SYMMETRY OF NONNEGATIVE SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION WITH SINGULAR NONLINEARITY

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ABSTRACT. We use the Method of Moving Plane (MMP) to obtain necessary and sufficient conditions for the radial symmetry of positive solutions of the following semilinear elliptic equation with singular nonlinearity:

$$\Delta u - \frac{1}{u^\nu} = 0 \text{ in } \mathbb{R}^n, n \geq 2$$

where $\nu > 0$. In order to apply MMP, it is crucial to obtain the asymptotic expansion of u at ∞.

1. INTRODUCTION

In this paper we investigate the symmetry and local behavior of nonnegative solutions of the equation

$$\Delta u - \frac{1}{u^\nu} = 0, \quad x \in \mathbb{R}^n, \quad n \geq 2, \quad \nu > 0. \tag{I}$$

We call u a nonnegative (positive) solution of (I) if $u \in C^0(\mathbb{R}^n)$, $u \geq 0$ ($u > 0$), $u \not\equiv 0$ in $\mathbb{R}^n$ and u satisfies (I) a.e. in $\mathbb{R}^n$. (Clearly $u \equiv 0$ is not a solution of (I).)

Problem (I) arises in the study of steady states of thin films. Equations of the type

$$u_t = -\nabla \cdot (f(u)\nabla \Delta u) - \nabla \cdot (g(u)\nabla u) \tag{1.1}$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set $\mathbf{\Sigma}_u = \{u = 0\}$ is the liquid/solid interface and is sometimes called set of **ruptures**. Ruptures play a very important role in the study of thin films. The coefficient $f(u)$ reflects surface tension effects-a typical choice is $f(u) = u^3$. The coefficient of the second-order term can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m$, $m < 0$. For backgrounds of (1.1), we refer to [BP1, BP2, LP1, LP2, LP3, WB] and the references therein.

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In general, let us assume that $f(u)=u^p$, $g(u)=u^m$, where p, $m\in\mathbb{R}$. Then if we consider the steady-state of (1.1), we see that u satisfying

$$u^p\nabla\Delta u+u^m\nabla u=\mathcal{C}$$

is a steady state of (1.1). Where $\mathcal{C}=(C_1,C_2,\ldots,C_n)$ is some constant vector. By assuming $\mathcal{C}=\mathbf{0}$ (which prevents linear term on x), we obtain

$$\Delta u+\frac{u^q}{q}-C=0 \text{ in } \Omega, \tag{1.2}$$

where $q=m-p+1$ and C is some constant. (Here we have assumed that $q\neq 0$. If $q=0$, we have to replace $\frac{u^q}{q}$ by $\log u$.) Note that solutions to (I) are steady-states of (1.1) but the reverse is not true. For thin films under van der Waals forces, we have $f(u)=u^3$, $g(u)=u^m$, $q=m-2<-2$. The one-dimensional steady-state problem of (1.1) has been studied thoroughly in [LP1, LP3] and the references therein. It is found that ruptures **never** occur in one-dimensional case. On the other hand, numerical works on van der Waals driven rupture for (1.1) in two-dimension suggested that the rupture can occur in points [BBD, HLU] or rings [WB, YD, YH].

In this paper, we consider problem (1.2) in $\mathbb{R}^n$ for $n\geq 2$ and assume that the constant $C=0$. Problem (1.2) becomes (I) with a simple scale of u. It is easy to see that if $u\in C^0_{loc}(\mathbb{R}^n\backslash\mathbf{\Sigma}_u)$, then $u\in C^\infty_{loc}(\mathbb{R}^n\backslash\mathbf{\Sigma}_u)$.

The structure of nonnegative solutions of (I) can be complicated since if u is a nonnegative solution of (I), then the rupture set $\mathbf{\Sigma}_u$ can be nonempty and with a positive Hausdorff dimension (see [GW]). In this paper we are interested in symmetry property of positive solutions u of (I), i.e., $\mathbf{\Sigma}_u=\emptyset$. Notice that $\mathbf{\Sigma}_u$ can contain at most one element if u is radially symmetric. Indeed, if $\mathbf{\Sigma}_u$ contains more than one element, then we claim that u can not be radially symmetric about some point $x_0\in\mathbb{R}^n$. In fact, suppose on the contrary, we have a radially symmetric nonnegative solution $u\in C^2(\mathbb{R}^n)$ of (I) with z_0, $z_1\in\mathbf{\Sigma}_u$ $(z_0\neq z_1)$. We assume that u is radially symmetric about a point x_0. Then there are three cases here: $x_0=z_0$; $x_0=z_1$ and $x_0\neq z_i$ $(i=0,1)$. Now, setting $r=|x-x_0|$, we easily know that $u(x):=u(r)$ satisfies the equation

$$(r^{n-1}u')'=r^{n-1}u^{-\nu},\ \ 0<r<\infty. \tag{1.4}$$

This implies that $u'(r)\geq 0$ for $r\in(0,\infty)$. Now for the first case, we have that $u(r_1)=0$ with $r_1:=|z_1-x_0|$, and hence $u\equiv 0$ in $B_{r_1}(x_0)$, which is impossible. We can derive contradictions for other two cases similarly. This implies that our claim holds. On the other hand, the same arguments imply that if $\Sigma_u = \{\text{a single point}\}$, then u must be radially symmetric about this point.

Define

$$\alpha = \frac{2}{\nu+1}, \quad \lambda = [\alpha(n-2+\alpha)]^{-1/(\nu+1)}. \tag{1.5}$$

It is easy to see that

$$u_0(x) = \lambda |x|^\alpha \tag{1.6}$$

is a nonnegative radially symmetric solution of (I). For this solution, it is clear that $\Sigma_{u_0} = \{0\}$ and $|x|^{-\alpha} u_0(x) = \lambda$. It will be seen that the limit

$$\lim_{|x|\to+\infty} |x|^{-\alpha} u(x) = \lambda \tag{1.7}$$

plays an important role in the radially symmetric properties of nonnegative solutions of (I).

Not all solutions of (I) with a single rupture point are radially symmetric: in fact, let $n = 2$ and $\nu = 3$, then the following solution

$$u_\epsilon(x) = \sqrt{2|x|}\left(\epsilon(\cos\frac{\theta}{2})^2 + \epsilon^{-1}(\sin\frac{\theta}{2})^2\right)^{\frac{1}{2}}, \epsilon > 0 \tag{1.8}$$

satisfies (I) and has one single point rupture 0. But certainly u_ϵ is not radially symmetric.

Our main goal of this paper is to find **necessary and sufficient** conditions for which solutions of (I) are radially symmetric.

It is very interesting to see that symmetry properties of positive solutions of (I) are related to those of positive solutions of the Lane-Emden equation with positive supercritical exponent

$$\Delta u + u^p = 0, \quad x \in \mathbb{R}^n, \;\; p > (n+2)/(n-2). \tag{1.9}$$

The symmetry and local behavior of positive C^2-solutions of (1.9) were studied by H. Zou [Zou]. He showed that for $n \geq 3$ and $(n+2)/(n-2) < p < m$, where

$$m = \begin{cases} \infty, & n = 3, \\ (n+1)/(n-3), & n > 3, \end{cases}$$

a solution u of (1.9) is radially symmetric about some point, provided that u has the following decay

$$u(x) = O(|x|^{-\frac{2}{p-1}}) \text{ at } +\infty. \tag{1.10}$$

In a more recent paper [Gu], the first author extended Zou's result to the cases $m \leq p < \infty$ if $n = 4$ and $p \geq n/(n-4)$ if $n \geq 5$. More precisely, he showed that for $n \geq 5$ and $p \geq n/(n-4)$, a nonnegative C^2 solution of (1.9) is radially symmetric about some point in $\mathbb{R}^n$ if and only if $\lim_{|x|\to+\infty} |x|^{\frac{2}{p-1}} u(x) = \lambda$ for some $\lambda > 0$. Furthermore, for $n \geq 4$ and $(n+1)/(n-3) \leq p < n/(n-4)$, u is radially symmetric about some point in $\mathbb{R}^n$ if and only if $\lim_{|x|\to+\infty} |x|^{\frac{2}{p-1}} u(x) = \lambda$ and $\lim_{|x|\to+\infty} |x|^{1-(\mu+n)/2}(|x|^{\frac{2}{p-1}} u(x) - \lambda) = 0$ where $\mu = \frac{4}{p-1} + 4 - 2n$.

Unlike those in [Zou] and [Gu], positive solutions of (I) do not decay as $|x| \to \infty$. In fact, because of the negative power of u, u grows as $|x| \to +\infty$. We will establish strong asymptotic estimates for the positive solutions of (I) satisfying (1.7), which are *good* enough to obtain their radially symmetric properties.

We remark that the negative exponent $u^{-\nu}$ can be considered as negative **supercritical** in $\mathbb{R}^n, n \geq 2$. In [ACW], it is found that in $\mathbb{R}$, $u^{-\nu}$ is subcritical if $\nu < 3$, critical if $\nu = 3$, and supercritical if $\nu > 3$. Thus all $u^{-\nu}$ with $\nu > 0$ can be considered supercritical in $\mathbb{R}^n, n \geq 2$.

In this paper, we will use the devices introduced in [Zou] and [Gu]. In particular, our arguments in the proofs below are closely related to those of [Gu]. The key ingredient of our arguments in this paper is the powerful Alexandroff-Serrin moving-plane method (MMP), which was first developed by Serrin in PDE theory, later extended and generalized by Gidas, Ni and Nirenberg [GNN1-2] and used by many authors. In contrast to the case of bounded domains or subcritical (critical) nonlinearities, where Hopf boundary Lemma or the Kelvin transform is available to start MMP, appropriately strong asymptotic estimates of solutions at infinity, replacing boundary lemmas or Kelvin transform and providing a starting point for the method, are crucial for the moving-plane procedure in the case of the entire space with supercritical nonlinearities.

Define $\mu = 2(\alpha + n - 2)$. We will establish the strong asymptotic estimates for positive solutions $u(x)$ of (I) satisfying (1.7) at infinity for the two cases below:

$$(a)\ \mu - n \geq 0, \quad (b)\ -1 < \mu - n < 0. \tag{1.11}$$

It is easily seen that (a) holds if $n \geq 4$ and $\nu > 0$; $n = 3$ and $0 < \nu \leq 3$; $n = 2$ and $0 < \nu \leq 1$, (b) holds if $n = 3$ and $\nu > 3$; $n = 2$ and $1 < \nu < 3$. For Case (a), the asymptotic estimate (1.7) of u is good enough for us to obtain its symmetry by the moving-plane method. For Case (b), the asymptotic estimate (1.7) of u is not good enough to do so. We need to obtain better asymptotic estimates for it.

Our main global results read

Theorem 1.1. *Let $n \geq 2$ be an integer, $\nu > 0$ and $u(x)$ a positive C^0-solution of (I). Suppose that*

$$\mu - n \geq 0, \quad \textit{i.e. } n \geq 4 \textit{ and } \nu > 0;\ n = 3 \textit{ and } 0 < \nu \leq 3;\ n = 2 \textit{ and } 0 < \nu \leq 1. \tag{1.12}$$

Then u is radially symmetric about some point $x_0 \in \mathbb{R}^n$ if and only if

$$\lim_{|x| \to +\infty} |x|^{-\alpha} u(x) = \lambda. \tag{1.13}$$

Theorem 1.2. *Assume that $n = 3$ and $\nu > 3$ or $n = 2$ and $\nu > 1$ (note that $-2 < \mu - n < 0$), $u(x)$ is a positive C^0-solution of (I). Then u is radially symmetric about some point $x_0 \in \mathbb{R}^n$ if and only if*

$$\lim_{|x| \to +\infty} |x|^{-\alpha} u(x) = \lambda \tag{1.14}$$

and

$$\lim_{|x| \to +\infty} |x|^{1+(\mu-n)/2} (|x|^{-\alpha} u(x) - \lambda) = 0. \tag{1.15}$$

We remark that in [Zou], it is only assumed that

$$u(x) = O(|x|^{-\frac{2}{p-1}}) \text{ at } +\infty.$$

Here in this paper we need the **exact** asymptotics. Example (1.8) shows that it is not enough to just assume that

$$u(x) = O(|x|^{\frac{2}{\nu+1}}) \text{ at } +\infty.$$

Example (1.8) also implies that the assumption (1.15) is needed in Theorem 1.2 at least for some ν.

However, in the case $n = 2$, $\nu \neq 3$, we can show the following:

Theorem 1.3. *If $u(x)$ satisfies (I) and the following growth condition:*

$$u(x) \geq C|x|^{\alpha} \quad \textit{at } +\infty. \tag{1.16}$$

Then (1.7) holds.

The radially symmetric solutions of (I) can be classified as follows:

Theorem 1.4. *All radially symmetric solutions of (I) can be classified as follows:*

(a) the first solution is a solution with a single rupture: $u_0(r) = (\frac{\nu+1}{2})^{\frac{2}{\nu+1}} r^{\frac{2}{\nu+1}}$;

(b) the other solutions form a one-parameter family $\{u_\eta\}_{\eta>0}$ with $u_\eta(r)$, $r = |x|$, strictly increasing in r, $u_\eta(0) = \eta > 0$, $u_\eta(r) = \eta u_1(\eta^{-(\nu+1)/2} r)$ and, as $r \to +\infty$,

$$r^{-\frac{2}{\nu+1}} u_\eta(r) \to \lambda.$$

As far as we know, our result is the first of its kind in dealing with radial symmetry of nonnegative solutions for semilinear elliptic equations with **negative** power. This paper is organized as follows:

Sections 2 and 3 provide some key inequalities for the difference $v(y) = r^{-\alpha}u(x) - \lambda, y = \frac{x}{r^2}$.

Section 4 and Section 5 study the Lipschitz and the Hölder continuity of v near 0.

Section 6 provides a key auxiliary lemma–Lemma 6.2, which is needed for using MMP.

In Section 7, we prove the necessary part of Theorems 1.1, 1.2 and Theorems 1.3, 1.4.

Finally in Section 8, we use MMP to finish the proofs of sufficient part of Theorems 1.1 and 1.2.

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2. Preliminaries

Let $n \geq 2$ be a positive integer and $\mathbb{R}^n$ be the n-Euclidean space. For $\nu > 0$, consider the equation

$$\Delta u = u^{-\nu}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

We are interested in nonnegative C^0-solutions u of (2.1) satisfying (1.7).

We begin with notation and definitions. Let

$$\alpha = \frac{2}{\nu + 1}, \quad \lambda = [\alpha(\alpha + n - 2)]^{-1/(\nu+1)}. \tag{2.2}$$

Throughout the paper, we shall assume that $\nu > 0$ and

$$\lim_{|x| \to \infty} |x|^{-\alpha} u(x) = \lambda. \tag{2.3}$$

In what follows, we denote $M = M(\cdots)$ positive constants, besides the arguments inside the parenthesis, depending upon the structural numbers n and ν, which may vary line from line.

For any function $u(x)$ on $\mathbb{R}^n$, we introduce the following Kelvin type transform

$$v(y) = r^{-\alpha}u(x) - \lambda, \quad y = \frac{x}{r^2}, \quad r = |x| > 0. \tag{2.4}$$

Equation (2.1) is converted to an equation with singular coefficients at the origin under (2.4). In particular, we turn to study the new equation near the origin.

Lemma 2.1. *Let u be a nonnegative solution of (2.1), v given by (2.4). Suppose that (2.3) holds. Then v satisfies the equation*

$$\Delta v - \frac{\mu y \cdot \nabla v}{s^2} + \frac{\mu v}{s^2} - \frac{f(v)}{s^2} = 0, \quad y \in \mathbb{R}^n \backslash \{0\}, \tag{2.5}$$

where $s = |y|$ and

$$\mu = 2(\alpha + n - 2), \quad f(t) = (t+\lambda)^{-\nu} - \lambda^{-\nu} + \nu \lambda^{-(\nu+1)} t.$$

Note that f is real analytic at $t = 0$ and satisfies

$$f(0) = f'(0) = 0, \quad f''(0) = \nu(\nu+1)\lambda^{-(\nu+2)} > 0.$$

Moreover, for any integer $\tau \geq 0$ there exists a constant $M = M(u) > 0, s_0 = s_0(u) > 0$ such that

$$\lim_{s \to 0} v(y) = 0, \quad |\nabla^\tau v(y)| \leq \frac{M}{s^\tau}, \quad \text{for } s = |y| \leq s_0. \tag{2.6}$$

Proof. Using (2.1) and (2.4), the equation (2.5) is obtained by direct calculations. The estimates (2.6) can be obtained by arguments exactly same as those in the proof of Lemma 2.1 of [Zou]. □

By Lemma 2.1, it amounts to studying solutions of (2.5) satisfying (2.6). Therefore, in the sequel, we shall assume that (2.6) is satisfied.

We introduce the function

$$w(s,\theta) = v(s,\theta) - \overline{v}(s), \tag{2.7}$$

where

$$\overline{v}(s) = \frac{1}{\omega_n} \int_{S^{n-1}} v(s,\theta) d\theta, \quad \omega_n = |S^{n-1}|.$$

Lemma 2.2. *Let v be a solution of (2.5) and Δ_θ the Laplace Beltrami operator on S^{n-1}. Then v, and $\overline{v}$ and w satisfy the following equations respectively*

$$v'' + \frac{\Delta_\theta v}{s^2} - \frac{\mu - n + 1}{s} v' + \frac{\mu v}{s^2} - \frac{f(v)}{s^2} = 0, \tag{2.8}$$

$$\overline{v}'' - \frac{\mu - n + 1}{s} \overline{v}' + \frac{\mu \overline{v}}{s^2} - \frac{\overline{f(v)}}{s^2} = 0 \tag{2.9}$$

and

$$w'' + \frac{\Delta_\theta w}{s^2} - \frac{\mu - n + 1}{s} w' + \frac{\mu w}{s^2} - \frac{f(v) - \overline{f(v)}}{s^2} = 0, \tag{2.10}$$

where the prime is the derivative with respect to the radius s.

Proof. Equation (2.8) follows directly from (2.5) and the formulas

$$\Delta v = v'' + \frac{\Delta_\theta v}{s^2} + \frac{n-1}{s} v', \quad \nabla v \cdot y = v' s.$$

Integrating (2.8) over S^{n-1} yields (2.9) since

$$\overline{\Delta_\theta v} = \frac{1}{\omega_n} \int_{S^{n-1}} \Delta_\theta v(s, \theta) d\theta = 0.$$

Finally, subtracting (2.9) from (2.8) gives (2.10). □

3. A Fundamental Inequality

The Lipschitz continuity of w at the origin is crucial in proving the expansion of u near ∞, which can be used to obtain the symmetry of u by MMP. To this end, we first obtain the Hölder type estimate for v. The function

$$W(s) = \Big(\int_{S^{n-1}} w^2(s, \theta) d\theta \Big)^{1/2}, \tag{3.1}$$

plays an important role in achieving our goal.

Theorem 3.1. *Let W be given by (3.1). Then there exist $s_0 > 0$ and a positive constant $K = K(v, \nu, n, s_0)$ such that*

$$W(s) \le K s^{1+\mu-n} \quad \textit{for } 0 < s < s_0 \tag{3.2}$$

if $-1 < \mu - n < 0$ and

$$W(s) \le Ks \ \ \textit{for } 0 < s < s_0 \tag{3.3}$$

if $\mu - n \ge 0$.

The proof of this theorem is related to that of Theorem 3.1 of [Gu]. We first obtain the following lemma.

Lemma 3.2. *For any $0 < \epsilon < \min\{(1+\mu-n)/2, 1/2\}$, there exist $\hat{\delta} = 1+\mu-n-\epsilon > 0$ for $-1 < \mu - n < 0$; $\hat{\delta} = 1 - \epsilon$ for $\mu - n \ge 0$; $s_0 = s_0(\epsilon) > 0$ and a positive constant $K = K(v, \hat{\delta}, s_0)$ such that*

$$W(s) \le K s^{\hat{\delta}}, \quad 0 < s < s_0. \tag{3.4}$$

Proof. Let $g(v) = f(v) - \overline{f(v)}$. Then, it is known from (2.10) that w satisfies

$$w'' + \frac{\Delta_\theta w}{s^2} - \frac{\mu - n + 1}{s} w' + \frac{\mu w}{s^2} - \frac{g(v)}{s^2} = 0. \tag{3.5}$$

It is known from [Rey] that the eigenvalues of the problem

$$-\Delta_\theta Q = \sigma Q, \quad \theta \in S^{n-1}$$

are

$$\sigma_k = k(n+k-2), \quad k \ge 0$$

with multiplicity $m_k = \frac{(n-3+k)!(n-2+2k)}{k!(n-2)!}$. In particular, we have

$$\sigma_0 = 0, \quad m_0 = 1, \quad Q_0 \equiv 1$$

$$\sigma_1 = n-1, \quad m_1 = n, \quad Q_i(\theta) = x_i, \quad 1 \le i \le n$$

$$\sigma_2 = 2n,$$

here $Q_i(\theta)$ denote the associate eigenvectors. Therefore, if $u \in L^2(S^{n-1})$ is orthogonal to Q_0, i.e., $\overline{u} = 0$, we have that

$$\int_{S^{n-1}} |\nabla_\theta u|^2 d\theta \ge (n-1) \int_{S^{n-1}} u^2 d\theta.$$

Moreover, if u is orthogonal to Q_0, Q_i $(i = 1, 2, \ldots, n)$, we have that

$$\int_{S^{n-1}} |\nabla_\theta u|^2 d\theta \ge 2n \int_{S^{n-1}} u^2 d\theta.$$

Since $w(s, \cdot) \in L^2(S^{n-1})$ and $\overline{w} = 0$, we have that $w(s,\theta) = w_1(s,\theta) + w_2(s,\theta)$, where $w_1(s,\theta) = \Sigma_{i=1}^n w_i(s) Q_i(\theta)$, $\{Q_1(\theta), \ldots, Q_n(\theta)\}$ is the basis of the eigenspace H_1 of $-\Delta_{S^{n-1}}$ corresponding to the eigenvalue $n-1$, $w_2(s, \cdot) \in H_1^\perp$. Now, it follows from (3.5) that $w_i(s)$ satisfies the equation

$$w_i''(s) - \frac{\mu - n + 1}{s} w_i'(s) + \frac{\mu - n + 1}{s^2} w_i(s) - \frac{g_i(s)}{s^2} = 0 \tag{3.6}$$

for $i = 1, 2, \ldots, n$, where

$$g_i(s) = \int_{S^{n-1}} f'(\xi(s,\theta)) \Sigma_{j=1}^n w_j(s) Q_j(\theta) Q_i(\theta) d\theta + \int_{S^{n-1}} f'(\xi(s,\theta)) w_2(s,\theta) Q_i(\theta) d\theta,$$

$\xi(s,\theta) = \eta v(s,\theta) + (1-\eta)\overline{v}$, $\eta \in (0,1)$ (see [Gu]).

Let $t = -\ln s$, $z_i(t) = w_i(s)$, $\tilde{\xi}(t,\theta) = \xi(s,\theta)$ and $z_2(t,\theta) = w_2(s,\theta)$. Then $z_i(t)$ satisfies the equation

$$z_i''(t) + (\mu - n + 2) z_i'(t) + (\mu - n + 1) z_i(t) - \tilde{g}_i(t) = 0 \tag{3.7}$$

where

$$\tilde{g}_i(t) = \int_{S^{n-1}} f'(\tilde{\xi}) \Sigma_{j=1}^n z_j(t) Q_j(\theta) Q_i(\theta) d\theta + \int_{S^{n-1}} f'(\tilde{\xi}) z_2(t,\theta) Q_i(\theta) d\theta.$$

We first study solutions of the equation

$$y''(t) + (\mu - n + 2) y'(t) + (\mu - n + 1) y(t) = 0. \tag{3.8}$$

A simple calculation implies that (3.8) admits two linearly independent positive solutions

$$y_1(t) = e^{-(1+\mu-n)t}, \qquad y_2(t) = e^{-t}.$$

Let $\delta_1 = 1 + \mu - n$, $\delta_2 = 1$. Then if $\mu - n > -1$, we see $\delta_1 > 0$ and $\delta_2 > 0$. By the ordinary differential equation theory, we have that

$$z_i(t) = M_1 e^{-\delta_1 t} + M_2 e^{-\delta_2 t} + M_3 \int_{t_0}^t \frac{e^{-\delta_1 s} e^{-\delta_2 t} - e^{-\delta_1 t} e^{-\delta_2 s}}{e^{-(\delta_1 + \delta_2) s}} \tilde{g}_i(s) ds \tag{3.9}$$

where M_j $(j = 1, 2)$ are constants depending upon t_0, δ_1 and δ_2, $|M_3| = |\frac{1}{\mu - n}|$ is a constant independent of t_0. Now we only consider the case that $\mu - n \ne 0$, the case that $\mu - n = 0$, i.e. $n = 3$ and $\nu = 3$ or $n = 2$ and $\nu = 1$ will be studied later. It follows from (3.9) that, for t sufficiently large,

$$|z_i(t)| \leq Ce^{-\delta t} + C_1 e^{-\delta t} \int_{t_0}^{t} e^{\delta s} |\tilde{g}_i(s)| ds \tag{3.10}$$

where $\delta = \min\{\delta_1, \delta_2\} > 0$, C_1 is independent of t_0. This implies that

$$|z_i(t)| \leq Ce^{-\delta t} + C_1 e^{-\delta t} \int_{t_0}^{t} e^{\delta s} \Big[\Sigma_{j=1}^{n} |z_j(s)||F_j(s)| + |G_i(s)|\Big] ds, \tag{3.11}$$

where

$$|F_j(s)| = |\int_{S^{n-1}} f'(\tilde{\xi}) Q_i(\theta) Q_j(\theta) d\theta|,$$

$$|G_i(s)| = |\int_{S^{n-1}} f'(\tilde{\xi}) z_2(t,\theta) Q_i(\theta) d\theta|.$$

Thus,

$$\Sigma_{i=1}^{n} |z_i(t)| \leq C_2 e^{-\delta t} + C_3 e^{-\delta t} \int_{t_0}^{t} e^{\delta s} \Big[\Sigma_{j=1}^{n} |z_j(s)||F_j(s)| + \Sigma_{i=1}^{n} |G_i(s)|\Big] ds, \tag{3.12}$$

where C_3 is independent of t_0. Set $F(t) = \max_{1 \leq j \leq n} |F_j(t)|$, $G(t) = \Sigma_{i=1}^{n} |G_i(t)|$ and $Z(t) = \Sigma_{i=1}^{n} |z_i(t)|$. Then

$$Z(t) \leq C_2 e^{-\delta t} + C_3 e^{-\delta t} \int_{t_0}^{t} e^{\delta s} [F(s)Z(s) + G(s)] ds. \tag{3.13}$$

Denote $d(t_0) = \max_{t \geq t_0} F(t)$. Using the fact that $f'(\tilde{\xi}(t,\theta)) \to 0$ as $t \to \infty$, we have that $d(t_0) \to 0$ as $t_0 \to \infty$. Thus,

$$e^{\delta t} Z(t) \leq C_2 + C_3 d(t_0) \int_{t_0}^{t} e^{\delta s} Z(s) ds + C_4 \int_{t_0}^{t} \tilde{F}(s) Z_2(s) e^{\delta s} ds, \tag{3.14}$$

where

$$\tilde{F}(t) = \Sigma_{i=1}^{n} \Big[\int_{S^{n-1}} \Big(f'(\tilde{\xi}) Q_i(\theta)\Big)^2 d\theta \Big]^{1/2},$$

$$Z_2(t) = \Big(\int_{S^{n-1}} z_2^2(t,\theta) d\theta \Big)^{1/2}.$$

Let $e^{\delta t} Z(t) = h(t)$. Then

$$h(t) \leq C_2 + C_3 d(t_0) \int_{t_0}^{t} h(s) ds + C_4 \int_{t_0}^{t} \tilde{F}(s) Z_2(s) e^{\delta s} ds.$$

Set $R(t) = \int_{t_0}^{t} h(s) ds$ and $l(t) = C_2 + C_4 \int_{t_0}^{t} \tilde{F}(s) Z_2(s) e^{\delta s} ds$. We have that

$$R'(t) \leq l(t) + C_3 d(t_0) R(t).$$

This implies that

$$R(t) \leq e^{C_3 d(t_0) t} \int_{t_0}^{t} e^{-C_3 d(t_0) s} l(s) ds.$$

It follows from the integration by parts that

$$\begin{aligned} h(t) &\le l(t) + C_3 d(t_0) e^{C_3 d(t_0) t} \int_{t_0}^t e^{-C_3 d(t_0) s} l(s) ds \\ &= l(t) - e^{C_3 d(t_0) t} \int_{t_0}^t l(s) \frac{d}{ds} (e^{-C_3 d(t_0) s}) \\ &= e^{C_3 d(t_0)(t - t_0)} l(t_0) + \int_{t_0}^t e^{C_3 d(t_0)(t-s)} l'(s) ds. \end{aligned}$$

Therefore,

$$Z(t) \le C_5 e^{-\tilde{\delta} t} + C_6 \int_{t_0}^t \tilde{F}(s) Z_2(s) e^{-\tilde{\delta}(t-s)} ds, \tag{3.15}$$

where $\tilde{\delta} = \delta - C_3 d(t_0)$. Since $d(t_0) \to 0$ as $t_0 \to \infty$, we can choose t_0 sufficiently large such that $\tilde{\delta} > 0$.

On the other hand, we know that $w_2(s, \theta)$ satisfies the equation

$$w_2''(s,\theta) + \frac{\Delta_\theta w_2(s,\theta)}{s^2} - \frac{\mu - n + 1}{s} w_2'(s,\theta) + \frac{\mu}{s^2} w_2(s,\theta) - \frac{g_2(s,\theta)}{s^2} = 0, \tag{3.16}$$

where

$$\begin{aligned} &\int_{S^{n-1}} g_2(s,\theta) w_2(s,\theta) d\theta \\ &= \int_{S^{n-1}} f'(\xi(s,\theta)) w(s,\theta) w_2(s,\theta) d\theta \\ &= \int_{S^{n-1}} f'(\xi(s,\theta)) w_1(s,\theta) w_2(s,\theta) d\theta + \int_{S^{n-1}} f'(\xi) w_2^2(s,\theta) d\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \int_{S^{n-1}} g_2(s,\theta) w_2(s,\theta) d\theta \right| &\le \left[\int_{S^{n-1}} (f'(\xi) w_1(s,\theta))^2 \right]^{1/2} \left[\int_{S^{n-1}} (w_2(s,\theta))^2 \right]^{1/2} \\ &+ H(s) \int_{S^{n-1}} w_2^2(s,\theta) d\theta, \end{aligned}$$

where $H(s) = \max_{\theta \in S^{n-1}} |f'(\xi(s,\theta))|$ and $H(s) \to 0$ as $s \to 0$.

Let $W_2(s) = (\int_{S^{n-1}} w_2^2(s,\theta) d\theta)^{1/2}$. By arguments similar to those in [Zou] and [Gu], we obtain that W_2 satisfies

$$W_2''(s) - \frac{\mu - n + 1}{s} W_2'(s) + \frac{\mu - 2n + H(s)}{s^2} W_2(s) + \frac{[\int_{S^{n-1}} (f'(\xi) w_1(s,\theta))^2]^{1/2}}{s^2} \ge 0 \tag{3.17}$$

for $s \in (0, S)$ and some $S > 0$. Here we use the inequality

$$\int_{S^{n-1}} |\nabla_\theta w_2(s,\theta)|^2 d\theta \ge 2n \int_{S^{n-1}} w_2^2(s,\theta) d\theta.$$

Making the transformations $t = -\ln s$, $w_2(s,\theta) = z_2(t,\theta)$ and $Z_2(t) = W_2(s)$, we have that $Z_2(t)$ satisfies the equation

$$Z_2''(t) + (\mu - n + 2) Z_2'(t) + (\mu - 2n + H^*(t)) Z_2(t) + H_1^*(t) Z(t) \ge 0, \tag{3.18}$$

where $H^*(t) = H(s)$ and $H^*(t) \to 0$ as $t \to \infty$. To obtain $H_1^*(t)$, we notice that

$$\begin{aligned}\int_{S^{n-1}} (f'(\xi)w_1(s,\theta))^2 d\theta &= \int_{S^{n-1}} \left(\Sigma_{i=1}^n f'(\xi)w_i(s)Q_i(\theta)\right)^2 d\theta \\ &\leq C(\Sigma_{i=1}^n |w_i(s)|)^2 H_1^2(s),\end{aligned}$$

where $H_1^2(s) = \int_{S^{n-1}}(f'(\xi))^2\Sigma_{i=1}^n Q_i^2(\theta)d\theta$ and $H_1(s) \to 0$ as $s \to 0$. Under the transformation, we have that $H_1^*(t) = H_1(s)$ and $Z(t) = \Sigma_{i=1}^n |w_i(s)|$. Thus, $H_1^*(t) \to 0$ as $t \to \infty$. Using (3.18) and (3.15), we obtain that

$$\begin{aligned}&Z_2''(t) + (\mu - n + 2)Z_2'(t) + (\mu - 2n + H^*(t))Z_2(t) \\ &+C_5 H_1^*(t)e^{-\tilde{\delta}t} + C_6 H_1^*(t)\int_{t_0}^t \tilde{F}(s)Z_2(s)e^{-\tilde{\delta}(t-s)}ds \geq 0. \qquad (3.19)\end{aligned}$$

For any $0 < \hat{\delta} < \tilde{\delta}$, choose $t^* > t_0$ such that $\mu - 2n + H^*(t) = 2(\frac{2}{\nu+1} - 2) + H^*(t) < 0$ for $t \geq t^*$, and $K = K(t_0, t^*) > 1$ such that

$$Z_2(t) \leq Ke^{-\hat{\delta}t}, \quad t \in [t_0, t^*].$$

Let $\zeta(t) = Ke^{-\hat{\delta}t}$. We claim that

$$\begin{aligned}&\zeta''(t) + (\mu - n + 2)\zeta'(t) + (\mu - 2n + H^*(t))\zeta(t) \\ &+C_5 H_1^*(t)e^{-\tilde{\delta}t} + C_6 H_1^*(t)\int_{t_0}^t \tilde{F}(s)\zeta(s)e^{-\tilde{\delta}(t-s)}ds \leq 0, \qquad (3.20)\end{aligned}$$

for $t \geq t^*$. In fact, a simple calculation implies that

$$\begin{aligned}&\zeta''(t) + (\mu - n + 2)\zeta'(t) + (\mu - 2n + H^*(t))\zeta(t) \\ &\quad +C_5 H_1^*(t)e^{-\tilde{\delta}t} + C_6 H_1^*(t)\int_{t_0}^t \tilde{F}(s)\zeta(s)e^{-\tilde{\delta}(t-s)}ds \\ &= K[\hat{\delta}^2 - (\mu - n + 2)\hat{\delta} + (\mu - 2n + H^*(t))]e^{-\hat{\delta}t} \\ &\quad +C_5 H_1^*(t)e^{-\tilde{\delta}t} + C_6 H_1^*(t)\int_{t_0}^t \tilde{F}(s)Ke^{-\hat{\delta}s}e^{-\tilde{\delta}(t-s)}ds \\ &= e^{-\hat{\delta}t}\Big[K\left(\hat{\delta}^2 - (\mu - n + 2)\hat{\delta} + (\mu - 2n + H_1^*(t))\right) \\ &\quad +C_5 H_1^*(t)e^{-(\tilde{\delta}-\hat{\delta})t} + C_6 H_1^*(t)\int_{t_0}^t \tilde{F}(s)Ke^{-(\tilde{\delta}-\hat{\delta})(t-s)}ds\Big].\end{aligned}$$

Since $\hat{\delta} < \tilde{\delta}$, we easily know that

$$C_5 H_1^*(t)e^{-(\tilde{\delta}-\hat{\delta})t} + C_6 H_1^*(t)\int_{t_0}^t \tilde{F}(s)Ke^{-(\tilde{\delta}-\hat{\delta})(t-s)}ds \to 0$$

as $t \to \infty$. On the other hand, since for $\nu > 0$,

$$\mu - 2n = 2(2/(\nu + 1) - 2) < 0, \quad \hat{\delta} - \mu + n - 2 < -1,$$

we easily know that our claim holds.

Let $X(t) = Z_2(t) - \zeta(t)$. We know that

$$X''(t) + (\mu - n + 2)X'(t) + (\mu - 2n + H^*(t))X(t)$$

$$+C_6H_1^*(t)\int_{t_0}^t \tilde{F}(s)X(s)e^{-\tilde{\delta}(t-s)}ds \geq 0. \tag{3.21}$$

Since $X(t) \to 0$ as $t \to \infty$ and $X(t^*) \leq 0$, the maximum principle implies that

$$X(t) \leq 0 \quad \text{for } t \geq t^*.$$

This implies that

$$Z_2(t) \leq Ke^{-\hat{\delta}t} \ \text{ for } t \geq t_0. \tag{3.22}$$

It follows from (3.15) that

$$Z(t) \leq Ke^{-\hat{\delta}t} \quad \text{for } t \geq t_0. \tag{3.23}$$

This implies that

$$W(s) := \Big(\int_{S^{n-1}} w^2(s,\theta)d\theta\Big)^{1/2} \leq Ks^{\hat{\delta}} \tag{3.24}$$

for $s \in (0, s_0)$, where $s_0 = e^{-t_0}$.

When $n = 3$ and $\nu = 3$ or $n = 2$ and $\nu = 1$, i.e., $\mu - n = 0$, equation (3.8) has only one characteristic value -1. By a different variation of constants formula and the same steps as above, we also obtain that for any $0 < \hat{\delta} < 1$,

$$W(s) := \Big(\int_{S^{n-1}} w^2(s,\theta)d\theta\Big)^{1/2} \leq Ks^{\hat{\delta}} \tag{3.25}$$

for $s \in (0, s_0)$, where $s_0 = e^{-t_0}$.

It is clear that for any $\epsilon > 0$, we can choose $t_0 = t_0(\epsilon)$ sufficiently large such that $0 < \hat{\delta} := \delta - \epsilon < \tilde{\delta}$. Since $\delta = 1 + \mu - n$ for $-1 < \mu - n < 0$ and $\delta = 1$ for $\mu - n \geq 0$, we have obtained our conclusion. This completes the proof of Lemma 3.2. □

We fix the ϵ in Lemma 3.2 for the proofs below.

Now we study the Hölder estimate for $\overline{v}$. Let $\sigma \in \mathbb{R}$ and

$$\rho(s) = s^{-\sigma}\overline{v}(s).$$

Lemma 3.3. *For any $0 < \sigma < \hat{\delta}$, there exists a positive constant $M = M(v)$ such that*

$$\rho(s) \leq M, \quad |\rho'(s)| \leq M/s, \quad 0 < s < s_0. \tag{3.26}$$

Proof. It is easily obtained from the equation of $\overline{v}$ and a simple calculation that $\rho(s)$ satisfies the equation

$$\rho''(s) + \frac{2\sigma - \mu + n - 1}{s}\rho' + \frac{\mu_1}{s^2}\rho - \frac{f_1(\overline{v})}{s^2}\rho = g(s), \tag{3.27}$$

where

$$\mu_1 = \sigma(\sigma + n - \mu - 2) + \mu; \quad f_1(t) = f(t)/t, \ \ t \neq 0$$

and

$$g(s) = \frac{\overline{f(v)} - f(\overline{v})}{s^{2+\sigma}} = o(s^{\hat{\delta}-\sigma-2}).$$

The last identity can be obtained from Lemma 3.2 and the fact that for s small

$$f(v) - f(\overline{v}) = o(|v - \overline{v}|),$$

and so,

$$|\overline{f(v)} - f(\overline{v})| \leq \frac{1}{\omega_n}\int_{S^{n-1}} |f(v) - f(\overline{v})| = o(W) = o(s^{\hat{\delta}}).$$

We first claim that there exist two positive constants $T = T(v)$ and $M = M(v)$ such that

$$\int_t^T \rho^2/s \leq M\Big[1 + (\rho(t))^2 + (\rho'(t))^2 t^2 + \int_t^T s(\rho')^2\Big] \tag{3.28}$$

for all $0 < t < T$. To see this, fix T and multiply (3.27) by $s\rho(s)$ and integrate from t to T,

$$\mu_1 \int_t^T \rho^2/s = \int_t^T s\rho g(s) + \int_t^T f_1(\overline{v})\rho^2/s$$

$$-(s\rho'\rho)|_t^T + \int_t^T s(\rho')^2 - \frac{2\sigma - \mu + n - 2}{2}\rho^2|_t^T. \tag{3.29}$$

Since $\mu_1 = \mu_1(\sigma) = \sigma^2 + (n - \mu - 2)\sigma + \mu$, we have that $\mu_1 > 1/2$ for $0 < \sigma < \hat{\delta}$. Indeed, we know that $\mu_1(\sigma)$ attains its minimum at $\sigma = 1 + (\mu - n)/2$. On the other hand, $\hat{\delta} < 1 + (\mu - n)/2$ for both $-1 < \mu - n < 0$ and $\mu - n \geq 0$. (Note that $\hat{\delta} < 1 + \mu - n < 1 + (\mu - n)/2$ if $-1 < \mu - n < 0$, $\hat{\delta} < 1 \leq 1 + (\mu - n)/2$ if $\mu - n \geq 0$.) These imply that $\mu_1(\sigma)$ is decreasing in $(0, 1 + (\mu - n)/2)$ and

$$\mu_1(\sigma) \geq \mu_1(\hat{\delta}) \geq \mu_1(1 + \mu - n) = n - 1, \quad \text{for } -1 < \mu - n < 0,\ \sigma \in (0, \hat{\delta}),$$

$$\mu_1(\sigma) \geq \mu_1(\hat{\delta}) \geq \mu_1(1) = n - 1 \quad \text{for } \mu - n \geq 0,\ \sigma \in (0, \hat{\delta}).$$

Thus, to obtain (3.28), it suffices to bound the right-hand side of (3.29) in terms of the right-hand side of (3.28). By the condition of f_1, one has

$$\lim_{s \to 0} f_1(\overline{v}) = 0.$$

Hence by fixing T small enough, we may bound

$$\Big|\int_t^T f_1(\overline{v})\rho^2/s\Big| \leq \frac{\mu_1}{4}\int_t^T \rho^2/s.$$

By the Schwartz inequality and the Young inequality, one has

$$\Big|\int_t^T s\rho g(s)\Big| \leq \Big(\int_t^T \rho^2/s\Big)^{1/2}\Big(\int_t^T s^3 g^2(s)\Big)^{1/2} \leq \frac{\mu_1}{4}\int_t^T \rho^2/s + M\int_t^T s^{2\hat{\delta}-2\sigma-1},$$

since

$$|g(s)| = o(s^{\hat{\delta}-\sigma-2}), \quad s \to 0.$$

Therefore,

$$\int_t^T f_1(\overline{v})\rho^2/s + \int_t^T s\rho g(s) \leq \frac{\mu_1}{2}\int_t^T \rho^2/s + MT^{2\hat{\delta}-2\sigma},$$

since $\hat{\delta} > \sigma$. Inserting this into (3.29), we obtain (3.28) immediately since the last three terms in (3.29) are bounded by the right-hand side of (3.28).

Notice that $\sigma < \hat{\delta} < \delta$ implies that $2\sigma < \mu - n + 2$ for both $-1 < \mu - n < 0$ and $\mu - n \geq 0$. Indeed,

$$2\sigma < 2\delta = \begin{cases} 2(\mu - n + 1) < \mu - n + 2 & \text{if } -1 < \mu - n < 0 \\ 2 \leq \mu - n + 2 & \text{if } \mu - n \geq 0. \end{cases}$$

The remainder of the proof of this lemma is a little variant of the proof of Lemma 4.2 of [Zou]. □

As an immediate corollary of Lemma 3.3, we obtain the following Hölder type estimate of $\overline{v}$ and $\overline{v}'$ near $s = 0$.

Lemma 3.4. *Let $\hat{\delta}$ be given in Lemma 3.2 and v a solution of (2.8). Then there exists a constant $M = M(v) > 0$ such that*

$$|\overline{v}(s)| \leq M s^{\hat{\delta}}, \quad |\overline{v}'(s)| \leq M s^{\hat{\delta}-1}, \tag{3.30}$$

and

$$\int_{S^{n-1}} v^2(s, \theta) \leq M s^{2\hat{\delta}}. \tag{3.31}$$

Proof. We only show $(3.30)_1$ and (3.31). The proof of $(3.30)_2$ is left to the readers. We first make the change of variables

$$t = -\ln s, \quad v_1(t) = \overline{v}(s).$$

Then v_1 satisfies the equation

$$v_1''(t) + (\mu - n + 2)v_1'(t) + \mu v_1 = g_1(t), \quad t > 0, \tag{3.32}$$

where

$$g_1(t) = \overline{f(v)} = f(v_1) + (\overline{f(v)} - f(v_1)) = O(|v_1|^2) + o(W) = o(e^{-\hat{\delta} t})$$

for $2\sigma > \hat{\delta}$ (see Lemma 3.3). The two characteristic values of the equation (3.32) are

$$k_1 = \frac{n - \mu - 2}{2} + \frac{[(\mu - n)^2 - 4(n-1)]^{1/2}}{2},$$

$$k_2 = \frac{n - \mu - 2}{2} - \frac{[(\mu - n)^2 - 4(n-1)]^{1/2}}{2}.$$

When $(\mu - n)^2 \leq 4(n-1)$ (note that this covers the case $-1 < \mu - n < 0$), we have that the equation (3.32) has two conjugate characteristic values

$$k_1 = -\sigma_0 + \sigma_1 i, \quad k_2 = -\sigma_0 - \sigma_1 i$$

with $\sigma_0 = 1 + (\mu - n)/2$, $\sigma_1 \geq 0$. It follows, by the variation of constants formula, that there exists a positive constant $M = M(v_1)$ such that

$$|v_1(t)| \leq M e^{-\sigma_0 t}\Big(1 + \int_{t_0}^{t} |g_1(s)| e^{\sigma_0 s}\Big) ds \leq M e^{-\hat{\delta} t}.$$

(Note that $\hat{\delta} < 1 + (\mu - n)/2$ no matter $\mu - n \geq 0$ or $-1 < \mu - n < 0$.)

When $(\mu - n)^2 > 4(n-1)$, we have that $\mu - n > 0$ and $\hat{\delta} < 1$. Moreover, the two characteristic values of (3.32) satisfy

$$k_1 < -1 \quad \text{and} \quad k_2 < -1.$$

Therefore, by arguments similar to the above, we obtain that

$$|v_1(t)| \leq Me^{-t} \leq Me^{-\hat{\delta}t}.$$

Since $v(s,\theta) = w(s,\theta) + \overline{v}$, it is easy to see that (3.31) follows from Lemma 3.2 and (3.30). This completes the proof. □

By arguments similar to those in the proof of Theorem 5.2 of [Zou], we obtain the following proposition from Lemmas 3.2 and 3.4.

Proposition 3.5. *Let $\tau \geq 0$ be an integer and v a solution of (2.8). Then for the ϵ given in Lemma 3.2, there exists a constant $M = M(v, \epsilon, \tau) > 0$ (independent of s) such that*

$$\max_{|y|=s} |D^\tau v(y)| \leq M s^{1+\mu-n-\epsilon-\tau} \ \text{for } 0 < s < s_0 \tag{3.33}$$

if $-1 < \mu - n < 0$ and

$$\max_{|y|=s} |D^\tau v(y)| \leq M s^{1-\epsilon-\tau} \ \text{for } 0 < s < s_0 \tag{3.34}$$

if $\mu - n \geq 0$.

Proof of Theorem 3.1

By Proposition 3.5, we know that

$$|f'(\tilde{\xi})| \leq Me^{-\hat{\delta}t}.$$

Let δ and G be as in the proof of Lemma 3.2. We have that

$$G(t) \leq Me^{-2\hat{\delta}t} = Me^{-2(\delta-\epsilon)t}.$$

Choose ϵ sufficiently small, it follows from (3.13) that

$$Z(t) \leq Me^{-\delta t} + Me^{-\delta t}\int_{t_0}^{t} e^{(\delta-\hat{\delta})s} Z(s)ds. \tag{3.35}$$

Let $R(t) = e^{\delta t}Z(t)$. Then the Gronwall's inequality implies that

$$R(t) \leq M.$$

Thus,

$$Z(t) \leq Me^{-\delta t}. \tag{3.36}$$

Arguments similar to those in the proof of Lemma 3.2 imply that

$$Z_2(t) \leq Me^{-\delta t}.$$

This completes the proof. □

Corollary 3.6. *Let v be a solution of (2.8). Then there exists a constant $M = M(v) > 0$ such that*

$$\overline{v}(s) \le M s^{1+(\mu-n)}, \quad |\overline{v}'(s)| \le M s^{\mu-n} \quad \text{for } 0 < s < s_0$$

if $-1 < \mu - n < 0$ and

$$|\overline{v}(s)| \le M s, \quad |\overline{v}'(s)| \le M \quad \text{for } 0 < s < s_0$$

if $\mu - n \ge 0$.

Proof. Since $2\hat{\delta} > \delta$ (by choosing ϵ small), the proof is similar to Lemma 3.4. □

Now we can use the estimates obtained in Theorem 3.1; Corollary 3.6 and arguments similar to those in the proof of Theorem 5.2 of [Zou] to obtain the following theorem.

Theorem 3.7. *Let $\tau \ge 0$ be an integer and v a solution of (2.8). Then there exist $s_0 > 0$ and $M = M(v, \tau) > 0$ (independent of s) such that*

$$\max_{|y|=s} |D^\tau v(y)| \le M s^{1+\mu-n-\tau} \quad \text{for } 0 < s < s_0 \tag{3.37}$$

if $-1 < \mu - n < 0$ and

$$\max_{|y|=s} |D^\tau v(y)| \le M s^{1-\tau} \quad \text{for } 0 < s < s_0 \tag{3.38}$$

if $\mu - n \ge 0$.

4. Local Lipschitz Type Estimates and Asymptotic Expansions for $\mu - n \ge 0$

In this section we will obtain the local Lipschitz type estimate for w. This yields a desired expansion for the application of the moving-plane method. Our main result in this section is the following theorem.

Theorem 4.1. *Let $\tau \ge 0$ be an integer. Then there exist $s_0 > 0$ and a constant $M = M(v, \tau) > 0$ (independent of s) such that*

$$\max_{|y|=s} |D^\tau w(y)| \le M s^{1-\tau} \quad \text{for } 0 < s < s_0 \tag{4.1}$$

where w is given by (2.7).

The proof of Theorem 4.1 is exactly the same as that of Theorem 3.7 and the local maximal principle as Lemma 5.1 of [Zou] plays a role. As before, we first establish a local L^2-estimate for w near the origin and then the rest is routine.

Proof. We only show the case that $\tau = 0$, the rest is left to the readers. Define $\tilde{w}(s, \theta) = w(s, \theta)/s$. Then $\tilde{w}$ satisfies the equation

$$\tilde{w}''(s, \theta) + \frac{\Delta_\theta \tilde{w}}{s^2} - \frac{\mu - n - 1}{s}\tilde{w}' + \frac{n-1}{s^2}\tilde{w} - \frac{f(v) - \overline{f(v)}}{s^3} = 0. \tag{4.2}$$

As in the proof of Lemma 3.2, we define

$$\tilde{w}(s,\theta) = \tilde{w}_1(s,\theta) + \tilde{w}_2(s,\theta)$$

and

$$\tilde{w}_1(s,\theta) = \Sigma_{i=1}^n \tilde{w}_i(s) Q_i(\theta).$$

Then $\tilde{w}_i(s)$ satisfies the equation

$$\tilde{w}_i''(s) - \frac{\mu - n - 1}{s}\tilde{w}_i'(s) - \frac{g_i(s)}{s^2} = 0 \tag{4.3}$$

for $i = 1, 2, \ldots, n$, where

$$g_i(s) = \int_{S^{n-1}} f'(\xi(s,\theta))\Sigma_{j=1}^n \tilde{w}_j(s) Q_j Q_i + \int_{S^{n-1}} f'(\xi)\tilde{w}_2(s,\theta) Q_i,$$

$\xi(s,\theta) = \rho v(s,\theta) + (1-\rho)\overline{v}$, $\rho \in (0,1)$. Let $t = -\ln s$, $z_i(t) = \tilde{w}_i(s)$, $\tilde{\xi}(t,\theta) = \xi(s,\theta)$ and $z_2(t,\theta) = \tilde{w}_2(s,\theta)$. Then $z_i(t)$ satisfies the equation

$$z_i''(t) + (\mu - n) z_i'(t) - \tilde{g}_i(t) = 0 \tag{4.4}$$

where

$$\tilde{g}_i(t) = \int_{S^{n-1}} f'(\tilde{\xi})\Sigma_{j=1}^n z_j(t) Q_j Q_i + \int_{S^{n-1}} f'(\tilde{\xi}) z_2(t,\theta) Q_i.$$

The two characteristic values of the equation

$$y''(t) + (\mu - n) y'(t) = 0$$

are $\lambda_1 = -(\mu - n)$, $\lambda_2 = 0$. Note that $\mu - n \geq 0$. Arguments similar to those in the proof of Lemma 3.2 imply that

$$\Sigma_{i=1}^n |z_i(t)| \leq C_7 + C_8 \int_{t_0}^t \left[\Sigma_{j=1}^n |z_j(s)||F_j(s)| + \Sigma_{i=1}^n |G_i(s)|\right] ds \tag{4.5}$$

where C_8 is independent of t_0. Since $|f'(\tilde{\xi})| = O(e^{-t})$ and $\Sigma_{i=1}^n |z_i(t)|$ is bounded (see Theorem 3.7 and Corollary 3.6), we have that

$$Z(t) \leq C_9 + C_{10} \int_{t_0}^t e^{-s} Z_2(s) ds,$$

where $Z(t)$ and $Z_2(t)$ are the same as that in the proof of Lemma 3.2. On the other hand, we know that Z_2 satisfies the equation

$$Z_2''(t) + (\mu - n) Z_2'(t) - (n + 1 - e^{-t}) Z_2(t) + e^{-t} Z(t) \geq 0. \tag{4.6}$$

By the same idea as in the proof of Lemma 3.2 we have that

$$Z_2(t) \leq M e^{-t} \quad \text{for } t \geq t_0.$$

This implies that $Z(t) \leq M$ for $t \geq t_0$. Thus, $\tilde{W}(s) \leq M$ for $s \in (0, s_0)$, where $\tilde{W}(s) = W(s)/s$, $s_0 = e^{-t_0}$.

By arguments same as those in the proof of Theorem 6.1 in [Zou], we obtain our conclusion. This completes the proof of Theorem 4.1. □

Let

$$\tilde{w}(s,\theta) = \frac{1}{s} w(s,\theta), \tag{4.7}$$

where w is given by (2.7). We view s as a parameter and show that $\tilde{w}$ tends to one of the first eigenfunctions or zero uniformly in $C^{\tau}(S^{n-1})$ as $s \to 0$ for any $\tau \geq 0$. We also obtain an expansion of v in terms of $\overline{v}$ with *good* remainder of w. The following lemma can be obtained from Theorem 4.1 and Lemma 7.1 of [Zou].

Lemma 4.2. *Let v be a solution of (2.8), $\tilde{w}$ be given by (4.7) and $\mu - n \geq 0$. Then for any non-negative integers τ and τ_1, there exists a constant $M = M(v,\tau,\tau_1) > 0$ such that*

$$|s^{\tau} D_{\theta}^{\tau_1} D_s^{\tau} \tilde{w}| \leq M, \quad y \in B_{s_0}(0), \quad y \neq 0. \tag{4.8}$$

Moreover, $\tilde{w}$ satisfies the equation

$$\tilde{w}''(s,\theta) + \frac{\Delta_\theta \tilde{w}}{s^2} - \frac{\mu - n - 1}{s}\tilde{w}' + \frac{n-1}{s^2}\tilde{w} = \frac{f(v) - \overline{f(v)}}{s^3}, \tag{4.9}$$

where

$$|g(s)| = \left|\frac{f(v) - \overline{f(v)}}{s^3}\right| \leq M s^{-1}. \tag{4.10}$$

Now we show the following theorem.

Theorem 4.3. *Let $\tilde{w}$ be a solution of (4.9). Then necessarily*

$$\lim_{s \to 0} \tilde{w}(s,\theta) = V(\theta), \tag{4.11}$$

where V is zero or one of the first eigenfunctions of $-\Delta$ on S^{n-1}, i.e.,

$$\Delta_\theta V + (n-1)V = 0, \quad \overline{V} = 0. \tag{4.12}$$

Proof. Let $\tilde{w}(s,\theta) = \tilde{w}_1(s,\theta) + \tilde{w}_2(s,\theta)$ be as in the proof of Theorem 4.1. It is easily known from the proof of Theorem 4.1 that $\tilde{w}_2(s,\theta) \to 0$ as $s \to 0$. (We know that $Z_2(t) \to 0$ as $t \to +\infty$.) On the other hand, we know from the proof of Theorem 4.1 that

$$\tilde{w}_1(s,\theta) = \Sigma_{i=1}^n \tilde{w}_i(s) Q_i(\theta)$$

and $\tilde{w}_i(s)$ satisfies the equation (4.3). Let $t = -\ln s$, $z_i(t) = \tilde{w}_i(s)$ and $\tilde{g}_i(t) = g_i(s)$. Then z_i satisfies the equation (4.4). We easily know that $\tilde{g}_i(t) \leq Me^{-t}$ and $z_i(t)$ is bounded for t sufficiently large. Then z_i''', z_i'' and z_i' remain also bounded when t is sufficiently large.

If $\mu - n = 0$, we easily obtain that

$$z_i'(t) \to 0, \quad z_i''(t) \to 0 \text{ as } t \to +\infty. \tag{4.13}$$

If $\mu - n > 0$, it follows from (4.4) that

$$(\mu - n)(z_i'(t))^2 = \tilde{g}_i(t) z_i'(t) - \Big(\frac{1}{2}(z_i'(t))^2\Big)'. \tag{4.14}$$

This implies that

$$\int_t^{\infty} (z_i'(s))^2 ds < \infty, \tag{4.15}$$

which implies that (4.13) still holds. Therefore, it follows easily from the equation that

$$z_i(t) \to z_0^i \quad \text{as } t \to \infty, \tag{4.16}$$

for $i = 1, 2, \ldots, n$. Where $z_0 = (z_0^1, z_0^2, \ldots, z_0^n)$ is a point in $\mathbb{R}^n$, z_0 may equals to $0 \in \mathbb{R}^n$. This also implies our conclusion. □

Combining Theorems 4.1 and 4.3, we establish the following asymptotic expansion at the origin for solutions of (2.8).

Theorem 4.4. *(Asymptotic Expansion) Let $\mu - n \geq 0$ and v a solution of (2.8). Then*

$$v(y) = \overline{v}(s) + s\tilde{w}(s, \theta), \tag{4.17}$$

where

$$\overline{v}(s) = O(s), \quad \overline{v}'(s) = O(1).$$

Moreover, for any integer $\tau \geq 0$, we have

$$\tilde{w}(s, \theta) \to V(\theta) \quad \textit{as } s \to 0 \tag{4.18}$$

uniformly in $C^\tau(S^{n-1})$, where V is zero or one of the first eigenfunction of $(-\Delta)$ on S^{n-1}, namely,

$$\Delta_\theta V + (n-1)V = 0, \quad \overline{V} = 0. \tag{4.19}$$

5. Local Hölder Type Estimates and Asymptotic Expansions for $-1 < \mu - n < 0$

In this section we study the local Hölder type estimates and asymptotic expansions for the solutions v of (2.8) when $-1 < \mu - n < 0$. We will see that we can not obtain local Lipschitz type estimates in this case without extra conditions on v. Our main ideas in all the proofs in this section are similar to those in Section 4. We first show the following theorem which is similar to Theorem 4.1.

Theorem 5.1. *Let $\tau \geq 0$ be an integer, $-1 < \mu - n < 0$ and v a solution of (2.8). Then there exist $s_0 > 0$ and a constant $M = M(v, \tau) > 0$ (independent of s) such that*

$$\max_{|y|=s} |D^\tau w(y)| \leq M s^{1+\mu-n-\tau} \quad \textit{for } 0 < s < s_0 \tag{5.1}$$

where w is given by (2.7).

Proof. The proof is similar to that of Theorem 4.1. Define

$$\tilde{w} = \frac{w}{s^{1+\mu-n}} \quad \text{and} \quad \tilde{w}(s, \theta) = \Sigma_{i=1}^n \tilde{w}_i(s) Q_i(\theta) + \tilde{w}_2(s, \theta), \tag{5.2}$$

which are similar to that in the proof of Theorem 4.1. We have that $\tilde{w}(s)$ satisfies the equation

$$\tilde{w}'' + \frac{\Delta_\theta \tilde{w}}{s^2} + \frac{1+\mu-n}{s}\tilde{w}' + \frac{n-1}{s^2}\tilde{w} = g(y) \tag{5.3}$$

where $|g(y)| = \left|\frac{\overline{f(v)} - f(v)}{s^{2+1+\mu-n}}\right|$ and $\tilde{w}_i(s)$ satisfies the equation

$$\tilde{w}_i''(s) + \frac{1+\mu-n}{s}\tilde{w}_i'(s) - \frac{g_i(s)}{s^2} = 0, \tag{5.4}$$

where

$$g_i(s) = \int_{S^{n-1}} f'(\xi)\left[\Sigma_{j=1}^n \tilde{w}_j(s)Q_jQ_i + \tilde{w}_2(s,\theta)Q_i\right]d\theta. \tag{5.5}$$

Let $t = -\ln s$, $z_i(t) = \tilde{w}_i(s)$ and $\tilde{g}_i(t) = g_i(s)$. Then

$$z_i'' - (\mu - n)z_i' - \tilde{g}_i(t) = 0. \tag{5.6}$$

The two characteristic values of the equation

$$y''(t) - (\mu - n)y'(t) = 0 \tag{5.7}$$

are $\lambda_1 = \mu - n$ and $\lambda_2 = 0$. We know that $\lambda_1 < 0$ if $-1 < \mu - n < 0$. Therefore, the exactly same arguments as those in the proof of Theorem 4.1 imply that

$$Z_2(t) \le Me^{-(1+\mu-n)t} \text{ for } t \ge t_0 \tag{5.8}$$

and

$$Z(t) \le M \text{ for } t \ge t_0. \tag{5.9}$$

These also imply that

$$\tilde{W}(s) \le M \text{ for } s \in (0, s_0) \tag{5.10}$$

where $\tilde{W}(s) = s^{-(1+\mu-n)}W(s)$. The rest of the proof is exactly same as that of Theorem 4.1. □

The following lemma which is similar to Lemma 4.2 can be obtained by Theorem 5.1 and Lemma 7.1 of [Zou].

Lemma 5.2. *Let v be a solution of (2.8), $\tilde{w}$ be given by (5.2). Then for any nonnegative integers τ and τ_1, there exists a constant $M = M(v,\tau,\tau_1) > 0$ such that*

$$|s^\tau D_\theta^{\tau_1} D_s^\tau \tilde{w}| \le M, \quad y \in B_{s_0}(0), \quad y \ne 0. \tag{5.11}$$

Moreover, $\tilde{w}$ satisfies the equation

$$\tilde{w}'' + \frac{\Delta_\theta \tilde{w}}{s^2} + \frac{1+\mu-n}{s}\tilde{w}' + \frac{n-1}{s^2}\tilde{w} = g(y) \tag{5.12}$$

where

$$|g(y)| = \left|\frac{\overline{f(v)} - f(v)}{s^{2+1+\mu-n}}\right| \le Ms^{\mu-n-1}. \tag{5.13}$$

Now we claim the following theorem.

Theorem 5.3. *Let $\tilde{w}$ be a solution of (5.12). Then necessarily*

$$\lim_{s\to 0} \tilde{w}(s,\theta) = V(\theta), \tag{5.14}$$

where V is zero or one of the first eigenfunctions of $-\Delta$ on S^{n-1} (with eigenvalue $(n-1)$), i.e.,

$$\Delta_\theta V + (n-1)V = 0, \quad \overline{V} = 0. \tag{5.15}$$

Proof. This theorem can be obtained by the same arguments as those in the proof of Theorem 4.3 or Theorem 7.1 of [Zou]. □

Combining Theorems 5.1 and 5.3, we establish the following asymptotic expansion at the origin for solutions of (2.8).

Theorem 5.4. *(Asymptotic Expansion) Let $-1 < \mu - n < 0$ and v a solution of (2.8). Then*

$$v(y) = \overline{v}(s) + s^{1+\mu-n}\tilde{w}(s,\theta), \tag{5.16}$$

where

$$\overline{v}(s) = O(s^{\hat{\sigma}}), \quad \overline{v}'(s) = O(s^{\hat{\sigma}-1}),$$

here $\hat{\sigma} = \min\{1 + (\mu - n)/2,\ 2(1 + \mu - n)\}$. Moreover, for any integer $\tau \geq 0$, we have

$$\tilde{w}(s,\theta) \to V(\theta) \quad \text{as } s \to 0 \tag{5.17}$$

uniformly in $C^\tau(S^{n-1})$, where V is zero or one of the first eigenfunctions of $-\Delta$ on S^{n-1}, namely,

$$\Delta_\theta V + (n-1)V = 0,\ \overline{V} = 0. \tag{5.18}$$

Remark 5.5. When $\mu - n = -1$, i.e. $n = 2$ and $\nu = 3$, we easily obtain the expansion of v of (2.8) as

$$v(y) = \overline{v}(s) + w(s,\theta), \tag{5.19}$$

where $w(s,\theta)$ is defined in (2.7) and $w(s,\theta) \to 0$ as $s \to 0$. Moreover, since $w(s,\theta)$ satisfies the equation

$$w'' + \frac{\Delta_\theta w}{s^2} + \frac{n-1}{s^2}w - \frac{f(v) - \overline{f(v)}}{s^2} = 0, \tag{5.20}$$

it follows by the same arguments as those in the proof of Lemma 7.3 of [Zou] that

$$\lim_{s\to 0} sw'(s,\theta) = 0, \quad \lim_{s\to 0} s^2 w''(s,\theta) = 0 \tag{5.21}$$

in $C^\tau(S^{n-1})$ uniformly for any integer $\tau \geq 0$.

6. An Auxiliary Lemma for $\mu - n \geq 0$

In this section we will obtain an auxiliary lemma for the moving-plane procedure. The main idea is similar to that of Section 8 of [Zou].

Using the transform (2.4), we immediately obtain an asymptotic expansion for nonnegative solutions of (I) at infinity by combining Theorem 4.4 and Lemma 8.1 of [Zou] under assumptions $\mu - n \geq 0$ and (2.3).

Theorem 6.1. *Let $\mu - n \geq 0$ and u be a nonnegative solution of (I). Suppose that the assumption (2.3) holds. Then we have the expansion*

$$u(x) = r^\alpha\Big(\lambda + \xi(r) + \frac{\eta(r,\theta)}{r}\Big), \tag{6.1}$$

where (r,θ) is the spherical coordinates with $r = |x|$. Furthermore the following properties are satisfied.

1. $\xi(r) = r^{-\alpha}\overline{u}(r) - \lambda$, *and there exist* R_0 $(= s_0^{-1}) > 0$ *and a constant* $M = M(u) > 0$ *such that*

$$|\xi(r)| \le Mr^{-1}, \quad |\xi'(r)| \le Mr^{-2} \quad \text{for } r > R_0. \tag{6.2}$$

2. Let τ *and* τ_1 *be two nonnegative integers. Then there exists a positive constant* $M = M(u, \tau, \tau_1)$ *such that*

$$|r^\tau D_\theta^{\tau_1} D_r^\tau \eta| \le M, \quad r > R_0. \tag{6.3}$$

3. Let τ *be a nonnegative integer. Then* $\eta(r,\theta)$ *tends to* $V(\theta)$ *uniformly in* $C^\tau(S^{n-1})$ *as* $r \to \infty$, *where*

$$V(\theta) = \theta \cdot x_0 \tag{6.4}$$

for some $x_0 \in \mathbb{R}^n$ *fixed and* $\theta = x/r \in S^{n-1}$.

The theorem enables us to establish the precise limit property below (Lemma 6.2) for nonnegative solutions of (I), which we need to begin the moving-plane procedure.

We first introduce some notation.

For $\gamma \in \mathbb{R}$, let Σ_γ be the hyperplane

$$\Sigma_\gamma = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \; x_1 = \gamma\}.$$

For $x \in \mathbb{R}^n$, denote x^γ the reflection point of x about Σ_γ, that is,

$$x^\gamma = (2\gamma - x_1, x_2, \dots, x_n).$$

As a corollary of the expansions (6.1)-(6.4), we have the following result.

Lemma 6.2. *Let* $\mu - n \ge 0$ *and* u *be a nonnegative solution of* (I). *Suppose that (2.3) holds. Then*

1. If $\gamma^j \in \mathbb{R} \to \gamma$ *and* $\{x^j\} \to \infty$, *with* $x_1^j < \gamma^j$, *then*

$$\lim_{j\to\infty} \frac{u(x^j) - u(x^{j^\gamma})}{(\gamma^j - x_1^j)|x^j|^{\alpha-2}} = -2\alpha\lambda\gamma - 2(x_0)_1, \tag{6.5}$$

where $(x_0)_1$ *is the first component of* x_0 *in (6.4).*

2. Denote

$$\gamma_0 = -(x_0)_1/(\alpha\lambda). \tag{6.6}$$

Then there exists a constant $M = M(u) > 0$ *such that*

$$u_1(x) \ge 0, \quad \text{if } x_1 \ge \gamma_0 + 1 \text{ and } |x| \ge M. \tag{6.7}$$

Proof. To prove (6.5), without loss of generality, we assume that

$$\lim_{j\to\infty} \frac{x^j}{|x^j|} = \overline{\theta} \in S^{n-1}.$$

For simplicity, we also assume that

$$\gamma^j \equiv \gamma, \quad j = 1, 2, \dots,$$

since the following arguments work equally well for the sequence $\{\gamma^j\}$.

Using the expansion (6.1), we have

$$\begin{aligned}\frac{u(x^j)-u(x^{j^\gamma})}{(\gamma-x_1^j)|x^j|^{\alpha-2}} &= \frac{\lambda}{(\gamma-x_1^j)|x^j|^{\alpha-2}}\Big(|x^j|^\alpha-|x^{j^\gamma}|^\alpha\Big)\\ &+ \frac{1}{(\gamma-x_1^j)|x^j|^{\alpha-2}}\Big(\xi(|x^j|)|x^j|^\alpha-\xi(|\xi^{j^\gamma}|)|x^{j^\gamma}|^\alpha\Big)\\ &+ \frac{1}{(\gamma-x_1^j)|x^j|^{\alpha-2}}\Big(|x^j|^{\alpha-1}\eta(|x^j|,\theta^j)-|x^{j^\gamma}|^{\alpha-1}\eta(|x^{j^\gamma}|,\theta^{j^\gamma})\Big).\end{aligned}$$

By the mean value theorem, one has

$$|x^j|^\alpha-|x^{j^\gamma}|^\alpha=-\frac{4\alpha\gamma\beta_j^{\alpha-1}(\gamma-x_1^j)}{|x^j|+|x^{j^\gamma}|}$$

where β_j is a number between $|x^j|$ and $|x^{j^\gamma}|$. Therefore,

$$\begin{aligned}&\frac{\lambda}{(\gamma-x_1^j)|x^j|^{\alpha-2}}\Big(|x^j|^\alpha-|x^{j^\gamma}|^\alpha\Big)\\ &=-\frac{4\lambda\alpha\gamma\beta_j^{\alpha-1}}{|x^j|^{\alpha-2}(|x^j|+|x^{j^\gamma}|)}\\ &=-4\alpha\lambda\gamma(1/2+o(1))\to-2\alpha\lambda\gamma \quad \text{as } j\to\infty,\end{aligned}$$

since $|x^j|/|x^{j^\gamma}|\to 1$. Similarly, we have for some β_j between $|x^j|$ and $|x^{j^\gamma}|$ that

$$\xi(|x^j|)|x^j|^\alpha-\xi(|\xi^{j^\gamma}|)|x^{j^\gamma}|^\alpha=[\alpha\beta_j^{\alpha-1}\xi(|x^j|)+|x^{j^\gamma}|^\alpha\xi'(\beta_j)]\frac{-4\gamma(\gamma-x_1^j)}{|x^j|+|x^{j^\gamma}|},$$

and in turn,

$$\begin{aligned}\frac{1}{|x^j|^{\alpha-2}(\gamma-x_1^j)}\Big(\xi(|x^j|)|x^j|^\alpha-\xi(|x^{j^\gamma}|)|x^{j^\gamma}|^\alpha\Big) &= O(|x^j|^{\alpha-2})\frac{4\gamma}{|x^j|^{\alpha-2}(|x^j|+|x^{j^\gamma}|)}\\ &= O(|x^j|^{-1})\to 0 \quad \text{as } j\to\infty.\end{aligned}$$

Here we have used the estimate (6.2). We write

$$\begin{aligned}&\frac{1}{|x^j|^{\alpha-2}(\gamma-x_1^j)}\Big(|x^j|^{\alpha-1}\eta(|x^j|,\theta^j)-|x^{j^\gamma}|^{\alpha-1}\eta(|x^{j^\gamma}|,\theta^{j^\gamma})\Big)\\ &=\frac{\eta(|x^{j^\gamma}|,\theta^{j^\gamma})}{|x^j|^{\alpha-2}(\gamma-x_1^j)}\Big(|x^j|^{\alpha-1}-|x^{j^\gamma}|^{\alpha-1}\Big)\\ &\quad+\frac{|x^j|}{\gamma-x_1^j}\Big(\eta(|x^j|,\theta^{j^\gamma})-\eta(|x^{j^\gamma}|,\theta^{j^\gamma})\Big)\\ &\quad+\frac{|x^j|}{\gamma-x_1^j}\Big(\eta(|x^j|,\theta^j)-\eta(|x^j|,\theta^{j^\gamma})\Big).\end{aligned}$$

As before, by (6.3) we bound

$$\frac{\eta(|x^{j^\gamma}|,\theta^{j^\gamma})}{|x^j|^{\alpha-2}(\gamma-x_1^j)}\Big(|x^j|^{\alpha-1}-|x^{j^\gamma}|^{\alpha-1}\Big)=O(|x^j|^{-1})\to 0 \ \text{ as } j\to\infty.$$

We can obtain the estimates

$$\frac{|x^j|}{\gamma - x_1^j}[\eta(|x^j|, \theta^{j^\gamma}) - \eta(|x^{j^\gamma}|, \theta^{j^\gamma})] = O(|x^j|^{-1}) \to 0 \text{ as } j \to \infty$$

$$\frac{|x^j|}{\gamma - x_1^j}[\eta(|x^j|, \theta^j) - \eta(|x^j|, \theta^{j^\gamma})] \to -2(x_0)_1$$

by the same idea as that in [Zou]. Thus,

$$\frac{1}{|x^j|^{\alpha-2}(\gamma - x_1^j)}\Big(|x^j|^{\alpha-1}\eta(|x^j|, \theta^j) - |x^{j^\gamma}|^{\alpha-1}\eta(|x^{j^\gamma}|, \theta^{j^\gamma})\Big) \to -2(x_0)_1, \text{ as } j \to \infty \tag{6.8}$$

These imply that (6.5) holds.

To prove (6.7), we may argue similarly as in proving (6.5). Indeed, suppose that (6.7) is false. Then there exists a sequence $\{x^j\} \to \infty$ such that

$$u_1(x^j) < 0, \quad x_1^j \geq \gamma_0 + 1, \quad j = 1, 2, \ldots.$$

It follows that there exists a sequence of *bounded positive* numbers $\{d_j\}$ such that

$$u(x^j) > u(x_{d_j}), \quad x_{d_j} = x^j + (2d_j, 0, \ldots, 0), \quad j = 1, 2, \ldots.$$

Denote

$$\gamma^j = x_1^j + d_j > x_1^j.$$

By assumption, one has

$$\frac{1}{(\gamma^j - x_1^j)|x^j|^{\alpha-2}}[u(x^j) - u(x^{j^\gamma})] > 0, j = 1, 2, \ldots \tag{6.9}$$

There are two possibilities, that is,

$$\liminf_{j\to\infty} \gamma^j < \infty, \quad \text{or} \quad \lim_{j\to\infty} \gamma^j = \infty.$$

If the first case occurs, we choose a convergent subsequence of $\{\gamma^j\}$ (with limit $\gamma \geq \gamma_0 + 1$, still denoted by $\{\gamma^j\}$) and apply (6.5)-(6.6), we obtain

$$\frac{1}{(\gamma^j - x_1^j)|x^j|^{\alpha-2}}[u(x^j) - u(x^{j^\gamma})] \to -2\alpha\lambda\gamma - 2(x_0)_1 \leq -2\alpha\lambda < 0.$$

This contradicts (6.9). We can derive a contradiction for the second case similarly. The proof is a little variant of the proof of Lemma 8.2 of [Zou]. Thus, neither the first and the second case can occur and (6.7) is shown. □

7. Necessary Conditions

In this section we will prove that, if u is a nonnegative radially symmetric solution of (I), then the limits (1.13) and (1.14)-(1.15) hold respectively for $\mu - n \geq 0$ and $-2 < \mu - n < 0$. Furthermore, we classify all radially symmetric solutions and prove Theorem 1.3 and Theorem 1.4.

Theorem 7.1. *Let $n \geq 2$, $\nu > 0$ and u be a nonnegative solution of (I). If u is radially symmetric about some point $x_0 \in \mathbb{R}^n$, then*

$$\lim_{|x|\to\infty} |x|^{-\alpha} u(x) = \lambda \tag{7.1}$$

where α and λ are as in (1.5). If $-2 < \mu - n < 0$, then

$$\lim_{|x|\to\infty} |x|^{1+(\mu-n)/2}(|x|^{-\alpha}u(x) - \lambda) = 0. \tag{7.2}$$

Proof. Without loss of generality, we assume $x_0 = 0$. First define a new independent variable $t = -\ln|x|$, $r = |x|$ and set

$$v(-\ln(|x|)) \equiv |x|^{-2/(\nu+1)}u(x). \tag{7.3}$$

Then the new function $v(t)$ satisfies

$$v''(t) - (n + 2\alpha - 2)v'(t) + \alpha(n + \alpha - 2)v(t) = v^{-\nu}. \tag{7.4}$$

Now look at the phase-plane portrait for this equation in the (v, v_t) plane. The only equilibrium point is $(v^*, 0)$ with $(v^*)^{-(\nu+1)} = \alpha(n + \alpha - 2)$; which is an unstable equilibrium. This implies that $v(t) \to v^*$ as $t \to -\infty$ and thus

$$\lim_{|x|\to\infty} |x|^{-\alpha}u(x) = \lambda.$$

To prove (7.2), we define

$$v(s) = |x|^{-\alpha}u(x) - \lambda, \quad s = 1/|x|, \quad \tilde{v}(s) = s^{-\sigma_0}v(s),$$

where $\sigma_0 = 1 + (\mu - n)/2$ with $-2 < \mu - n < 0$. Then, by (3.27), $\tilde{v}(s)$ satisfies the equation

$$\tilde{v}'' + \frac{1}{s}\tilde{v}' - \frac{((\mu - n)^2 - 4n + 4)/4}{s^2}\tilde{v} - \frac{f(v)}{s^{2+\sigma_0}} = 0. \tag{7.5}$$

Since $v(s) \to 0$ as $s \to 0$ (see (7.1)), by arguments similar to those in the proofs of Lemmas 3.3 and 3.4, we have that $\tilde{v}(s) \leq M$ for s sufficiently small. Indeed, if we use the notation in the proof of Lemmas 3.3 and 3.4, we claim that for any $0 < \sigma < \sigma_0$, $v(s) = O(s^\sigma)$. In fact, noticing that $g(s)$ in the proof of Lemma 3.3 is 0 here and $2\sigma < \mu - n + 2$ if $0 < \sigma < \sigma_0$, this claim can be obtained from a variant of the proof of Lemma 3.3 (since $\mu_1(\sigma_0) > 0$ for $-2 < \mu - n < 0$). This implies that the $g_1(t)$ in the proof of Lemma 3.4 satisfies

$$g_1(t) = O(|v_1|^2) = O(e^{-2\sigma t})$$

here. Choose $0 < \sigma < \sigma_0$ and $2\sigma > \sigma_0$. The proof of Lemma 3.4 shows that

$$|v_1(t)| \leq Me^{-\sigma_0 t}\Big(1 + \int_{t_0}^{t} |g_1(s)|e^{\sigma_0 s}\Big)ds \leq Me^{-\sigma_0 t}.$$

This implies $\tilde{v}(s) \leq M$. Let $t = -\ln s$, $\hat{v}(t) = \tilde{v}(s)$. Then $\hat{v}(t)$ satisfies the equation

$$\hat{v}'' - \frac{((\mu - n)^2 - 4n + 4)}{4}\hat{v} + O(e^{-\sigma_0 t})\hat{v} = 0 \tag{7.6}$$

and $\hat{v}$ is bounded for t is sufficiently large. Arguments same as those in the proof of Theorem 4.3 imply

$$\lim_{t\to\infty} \hat{v}'(t) = 0 = \lim_{t\to\infty} \hat{v}''(t).$$

This implies that

$$\lim_{t\to\infty} \hat{v}(t) = 0$$

(note that $(\nu - n)^2 - 4n + 4 < 0$ for $-2 < \mu - n < 0$, $\nu > 0$ and $n \geq 2$). This completes the proof. □

Proof of Theorem 1.3:

This follows from results in [CW].

Suppose that $u(x) \geq C|x|^{\frac{2}{\nu+1}}$ for $|x|$ large. We now consider the function v defined at (7.3) which satisfies (7.4). As $t \to -\infty$, $v(t,\theta) \geq C$ and $v^{-\nu} \leq C$. Hence by Harnack inequality, $v(t,\theta) \leq C$ as $t \to -\infty$. By the results of L. Simon [Si], $v(t,\theta) \to v(\theta)$, where $v(\theta)$ satisfies

$$v_{\theta\theta} + \frac{4}{(\nu+1)^2} v - \frac{1}{v^\nu} = 0, \ v \ \text{ is } 2\pi\text{-periodic.}$$

By Theorem 2.1 of [CW], $v(\theta) \equiv \textit{constant}$. This proves Theorem 1.3. □

Proof of Theorem 1.4.

Let $u = u(r)$ be a radially symmetric solution of (I). If $u(0) = 0$, then we have

$$(r^{n-1}u_r)_r = \frac{r^{n-1}}{u^\nu}$$

which implies that $u_r \geq 0$ and $r^{n-1}u_r \to 0$ as $r \to 0$. Hence

$$r^{n-1}u_r = \int_0^r \frac{s^{n-1}}{u^\nu(s)} ds \geq \frac{1}{nu^\nu(r)} r^n$$

which implies that

$$u(r) \geq Cr^\alpha \text{ for all } \ r \geq 0 \tag{7.7}$$

We now consider the function v defined at (7.3) which satisfies (7.4). As we know, $v(t) \to v^*$ as $t \to -\infty$. Next we consider the case when $t \to +\infty$. From (7.7), we see that $v(t) \geq C$ for all t. Since $e^{-\frac{2}{\nu+1}t} v(t) \to 0$ as $t \to +\infty$ and $v^{-\nu} \leq C$, a simple ODE theory shows that $v(t)$ is bounded as $t \to +\infty$ and $v(t) \to v^*$ as $t \to +\infty$ (since v^* is the only positive equilibrium point).

Now multiplying the equation for $v(t)$ by $v'(t)$ and integrating over $(-\infty, +\infty)$, we see that

$$-(n + 2\alpha - 2) \int_{-\infty}^{+\infty} (v'(t))^2 dt = 0$$

which implies that $v(t) \equiv v^*$. Thus, $u = u_0(r) = (\frac{\nu+1}{2})^{\frac{2}{\nu+1}} r^{\frac{2}{\nu+1}}$.

If $u(0) = \eta > 0$, by Theorems 1.1 and 1.2, we have that

$$\lim_{|x|\to+\infty} |x|^{-\alpha} u(x) = \lambda.$$

Then by scaling invariance, all solutions of (I) form a one-parameter family of solutions. □

8. The Moving-Plane Method: Proof of the Main Results

In this section we use the moving-plane method to give the proofs of Theorems 1.1 and 1.2.

The following special form of maximum principles is useful.

Lemma 8.1. *Let $\gamma \in \mathbb{R}^1$ and u be a positive solution of (I). Suppose that*

$$u(x) \le u(x^\gamma), \quad u(x) \not\equiv u(x^\gamma), \quad \text{if } x_1 < \gamma.$$

Then

$$u(x) < u(x^\gamma), \quad \text{if } x_1 < \gamma \tag{8.1}$$

and

$$u_1 > 0 \quad \text{on } x_1 = \gamma, \tag{8.2}$$

where x^γ is the reflection point of x with respect to Σ_γ.

Proof. Consider the function

$$v(x) = u(x) - u(x^\gamma) \le 0, \quad x_1 < \gamma.$$

Then v satisfies

$$\Delta v = -\nu h(x) v(x), \quad x_1 < \gamma,$$

where $h(x) = \int_0^1 \xi_\rho^{-(\nu+1)} d\rho$ and $\xi_\rho = \rho u(x) + (1-\rho) u(x^\gamma)$. Since $u(x^\gamma) > 0$ and $u(x) > 0$ for $x_1 < \gamma$, we have that $h(x) > 0$ for $x_1 < \gamma$. Hence by the strong maximum principle, v assumes nonnegative maximal values only on the boundary $x_1 = \gamma$, which implies (8.1), while (8.2) is a direct consequence of the Hopf's boundary lemma since $v = 0$ on $x_1 = \gamma$. □

Proof of Theorem 1.1

We only need to prove the sufficiency. We first claim that there exists $\gamma' > 0$ such that

$$u(x) < u(x^\gamma), \quad \text{if } x_1 < \gamma \text{ and } \gamma \ge \gamma'. \tag{8.3}$$

Suppose for contradiction that (8.3) is not true. Then there exist two sequence $\{\gamma^i\} \to \infty$ and $\{x^i\}$ with $x_1^i < \gamma^i$ such that

$$u(x^i) \ge u(y^i), \quad y^i = x^{i\gamma^i}, \quad i = 1, 2, \dots \tag{8.4}$$

Obviously y^i tends to infinity, so $u(y^i)$ tends to infinity. In turn $|x^i| \to \infty$. By Lemma 6.2, we must have

$$x_1^i \leq \gamma_0 + 1, \quad \text{for } i \text{ large.}$$

It follows that for any $\gamma_1 > \gamma_0 + 1$,

$$u(x^i) \geq u(y^i) \geq u(x^{i\gamma_1}), \quad \text{for } i \text{ large}$$

since $x_1^{i\gamma^i} >> x_1^{i\gamma_1}$ for i large and $u(x) \to \infty$ as $|x| \to \infty$. On the other hand, by Lemma 6.2 again, we conclude that

$$0 \leq \frac{1}{(\gamma_1 - x_1^i)|x^i|^{\alpha-2}}[u(x^i) - u(x^{i\gamma_1})] \to -2\alpha\gamma_1\lambda - 2(x_0)_1 < 0.$$

This is a contradiction and (8.3) follows.

Now let Γ be a subset of $\mathbb{R}$ defined by

$$\Gamma = \{\gamma \in (\gamma_0, \infty) : \ (8.3) \ \text{ holds}\}.$$

We shall prove that

$$\Gamma = (\gamma_0, \infty). \tag{8.5}$$

We first show that Γ is open. On the contrary, for some $\gamma \in \Gamma$, there exist two sequences $\{\gamma^i\} \to \gamma$ and $\{x^i\}$ with $x_1^i < \gamma^i$ such that (8.4) holds. Obviously there is a subsequence of $\{x^i\}$ tending to either infinity or $\hat{x} \in \mathbb{R}^n$ as $i \to \infty$. If the first case occurs, we simply use Lemma 6.2 and derive a contradiction, since $\gamma > \gamma_0$. If the second case occurs, we infer, from the definition of γ, that

$$\hat{x}_1 = \gamma.$$

It follows that

$$u_1(\hat{x}) \leq 0, \quad \hat{x}_1 = \gamma.$$

This simply cannot happen because of (8.2), that is, Γ is open.

Put

$$\tilde{\gamma} = \inf\{\gamma \in (\gamma_0, \infty) : \ \ (\gamma, \infty) \subset \Gamma\}.$$

We want to show that

$$\tilde{\gamma} = \gamma_0. \tag{8.6}$$

Suppose for contradiction this is not true, i.e., $\tilde{\gamma} > \gamma_0$. By continuity, one has

$$u(x) \leq u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}.$$

Thanks to Lemma 8.1, one sees that either

$$u(x) \equiv u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}$$

or

$$u(x) < u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}, \text{ i.e., } \tilde{\gamma} \in \Gamma.$$

The latter cannot occur because $(\tilde{\gamma}, \infty)$ is maximal and Γ is open. The former cannot occur either because it contradicts Lemma 6.2 since $\tilde{\gamma} > \gamma_0$. Thus $\tilde{\gamma} = \gamma_0$ and (8.5) is proved.

By continuity again, we have

$$u(x) \leq u(x^{\gamma_0}) \quad \text{for } x_1 < \gamma_0.$$

Reversing the x_1-axis, we conclude that

$$u(x) \leq u(x^{\gamma_0}) \quad \text{for } x_1 > \gamma_0.$$

That is, u is symmetric about the plane $x_1 = \gamma_0$. Since this argument applies for any direction, we finally obtain the radial symmetry of u about some point $x_0 \in \mathbb{R}^n$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2

It is enough to prove the sufficiency. First, we notice that the asymptotic expansion obtained in Theorem 5.4 is not good enough to use the moving-plane method. This implies that the assumption (1.7) is not enough to guarantee the symmetry of u, we need stronger assumptions.

The following lemma implies our conclusion.

Lemma 8.2. *Let $-2 < \mu - n < 0$, $v(s,\theta)$, $w(s,\theta)$ be defined as in (2.4) and (2.7). Assume that $v(s,\theta)$ satisfies that*

$$s^{-\sigma_0} v(s,\theta) \to 0 \quad \text{as} \quad s \to 0 \tag{8.7}$$

where $\sigma_0 = 1 + (\mu - n)/2$. Then v has local Lipschitz type estimate and the asymptotic expansion similar to (4.17).

Proof. Let $\tilde{w}(s,\theta) = s^{-\sigma_0} w(s,\theta)$. We have that $\tilde{w}$ satisfies the equation

$$\tilde{w}'' + \frac{1}{s}\tilde{w}' + \frac{\Delta_\theta \tilde{w}}{s^2} - \frac{((\mu-n)^2 - 4n + 4)/4)}{s^2}\tilde{w} - \frac{f(v) - \overline{f(v)}}{s^{2+\sigma_0}} = 0. \tag{8.8}$$

Define

$$\tilde{W}(s) = \Big(\int_{S^{n-1}} \tilde{w}^2(s,\theta) d\theta \Big)^{1/2}.$$

Arguments similar to those in the proof of Theorem 3.1 of [Zou] imply that $\tilde{W}(s)$ satisfies the inequality

$$\tilde{W}'' + \frac{1}{s}\tilde{W} - \frac{(\mu-n)^2/4 - F(s)}{s^2}\tilde{W} \geq 0, \tag{8.9}$$

and $\tilde{W}(s) \to 0$ as $s \to 0$, where

$$F(s) = \max_{\theta \in S^{n-1}} |f'(v(s,\theta))|.$$

Using the comparison principle as in [Zou], we obtain the following fundamental inequality similar to that in Theorem 3.1 of this paper or Theorem 3.2 of [Zou],

$$\tilde{W}(s) \le M s^{\tilde{\delta}}, \quad 0 < s < 1 \tag{8.10}$$

for any $0 < \tilde{\delta} < |\mu - n|/2 - F(s)$. (We know that $F(s) \to 0$ as $s \to 0$.) This implies that for any $0 < \tilde{\delta} < |\mu - n|/2$, there exist $s_0 = s_0(\tilde{\delta}) > 0$ sufficiently small and a positive constant $M = M(\tilde{\delta}, v) > 0$ such that

$$\tilde{W}(s) \le M s^{\tilde{\delta}}, \; 0 < s < s_0. \tag{8.11}$$

Let $W(s)$ be the same as that in Theorem 3.1. Note that $0 < -(\mu - n)/2 < 1$ for $n = 3$ and $\nu > 3$; $n = 2$ and $\nu > 1$. Then we can obtain the same conclusion as Theorem 3.2 of [Zou] (note $W(s) = s^{\sigma_0}\tilde{W}(s)$). That is, for any $0 < \max\{-(\mu - n)/2, (1 + \mu - n)\} < \delta < 1$, there exist $\hat{s}_0 = \hat{s}_0(\delta) > 0$ and a positive constant $M = M(\delta, v)$ such that

$$W(s) \le M s^{\delta}, \quad 0 < s < \hat{s}_0.$$

We can also obtain the same conclusion as Lemma 6.1 of [Zou]. Indeed, define $\hat{W}(s) = W(s)/s$. We infer by a same argument as that in the proof of Lemma 6.1 of [Zou] that there exists a constant $M = M(v) > 0$ such that $\hat{W}$ satisfies the equation

$$\hat{W}'' + \frac{1 - (\mu - n)}{s}\hat{W}' + M s^{\sigma_0 - 2}\hat{W} \ge 0, \quad 0 < s < \hat{s}_0. \tag{8.12}$$

We also have

$$\hat{W} \le M s^{\delta - 1}, \quad |\hat{W}'| \le M s^{\delta - 2}. \tag{8.13}$$

For any $T > 0$ ($T \le \hat{s}_0$), multiply (8.12) by $s^{1-(\mu-n)}$ and integrate from $T > t > 0$ to T to obtain

$$s^{1-(\mu-n)}\hat{W}'|_t^T + M \int_t^T \hat{W} s^{\sigma_0 - (\mu - n) - 1} ds \ge 0. \tag{8.14}$$

By (8.13), one sees that

$$\lim_{t \to 0} |t^{1-(\mu-n)}\hat{W}'(t)| = 0, \quad \int_t^T \hat{W} s^{\sigma_0 - (\mu-n) - 1} ds \le M T^{\delta + \sigma_0 - (\mu - n) - 1}, \tag{8.15}$$

since $\delta > 1 + \mu - n$ and $\sigma_0 - (\mu - n) - 1 = -(\mu - n)/2 > 0$. Thus letting t tend to 0 in (8.14) yields

$$\hat{W}'(T) + M T^{\delta + \sigma_0 - 2} \ge 0.$$

For any $s \le T$, integrate from s to T to obtain

$$\hat{W}(T) - \hat{W}(s) + M \int_s^T t^{\delta + \sigma_0 - 2} \ge 0,$$

that is,

$$\hat{W}(s) \le \hat{W}(T) + M T^{\delta + \sigma_0 - 1},$$

since $\delta + \sigma_0 - 1 > 0$ (note that $\delta > -(\mu - n)/2$). Now we obtain

$$W(s) \leq Ms, \quad 0 < s < \hat{s}_0.$$

This implies that the conclusion similar to Theorem 4.1 holds for our case here.

By the same procedure as in the proofs of Lemma 4.2, Theorem 4.3 and Theorem 4.4, we obtain the local Lipschitz estimate for v and the asymptotic expansion of v similar to that in (4.17). □

From Lemma 8.2 we obtain the conclusions similar to those in Theorem 6.1 for u. The proof of sufficiency is then obtained by moving-plane method as we did in Sections 6 and 8. This completes the proof of Theorem 1.2. □

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