}$ for $1 < a_1 < a_0$, we use a pointwise inequality from [8] originally due to Córdoba-Córdoba [10], which states that for any $\varphi \in C^2(\mathbb{R}^n)$ convex and u such that $(-\Delta)^s u$ and $(-\Delta)^s \varphi(u)$ both exist, there holds

$$(-\Delta)^s \varphi(u) \leq \varphi'(u)(-\Delta)^s u.$$

With $\varphi(v) = |v|^{a_1}$, we compute

$$(-\Delta)^s |v|^{a_1} \leq a_1 |v|^{a_1-2} v(-\Delta)^s v$$

and hence

$$
\begin{aligned}
\| \, |v|^{a_1} \, \|_{H^s}^2 &= \int_\Omega |v|^{a_1} \, (-\Delta)^s \, |v|^{a_1} \, dx \\
&\le a_1 \int_\Omega |v|^{2a_1-1} |(-\Delta)^s v| \, dx \\
&\le a_1 \int_\Omega |v|^{2a_1-1} |\bar f| \, dx + C \int_\Omega (\bar u^{p-1} |v|^{2a_1} + |v|^{2a_1-1+p}) \, dx \\
&\le C \Bigg[\left(\int_\Omega |v|^{(2a_1-1)\frac{2n}{n-2s}} \, dx \right)^{\frac{n-2s}{2n}} \left(\int_\Omega \bar f^{\frac{2n}{n+2s}} \, dx \right)^{\frac{n+2s}{2n}} \\
&\quad + \left(\int_\Omega \bar u^{\frac{q}{q-1}(p-1)} \, dx \right)^{\frac{q-1}{q}} \left(\int_\Omega |v|^{2a_1 q} \, dx \right)^{\frac{1}{q}} + \int_\Omega |v|^{2a_1-1+p} \, dx \Bigg].
\end{aligned}
$$

If we choose $1 < a_1 < a_0$ such that $2a_1 < a_0 + 1$ and $2a_1 - 1 + p \le \frac{2na_0}{n-2s}$, then using the fact that $\bar u \in L^q$ for $q < p^*$, $v \in L^{\frac{2na_0}{n-2s}}$ and $\bar f \in L^{q_0}$ for $\frac{2n}{n+2s} \le q_0 < \frac{n}{2s}$, we get that the above estimate is finite. Thus for some $a_1 > 1$, both $\| |v|^{a_1} \|_{H^s}$ and $\|v^{a_1}\|_{L^{\frac{2n}{n-2s}}}$ are finite. The proof is completed.

□

4. Proofs of Theorem 1.3 and 1.4

In this section we complete the proofs of Theorem 1.3 and 1.4. Here we also follow very closely the arguments in [9], making the necessary modifications to handle the non-local terms.

Proof of Theorem 1.3. First we give a proof of the existence of weak solutions with prescribed singular set.

If the singular set S consists of a finite number of points, then the existence of two distinct solutions is proved in Lemma 3.2.

Let $\{\bar x_1, \ldots, \bar x_k, \ldots\}$ be a countable dense subset of S, and let p_k be an increasing sequence satisfying $\lim_{k\to\infty} p_k = p^*$. For $\eta > 0$, by Lemma 3.1, we can construct a sequence of quasi-solutions $(\bar u_k, \bar f_k)$ with singular set $\{\bar x_1, \cdots, \bar x_k\}$ such that

$$
\int_\Omega |\bar u_{k+1} - \bar u_k|^{p_k} \, dx \le \frac{\eta}{2^k}, \quad \int_\Omega |\bar f_{k+1} - \bar f_k|^{q_0} \, dx \le \frac{\eta}{2^k},
$$

where q_0 is a fixed constant such that $\frac{2n}{n+2s} \le q_0 < \frac{n}{2s}$. Hence $\{\bar u_k\}$ converges strongly to $\bar u$ in $L^q(\Omega)$ for $q < p^*$. When η is small, by Lemma 3.3, we can find two sequences of solutions $\{u_k^i\}$ of (1.1) such that $u_k^i = \bar u_k + v_k^i$, $i = 1, 2$, satisfying

$$
\|v_k^1\|_{H^s}^2 \le \rho_0 \le \|v_k^2\|_{H^s}^2 \le \rho_1,
$$

where $\rho_1 > \rho_0$ are two constants independent of k. Letting v_k be one of these two solutions, we have

$$
\| \, |v_k|^{a_1} \, \|_{H^s}^2 \le C \tag{4.1}
$$

for some $a_1 > 1$ and $C > 0$ are independent of k. By Sobolev's embedding and Hölder's inequality, we may further assume that $\{v_k\}$ converges in $L^{\frac{2na}{n-2s}}$ and weakly converges to v in H_0^s for any $1 < a < a_1$. We wish to prove that $v_k \to v$ strongly in H_0^s. By elliptic estimates (see the summary in [14], for instance), it suffices to show that $|\bar u_k+v_k|^p-\bar u_k^p$ converges strongly in $L^{\frac{2n}{n+2s}}$. This statement is proved in the following two steps:

Step 1. $\{\bar u_k^{p-1}v_k\}$ strongly converges to $\bar u^{p-1}v$ in $L^{\frac{2n}{n+2s}}$.

We have

$$\begin{aligned}
&\int_\Omega |\bar u^{p-1}v_k-\bar u^{p-1}v|^{\frac{2n}{n+2s}}\,dx\\
&\leq C\int_\Omega |\bar u_k^{p-1}-\bar u^{p-1}|^{\frac{2n}{n+2s}}v_k^{\frac{2n}{n+2s}}\,dx+\int_\Omega \bar u^{(p-1)\frac{2n}{n+2s}}|v_k-v|^{\frac{2n}{n+2s}}\,dx\\
&\leq C\Bigg[\left(\int_\Omega |\bar u_k^{p-1}-\bar u^{p-1}|^{\frac{2nq'}{n+2s}}\,dx\right)^{\frac{1}{q'}}\left(\int_\Omega v_k^{\frac{2na}{n-2s}}\,dx\right)^{\frac{n-2s}{a(n+2s)}}\\
&\quad+\left(\int_\Omega \bar u^{(p-1)\frac{2nq'}{n+2s}}\,dx\right)^{\frac{1}{q'}}\left(\int_\Omega |v_k-v|^{\frac{2na}{n-2s}}\,dx\right)^{\frac{n-2s}{a(n+2s)}}\Bigg],
\end{aligned}$$

where

$$\frac{1}{q'}=1-\frac{n-2s}{a(n+2s)}>\frac{4s}{n+2s}.$$

Since $\bar u_k\to\bar u$ in $L^q(\Omega)$ as $k\to\infty$ for any $q<p^*=\frac{n(p-1)}{2s}$, and the exponent $\frac{2nq'}{n+2s}$ satisfies

$$(p-1)\frac{2nq'}{n+2s}<\frac{n(p-1)}{2s}=p^*,$$

we have

$$\int_\Omega |\bar u_k^{p-1}-\bar u^{p-1}|^{\frac{2nq'}{n+2s}}\,dx+\int_\Omega |v_k-v|^{\frac{2na}{n-2s}}\,dx\to 0$$

as $k\to\infty$. Step 1 is proved.

Step 2. Since

$$\Big||\bar u_k+v_k|^p-\bar u_k^p-p\bar u_k^{p-1}v_k\Big|^{\frac{p+1}{p}}\leq C(\bar u_k^{p-1}v_k^2+v_k^{p+1}),$$

by Step 1 and Lebesgue's dominated convergence theorem, we have

$$\int_\Omega \Big|(|\bar u_k+v_k|^p-\bar u_k^p-p\bar u_k^{p-1}v_k)-(|\bar u+v|^p-\bar u^p+p\bar u^{p-1}v)\Big|^{\frac{p+1}{p}}\,dx\to 0.$$

Moreover, since $\frac{p+1}{p}=1+\frac{1}{p}>\frac{2n}{n+2s}$ and $|\bar u_k+v_k|^p-\bar u_k^p$ can be written as the sum of $|\bar u_k+v_k|^p-\bar u_k^p-p\bar u_k^{p-1}v_k$ and $p\bar u_k^{p-1}v_k$, we conclude that $\{|\bar u_k+v_k|^p-\bar u_k^p\}$ strongly converges in $L^{\frac{2n}{n+2s}}(\Omega)$. By elliptic estimates, $\{v_k\}$ converges to v strongly in $H_0^s(\Omega)$, and the limits satisfy $v^1\neq v^2$.

Let us show that both solutions are singular in the set S. Set $u=\bar u+v$ where v is one of v^i obtained above. Then u is a solution of (1.1). It is obvious that $u\in\mathcal{C}^\infty(\Omega\backslash S)$. For any $x_0\in S$, any open neighborhood of x_0 contains $\bar x_k$ for some k. If u is bounded in this neighborhood,

$$|v|\geq\bar u-C\geq C_0|x-x_k|^{-\frac{2s}{p-1}}-C,$$

in view of *(i)* of Lemma 3.1, which is a contradiction to $v\in L^{p+1}$. Therefore, the singular set of u is exactly equal to S.

Finally, let $\eta_0 = \frac{1}{k}$ where k is a large positive integer. By the above argument, we can construct two sequences of solutions $\{u_k^i\}$, $i = 1, 2$, such that $u_k^i = \bar{u}_k^i + v_k^i$ with $v_k^i \in H_0^s(\Omega)$ which satisfies

$$\|\bar{u}_k\|_{L^{p_k}}^{p_k} \le \frac{1}{k}, \quad \|v_k^1\|_{H^s}^2 \le \rho_k \le \rho_0 \le \|v_k^2\|_{H^s}^2 \le \rho_1,$$

where $p_k \to p^*$ and $\lim_{k\to\infty} \rho_k = 0$; both ρ_0, ρ_1 are two constants independent of k. Thus $u_k^1 \to 0$ in $L^q(\Omega)$ for $q < p^*$.

On the other hand, as in the proof of Steps 1 and 2, $v_k^2 \to v^2$ in H_0^s and $\|v^2\|_{H^s}^2 \ge \rho_0$. This implies that v^2 is a weak solution of (1.1). Since $v^2 \in H_0^s(\Omega)$, by a bootstrap argument, we can show that $v^2 \in \mathcal{C}^\infty(\Omega)$.

The proof of Theorem 1.3 is finished.

□

Proof of Theorem 1.4. We follow the same procedure as in the proof of the previous theorems.

By the assumption **H**, let $m > \frac{n+2s}{2}$ be positive integer such that the following holds:

$$\frac{\Gamma(\frac{m+2s}{4})^2}{\Gamma(\frac{m-2s}{4})^2} > p\frac{\Gamma(\frac{m}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{m-2s}{2} - \frac{s}{p-1})}.$$

We let $p := \frac{n+2s}{n-2s}$ here and in the following.

Let $\mathbb{S}^{n-m}$ be a $(n-m)$-dimensional sphere in $\mathbb{R}^n$ and $p_0 \in \mathbb{S}^{n-m}$. Using the Kelvin transform for the solution of (1.7) above, we can construct a family of solutions $\{u_\alpha, \alpha > 0\}$ of

$$(-\Delta)^s u_\alpha = u_\alpha^p \quad \text{in } \mathbb{R}^n$$

satisfying properties (4.2)-(4.4) below:

$$\lim_{x\to\mathbb{S}^{n-m}} u_\alpha(x) d(x)^{\frac{n-2s}{2}} = A_{p,m} \tag{4.2}$$

uniformly in any compact set of $\mathbb{S}^{n-m}\backslash\{p_0\}$ where $d(x)$ denotes the distance from x to $\mathbb{S}^{n-m}$. Moreover,

$$\lim_{\alpha\to 0} u_\alpha(x) = 0 \quad \text{and} \quad \lim_{\alpha\to 0}\int_{\mathbb{R}^n} u_\alpha^p(x)\, dx = 0, \tag{4.3}$$

uniformly in any compact set of $(\mathbb{R}^n \cup \{\infty\})\backslash\mathbb{S}^{n-m}$.

Moreover, thanks to statement (3.5) in Lemma 3.1, for $\phi \in H^s(\mathbb{R}^n)$,

$$p\int_{\mathbb{R}^n} u_\alpha^{p-1}\phi^2\, dx \le (1-\varepsilon_0)\|\phi\|_{H^s}^2 \tag{4.4}$$

for some positive constant ε_0 depending on m, n.

Let us show the above claims. For this, let $\tilde{u}_\alpha(x, y) = \tilde{u}_\alpha(x)$ where $\tilde{u}_\alpha$ is the entire radial solution of

$$\begin{cases} (-\Delta_{\mathbb{R}^m})^s \tilde{u}_\alpha(x) = \tilde{u}_\alpha^{\frac{n+2s}{n-2s}}(x) & \text{in } \mathbb{R}^m, \\ \tilde{u}_\alpha(x) > 0 & \text{in } \mathbb{R}^m, \\ \lim\limits_{|x|\to 0} |x|^{\frac{2s}{p-1}}\, \tilde{u}_\alpha(x) = A_{p,m}, & \\ \lim\limits_{|x|\to\infty} |x|^{m-2s}\, \tilde{u}_\alpha(x) = \alpha, & \end{cases} \tag{4.5}$$

for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n-m}$. The existence comes from Theorem 1.2, since $2m > n+2s$, and **H**. is satisfied, we have $\frac{m}{m-2s} < \frac{n+2s}{n-2s} < p_1(m)$.

We will show that $\tilde{u}_\alpha(x)$ is a weak solution of

$$(-\Delta_{\mathbb{R}^m})^s \tilde{u}_\alpha(x) = \tilde{u}_\alpha^{\frac{n+2s}{n-2s}}(x) \quad \text{in } \mathbb{R}^m,$$

Indeed, following Lemma 2.1 in [5], for any $\varepsilon > 0$, we set

$$\zeta(x) = 1 - \left(1 + \left|\varepsilon^{-1}x\right|^2\right)^{-(m+2s)}.$$

so that $\zeta(x) \to 1$ for all $x \in \mathbb{R}^m \setminus \{0\}$ as $\varepsilon \to 0$. Let $\varphi \in C_c^\infty(B_1)$. Testing (4.5) with $\zeta\varphi$, we have

$$\begin{aligned}
\int_{B_1} \zeta\varphi \tilde{u}_\alpha^{\frac{n+2s}{n-2s}}\,dx &= \int_{B_1} \zeta\varphi(-\Delta)^s \tilde{u}_\alpha\,dx \\
&= \int_{\mathbb{R}^m} \tilde{u}_\alpha (-\Delta)^s(\zeta\varphi)\,dx \\
&= \int_{\mathbb{R}^m} \tilde{u}_\alpha \left(\zeta(-\Delta)^s\varphi + c_{n,s}\int_{\mathbb{R}^m} \varphi(y)\frac{\zeta(x)-\zeta(y)}{|x-y|^{n+2s}}\,dy\right)dx
\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^m} \zeta\left(\varphi\tilde{u}_\alpha^{\frac{n+2s}{n-2s}} - \tilde{u}_\alpha(-\Delta)^s\varphi\right)dx = \int_{\mathbb{R}^m} \tilde{u}_\alpha\left(c_{n,s}\int_{B_1}\varphi(y)\frac{\zeta(x)-\zeta(y)}{|x-y|^{n+2s}}\,dy\right)dx$$

A slight modification of the proof of Lemma 3.13 in [14] shows that

$$\left|\int_{B_1}\varphi(y)\frac{\zeta(x)-\zeta(y)}{|x-y|^{m+2s}}\,dy\right| \le C\varepsilon^{-2s}\left(1+\left|\varepsilon^{-1}x\right|^2\right)^{-(m+2s)},$$

which implies

$$\left|\int_{\mathbb{R}^m}\zeta\left(\varphi\tilde{u}_\alpha^{\frac{n+2s}{n-2s}} - \tilde{u}_\alpha(-\Delta)^s\varphi\right)dx\right| = C\int_{\mathbb{R}^m}\left|\tilde{u}_\alpha\right|\varepsilon^{-2s}\left(1+\left|\varepsilon^{-1}x\right|^2\right)^{-(m+2s)}dx$$

Note that $\tilde{u}_\alpha \in L^q(\mathbb{R}^m)$ where $q = \frac{n+2s}{n-2s} \in \left(\frac{m}{m-2s}, \frac{m+2s}{m-2s}\right)$. By Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^m}\zeta\left(\varphi\tilde{u}_\alpha^{\frac{n+2s}{n-2s}} - \tilde{u}_\alpha(-\Delta)^s\varphi\right)dx\right| \\
&\le C\left\|\tilde{u}_\alpha\right\|_{L^q(\mathbb{R}^m)}\left\|\varepsilon^{-2s}(1+\left|\varepsilon^{-1}x\right|)^{-(m+2s)}\right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)}
\end{aligned}$$

It suffices to show that the right hand side is $o(1)$ as $\varepsilon \to 0$. But it is clear that

$$\varepsilon^{-\frac{2sq}{q-1}}\int_{\mathbb{R}^m}\frac{1}{\left(1+|\varepsilon^{-1}x|^2\right)^{\frac{(m+2s)q}{q-1}}}\,dx = \varepsilon^{m-\frac{2sq}{q-1}}\int_{\mathbb{R}^m}\frac{1}{\left(1+|y|^2\right)^{\frac{(m+2s)q}{q-1}}}\,dy \to 0$$

as $\varepsilon \to 0$ since $m > \frac{n+2s}{2}$. This proves that $\tilde{u}_\alpha(x)$ is a weak solution of

$$(-\Delta_{\mathbb{R}^m})^s \tilde{u}_\alpha(x) = \tilde{u}_\alpha^{\frac{n+2s}{n-2s}}(x) \quad \text{in } \mathbb{R}^m,$$

and hence $\tilde{u}_\alpha(x,y)$ is a weak solution of

$$(-\Delta_{\mathbb{R}^n})^s \tilde{u}_\alpha(x,y) = \tilde{u}_\alpha^{\frac{n+2s}{n-2s}}(x,y) \quad \text{in } \mathbb{R}^n.$$

Let $e = (1, 0, \cdots, 0)$ and $u_\alpha(x) = \frac{1}{|x-x_0|^{n-2s}} \tilde{u}_\alpha\big(\frac{x-x_0}{|x-x_0|^2} + e\big)$ for $x \in \mathbb{R}^n$ and $x_0 \notin \mathbb{R}^{n-m}$. Then $u_\alpha(x)$ is a weak solution of

$$\begin{cases} (-\Delta)^s u_\alpha = u_\alpha^{\frac{n+2s}{n-2s}}(x) & \text{in } \mathbb{R}^n, \\ u_\alpha(x) = O(|x|^{-(n-2s)}) & \text{as } |x| \to \infty. \end{cases} \tag{4.6}$$

Let $\tilde{S}^{n-m}$ be the pre-image of $\mathbb{R}^{n-m}$ under the mapping $x \to \frac{x-x_0}{|x-x_0|^2} + e$. It is easy to see that $\tilde{S}^{n-m}$ is an $(n-m)$-dimensional sphere in $\mathbb{R}^n$ and $u_\alpha(x)$ is a family of weak solutions of

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}} \quad \text{in } \mathbb{R}^n$$

satisfying (4.2)-(4.4). For (4.4), it is easy to check with both sides of (4.4) are invariant under the Kelvin transformation. Since $\tilde{S}^{n-m}$ is congruent to any $(n-m)$-sphere, the above claim follows easily.

Step 1. Let $\mathbb{S}_1, \dots, \mathbb{S}_k$ be a sequence of disjoint $(n-m)$-dimensional spheres. Fix a small positive number $\eta > 0$ which will be determined later. We construct here a sequence of positive approximate solutions $\{(\bar{u}_k, \bar{f}_k)\}$ satisfying (4.7), (4.8) and (4.9).

Fix $p_k \in \mathbb{S}_k$, $k = 1, 2, \dots$, we have

$$\lim_{x \to \mathbb{S}_j, x \notin \mathbb{S}_j} \bar{u}_k \, d_j(x)^{\frac{2s}{p-1}} = A_{p,m} \tag{4.7}$$

uniformly in any compact set of $\mathbb{S}_j \backslash \{p_j\}$, $j = 1, \dots, k$, where $d_j(x)$ denotes the distance from x to $\mathbb{S}_j$.

Denote $\bar{f}_k = (-\Delta)^s \bar{u}_k - \bar{u}_k^p$. We have

$$\begin{cases} \int_{\mathbb{R}^n} \bar{u}_k^p \, dx < \eta, \ \int_{\mathbb{R}^n} \bar{f}^{\frac{2n}{n+2s}} \, dx < \eta, \\ \bar{u}_k \to \bar{u} \text{ in } L^p(\mathbb{R}^n), \\ \text{Supp}\{\bar{f}_k\} \subset \bigcup_{j=1}^k B(\mathbb{S}_j, r_j), \end{cases} \tag{4.8}$$

where $B(\mathbb{S}_j, r_j) = \{x \in \mathbb{R}^n \mid d_j(x) \le r_j\}$ and $\lim_{j \to \infty} r_j = 0$.

The quadratic form

$$Q(\phi) = \left(1 + \sum_{j=1}^{k} 3^{-j} \varepsilon_0 - \varepsilon_0\right) \|\phi\|_{H^s}^2 - p \int_{\mathbb{R}^n} \bar{u}_k^{p-1} \phi^2 \, dx \tag{4.9}$$

is positive definite and equivalent to the $H^s(\mathbb{R}^n)$-norm.

The construction of $\bar{u}_k$ is exactly the same as in Lemma 3.1 except that the cut-off function $\chi_k(x - \bar{x}_k)$ is replaced by $\chi_k(d_k(x))$. To prove (4.9), it suffices to note the inequality (3.9) becomes

$$p \int_{\mathbb{R}^n} \bar{u}_k^{p-1} \phi^2 \, dx \le \Big(1 + \sum_{j=1}^{k-1} 3^{-j} \varepsilon_0 - \varepsilon_0\Big) \|\phi\|_{H^s}^2 + C \int_K \phi^2 \, dx, \tag{4.10}$$

where C is a constant independent of α and K is a bounded set independent of α. Then the rest of the proof can go through without any modification.

Next, let

$$E(\phi) = \frac{1}{2} \|\phi\|_{H^s}^2 - \int_{\mathbb{R}^n} F(\bar{u}_k, \phi) + \int_{\mathbb{R}^n} \bar{f}_k \phi \, dx$$

for $\phi \in H^s(\mathbb{R}^n)$ where

$$F(s,t) = \frac{1}{p+1}\big\{|s+t|^p(s+p) - s^{p+1} - (p+1)s^p t\big\}.$$

It is not difficult to see that $E(\phi)$ is continuous in the strong topology of $H^s(\mathbb{R}^n)$.

Now, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x,t)| \leq \tfrac{p}{2}(1+\varepsilon)s^{p-1}t^2 + C_\varepsilon |t|^{p+1},$$

which yields

$$\begin{aligned} 2E(\phi) \geq \|\phi\|_{H^s}^2 - p(1+\varepsilon)\int_{\mathbb{R}^n} \bar{u}_k^{p-1}\phi^2\,dx - C_\varepsilon \int_{\mathbb{R}^n} |\phi|^{p+1}\,dx \\ - 2\left(\int_{\mathbb{R}^n} \bar{f}_k^{\frac{2n}{n+2s}}\,dx\right)^{\frac{n+2s}{2n}} \left(\int_{\mathbb{R}^n} |\phi|^{p+1}\,dx\right)^{\frac{1}{p+1}}. \end{aligned}$$

Fix $\varepsilon_1 > 0$ so that, by (4.10),

$$\|\phi\|_{H^s}^2 - p(1+\varepsilon_1)\int_{\mathbb{R}^n} \bar{u}_k^{p-1}\phi^2\,dx \geq \varepsilon_1 \|\phi\|_{H^s}^2.$$

By Sobolev's embedding theorem, we have

$$2E(\phi) \geq \varepsilon_1 \|\phi\|_{H^s}^2 - C_1\left(\|\phi\|_{H^s}^{p+1} + 2\eta\|\phi\|_{H^s}\right).$$

Therefore, there exists small $\rho = \rho(\eta)$ such that

$$\inf_{\|\phi\|_{H^s}=\rho} E(\phi) \geq \frac{\varepsilon_1}{4}\rho^2 > 0$$

with $\lim_{\eta\to 0}\rho(\eta) = 0$.

Step 2. We claim that there exists $v_0 \in H^s(\mathbb{R}^n)$ with $\|v_0\|_{H^s} < \rho$ such that

$$E(v_0) = \inf_{\|v\|_{H^s}\leq \rho} E(v) < 0. \tag{4.11}$$

Let $v_j \in H^s(\mathbb{R}^n)$ with $\|v_j\|_{H^s} < \rho$ and $\lim_{j\to\infty} E(v_j) = \inf_{\|v\|_{H^s}\leq\rho} E(v)$. Since $\{v_j\}$ is bounded in $H^s(\mathbb{R}^n)$, we can assume that $v_j \to v_0$ weakly for some $v_0 \in H^s(\mathbb{R}^n)$. If $v_j \to v_0$ strongly, we are done. Hence we may assume that $\phi_j = v_j - v_0$ is weakly convergent to 0 in $L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$ as well but $0 < \lim_{j\to\infty}\|v_j\|_{H^s} =: \bar{\rho}$. In order to obtain a contradiction, we compute,

$$E(v_0+\phi_j) - E(v_0) = \frac{1}{2}\|\phi_j\|_{H^s}^2 - \int_{\mathbb{R}^n} [F(\bar{u}, v_0+\phi_j) - F(\bar{u}, v_0)]\,dx + o(1).$$

Because $\bar{f}_k \in L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$, we decompose the second term of the right hand side above into two terms:

$$\begin{aligned} &F(\bar{u}, v_0+\phi_j) - F(\bar{u}, v_0) \\ &= \frac{1}{p+1}\Big\{|\bar{u}+v_0+\phi_j|^p(\bar{u}+v_0+\phi_j) - |\bar{u}+v_0|^p(\bar{u}+v_0) - (p+1)|\bar{u}+v_0|^p\phi_j\Big\} \\ &\quad + \Big\{|\bar{u}+v_0|^p - \bar{u}^p\Big\}\phi_j \\ &=: g_1 + g_2\phi_j. \end{aligned}$$

For g_1, we have that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}|g_1| &\le \frac{p+\varepsilon}{2}|\bar u + v_0|^{p-1}\phi_j^2 + C_\varepsilon |\phi_j|^{p+1}\\ &\le \frac{p+2\varepsilon}{2}\bar u^{p-1}\phi_j^2 + C_\varepsilon\left(|v_0|^{p-1}\phi_j^2 + |\phi_j|^{p+1}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n}|g_1|\,dx &\le \frac{p+2\varepsilon}{2}\int_{\mathbb{R}^n}\bar u^{p-1}\phi_j^2\,dx + C_\varepsilon\int_{\mathbb{R}^n}\left(|v_0|^{p-1}\phi_j^2 + |\phi_j|^{p+1}\right)dx\\ &\le \frac{p+2\varepsilon}{2}\int_{\mathbb{R}^n}\bar u^{p-1}\phi_j^2\,dx + C_\varepsilon \rho^{p-1}\|\phi_j\|^2_{H^s}.\end{aligned}$$

For g_2, we note that $p \le 2$ $(n \ge 4s)$, and

$$\left||\bar u + v_0|^p - \bar u^p - p\bar u^{p-1}v_0\right| \le C|v_0|^p$$

for some constant $C > 0$. By Sobolev's embedding, we have

$$|\bar u + v_0|^p - \bar u^p - p\bar u^{p-1}v_0 \in L^{\frac{p+1}{p}}(\mathbb{R}^n).$$

Therefore,

$$\int_{\mathbb{R}^n} g_2\phi_j\,dx = \int_{\mathbb{R}^n}(|\bar u + v_0|^p - \bar u^p - p\bar u^{p-1}v_0)\phi_j\,dx + p\int_{\mathbb{R}^n}\bar u^{p-1}v_0\phi_j\,dx = o(1).$$

Combining the above two estimates, we have

$$E(v_0+\phi_j) - E(v_0) \ge \frac{1}{2}\|\phi_j\|^2_{H^s} - \frac{p+2\varepsilon}{2}\int_{\mathbb{R}^n}\bar u^{p-1}\phi_j^2\,dx - C_\varepsilon\rho^{p-1}\|\phi_j\|^2_{H^s} + o(1).$$

Choose $\varepsilon = \frac{p\varepsilon_1}{2}$ and η small enough such that $C_\varepsilon\rho^{p-1} < \frac{\varepsilon_1}{4}$, then we have

$$\lim_{j\to\infty} E(v_0+\phi_j) > E(v_0)$$

which is a contradiction. Hence Step 2 is proved.

Finally, let v_k be a solution of $E(v_k) = \inf_{\|v\|_{H^s}\le\rho} E(v)$. Then by the maximum principle, we can show that $u_k = \bar u_k + v_k$ is a positive solution of

$$(-\Delta)^s u = u^p \text{ in } \mathbb{R}^n.$$

Since v_k is bounded in $H^s(\mathbb{R}^n)$, we can assume that, after passing to a subsequence, $v_k \to v \in H^s(\mathbb{R}^n)$ in $L^p(K)$ for any compact set K of $\mathbb{R}^n$. Hence $u = \bar u + v$ is a non-negative weak solution of (1.13). Since $v \in H^s(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, we have

$$v \in L^{\frac{n+2s}{n-2s}}(d\mu), \quad \text{hence} \quad u \in L^{\frac{n+2s}{n-2s}}(d\mu).$$

We claim that the singular set of u must include $\cup_{j=1}^\infty \mathbb{S}_j$. If, on the contrary, there exists a point $q \in \mathbb{S}_j\backslash\{p_j\}$, q outside the singular set of u, then there exists a neighborhood U of q such that $u(x) \le C$ in U. This implies

$$-\bar u \le v \le C - \bar u \tag{4.12}$$

or $|v| \ge \bar u - C \ge \bar u_j - C$. However, $v \in L^{\frac{n+2s}{n-2s}}(U)$ implies $\bar u_j \in L^{\frac{2n}{n-2s}}(U)$ which is impossible since $m < \frac{2s(p+1)}{p-1}$. Therefore, $\cup\mathbb{S}_j$ is contained in the singular set of u. Suppose that $\cup\mathbb{S}_j$ is dense in $\mathbb{R}^n$. Because the singular set of u is closed, we conclude that the singular set of u is the whole space $\mathbb{R}^n$.

□

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