The leapfrogging and the Vortex filament conjecture for Euler equations

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The Euler equation for the velocity of an incompressible fluid in $\mathbb{R}^n$, $n = 2, 3$.

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbb{R}^n \times (0, T) \end{cases} \tag{E}$$

$\mathbf{u}(x, t) : \mathbb{R}^n \times [0, T) \to \mathbb{R}^n$, $p(x, t) : \mathbb{R}^n \times [0, T) \to \mathbb{R}$.

The vorticity

$$\omega := \nabla \times \mathbf{u}$$

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The vorticity

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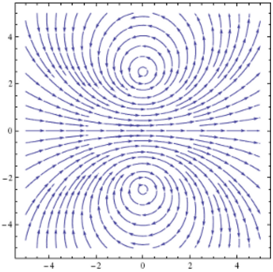
- Analysis of solutions to Euler Flows with singular concentrated vorticity $\omega(x, t) = \nabla \times \mathbf{u} \sim \delta_\Gamma$, $\Gamma = \{p_1, ..., p_N\}$ (in $\mathbb{R}^2$) or $\Gamma = \{(x(t), y(t), z(t))\}$ ($\mathbb{R}^3$);
- Introduce new Gluing Methods for Euler flows.

Analysis of vortices (and vortex lines): A mathematical subject with a long history. $1 million dollar Clay Problem.

$n = 2$

- Wolibner (1933), Yudovich (1963): there is global existence and uniqueness if $\omega_0 \in L^\infty$. If ω_0 is regular, then $\omega(t)$ is regular for all times .
- Kiselev-Sverak (2014): vorticity grows $e^{Ce^{Ct}}$.
- We are interested in describing the evolution of solutions to Euler equation with vorticities $\omega(x,t)$ being **very concentrated** around a finite number of points. (Concentrated vortices)



$n = 2$. If $\omega = \sum_{j=1}^{m} \kappa_j \delta_{\xi_j}$, then points evolve by the k-body Kirchoff-Routh law

$$(K) \qquad \dot{\xi}_j = \frac{1}{2\pi} \sum_{i \neq j} \kappa_i \frac{(\xi_i - \xi_j)^{\perp}}{|\xi_i - \xi_j|^2}, \quad t \in [0, T]$$

Marchioro-Pulvirenti (1993): If $\omega_\varepsilon(x, 0)$ is concentrated around $\xi_j(0)$, then

$$\omega_\varepsilon(x, t) \rightharpoonup \sum_{j=1}^{k} \kappa_j \delta_{\xi_j(t)}, \quad \text{as} \quad \varepsilon \to 0,$$

Ionescu-Jia (2020): Stability of singular solution $\omega = \delta_0$.

Bedrossian-Coti Zelati-Vicol (arXiv 2020): linear stability of the radially symmetric decreasing solution $\omega_0 = \frac{1}{(1+|x|^2)^2}$.

Traveling Wave Solutions

Ao-Davila-del Pino-Musso-Wei (2020): exact traveling wave solutions

$$\alpha \omega_x + \nabla^{\perp} \psi \nabla \omega = 0$$

$$\omega_\varepsilon(x,t) \rightharpoonup \sum_{j=1}^{N} \kappa_j \delta(x - (\xi_{j,1} - \alpha t, \xi_{j,2})),$$

$(\xi_1, ..., \xi_N)$ roots of Adler-Moser polynomials—Special solutions to the Tkachenko equation:

$$P''Q - 2P'Q' + PQ'' = 2\mu \left(P'Q - PQ'\right)$$

Dynamical Correspondence

Problem: (desingularization of vortices):

Given solutions $\xi_j = \xi_j(t)$ to the Hamiltonian system

$$\dot{\xi}_j = \nabla^{\perp}_{\xi_j} K(\xi), \quad j = 1, \ldots, k, \tag{K}$$

in $(0, T)$.
Find of a family of true, smooth solutions $\omega_\varepsilon(x, t)$ which in the *singular limit* $\varepsilon \to 0$ approaches a singular vortex solution supported on a given trajectory $\xi = \xi(t)$ of System (K).

Davila-del Pino-Musso-Wei (2020): Let $\xi(t) = (\xi_1(t), \ldots, \xi_N(t))$ be a collisionless solution of Kirchoff-Routh System *(K)* in the interval $(0, T)$. Then there exists a solution $(\omega_\varepsilon, \Psi_\varepsilon)$ of the 2D Euler of the form

$$\omega_\varepsilon(x,t) \sim \sum_{j=1}^{k} \frac{\kappa_j}{\varepsilon^2} U\left(\frac{x-\xi_j}{\varepsilon}\right), \quad U(y) = \frac{1}{\pi(1+|y|^2)^2}$$

$$\omega_\varepsilon(x,t) \rightharpoonup \sum_{j=1}^{k} \kappa_j \delta_{\xi_j(t)}, \quad \frac{1}{|\log \varepsilon|} |\mathbf{u}_\varepsilon|^2 \rightharpoonup \sum_{j=1}^{k} \kappa_j^2 \delta_{\xi_j(t)} \quad \text{as} \quad \varepsilon \to 0$$

Here U is the *Kaufmann-Scully vortex*: $U = \frac{1}{(1+|x|^2)^2}$.

Stream-vorticity formulation

In $\mathbb{R}^3$ the problem becomes

$$\begin{cases} \omega_t + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = 0 & , \\ u = \nabla\times\psi, \quad -\Delta\psi = \omega. \end{cases} \qquad \text{(SV)}$$

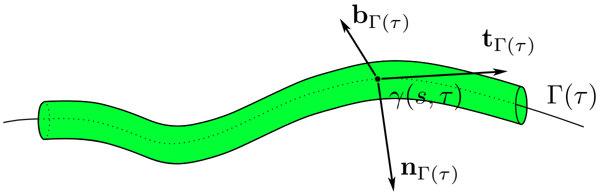
The vorticity is concentrated in an ε-neighbourhood of a time evolving curve (filament) $\Gamma(t)$ parametrized by arclength as $\gamma(s,t)$ in $\mathbb{R}^3$

$$\omega_\varepsilon(x,t) \rightharpoonup c\,\delta_{\Gamma(t)}\,\mathbf{T}_{\Gamma(t)}$$

Nearly singular solutions for Euler in $\mathbb{R}^3$?

Open question: Solutions with concentrated vorticities near curves (filaments): Vortex filament or Binormal Flow conjecture (Helmholtz, Da Rios, Levi-Civita 1858-1906-1931).

Let us consider a Frenet frame for $\gamma(\cdot, t)$,



$$\gamma_{ss} = \kappa\, \mathbf{n}, \quad \mathbf{b} = \gamma_s \times \mathbf{n}, \quad \gamma_s = \mathbf{t}.$$

n normal and **b** binormal vectors. κ curvature.

Binormal Flows

In a neighborhood of the curve, we introduce local Fermi coordinates

$$X(s, z_1, z_2, t) = \gamma(s,t) + z_i n_i(s,t), \quad |z_i| < \delta$$

We look for solutions of 3D Euler flow in the form

$$\boxed{\omega_\varepsilon(x, |\log \varepsilon|^{-1}\tau) \rightharpoonup 8\pi\delta_{\Gamma(\tau)}(x)\gamma_s(s,\tau), \qquad x = \gamma(s,\tau) + \sum_{i=1}^{2} z_i n_i}$$

where $\delta_{\Gamma(\tau)}$ denotes uniform Dirac's mass on the curve.

da Rios' formal computation (1904): γ evolves by binormal flow

$$\gamma_t = 2\,c|\log\varepsilon|\;(\gamma_s \times \gamma_{ss}) = 2\;c|\log\varepsilon|\,\kappa\,\mathbf{b}$$

Equivalently, $t = |\log\varepsilon|^{-1}\tau$,

$$\gamma_\tau = 2\;c\,\kappa\,\mathbf{b}$$

Levi-Civita (1908), Ricca (1991), Betchov (1965), Arms-Hana (1965), Ting-Klein (1991), Callegari-Ting (1996)

Vortex Filament Conjecture

Given a solution to the binormal flow

$$\gamma_\tau = 2\, c\, \kappa\, \mathbf{b} \quad \text{in} \quad [0, T]$$

Find a true solution of 3D Euler Flow satisfying

$$\vec{\omega}_\varepsilon(\cdot, |\log \varepsilon|^{-1}\tau) \rightharpoonup c\, \delta_{\Gamma(\tau)} \mathbf{T}_{\Gamma(\tau)}, \quad 0 \le \tau \le T$$

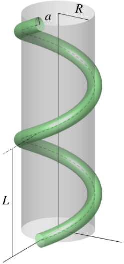
Helmholtz, Kelvin, Da Rios

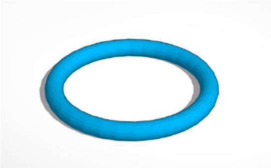
Benedetto-Caglioti-Marchioro (2015), Jerrard-Seis (2017)

Jerrard-Seis, 2017: **If** vorticities are concentrated around tubes $\gamma(t, s)$, they evolve in weak sense by binormal flow.

This Conjecture is unknown except for some **special cases**.

Examples of binormal flow: a helix whose horizontal section rotates at a constant angular speed and a vertically translating circle are solutions of the bi-normal flow of curves.





Exact solutions for 3D Euler with Helical Symmetry

One known solution of the binormal flow that does not change its form in time is the rotating-translating helix, the curve $\Gamma(\tau)$ parametrized as

$$\gamma(s,\tau) = \begin{pmatrix} R\cos\left(\frac{s-a_1\tau}{\sqrt{h^2+R^2}}\right) \\ R\sin\left(\frac{s-a_1\tau}{\sqrt{h^2+R^2}}\right) \\ \frac{hs+b_1\tau}{\sqrt{h^2+R^2}} \end{pmatrix},$$

$$a_1 = \frac{2ch}{h^2+R^2}, \quad b_1 = \frac{2cR^2}{h^2+R^2}.$$

Theorem (Davila-del Pino-Musso-Wei (2021))

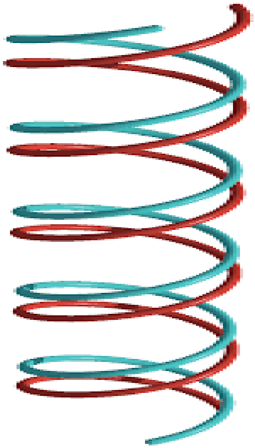
Let $\Gamma(\tau)$ be the helix. Then there exists a smooth solution $\vec{\omega}_\varepsilon(x,t)$ to 3D Euler, defined for $t \in (-\infty, \infty)$ that does not change form and follows the helix, such that for all τ,

$$\vec{\omega}_\varepsilon(x, \tau|\log\varepsilon|^{-1}) \rightharpoonup c\delta_{\gamma(s,\tau)}\gamma_s(s,\tau) \quad \text{as } \varepsilon \to 0.$$

This result extends to the situation of several helices symmetrically arranged: $\cup_{j=1}^{k}[R_{2\pi\frac{j-1}{k}}\gamma(s,\tau)]$, $k \geq 2$.

Dutrifoy (1999), Ettinger-Titi (2009), Bronzi-Lopes Filho-Nussenzveig Lopes (2015), Jiu-Li-Niu (2017), Lopes Filho-Mazzucato-Niu-Nussenzveig Lopes-Titi (2015)

Double Helices $k = 2$



Ettinger-Titi (2009) : Solutions $\vec{\omega}(x, y, z, t)$ of 3d-Euler with **Helicoidal symmetry** and **velocity orthogonal to the symmetry lines** of the Helix can be obtained by screw motion of vectors formed from a two-variable scalar function $\omega(x + iy, t)$ in the form

$$\vec{\omega}(x, y, z, t) = \omega(e^{i\frac{z}{h}}(x + iy), t) \begin{bmatrix} i(x + iy) \\ h \end{bmatrix}$$

where, for $t = \tau|\log\varepsilon|^{-1}$, $(x, y) \in \mathbb{R}^2$,

$$|\log\varepsilon|\omega_t + \nabla^{\perp}\psi \cdot \nabla\omega = 0, \qquad -\nabla \cdot (K\nabla\psi) = \omega$$

and

$$K(x, y) = \frac{1}{h^2 + x^2 + y^2} \begin{pmatrix} h^2 + y^2 & -xy \\ -xy & h^2 + x^2 \end{pmatrix}$$

Rotating helicoidal solutions, with velocity α

$$\omega(x+iy,t)=\omega(e^{-i\alpha t}(x+iy)),\quad \psi(x+iy,t)=\psi(e^{-i\alpha t}(x+iy))$$

The problem becomes

$$\nabla\omega\cdot\nabla^{\perp}\left(\psi-\frac{\alpha}{2}|\log\varepsilon|(x^2+y^2)\right)=0,\quad -\nabla\cdot(K\nabla\psi)=\omega.$$

Take $\boxed{\omega=f(\psi-\frac{\alpha}{2}|\log\varepsilon|(x^2+y^2))}$, for some f.

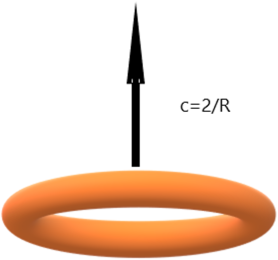
The problem reduces to the semilinear elliptic equation with anisotropic coefficients

$$-\nabla\cdot(K\nabla\psi)=f(\psi-\frac{\alpha}{2}|\log\varepsilon|(x^2+y^2))=\omega\quad \text{in } \mathbb{R}^2$$

Gluing methods can be used to construct concentrated solutions.

Vortex rings

Another known solution of the binormal flow that does not change its form in time is a circle traveling vertically with constant speed $\alpha = \frac{2}{R}$.



Solutions to Euler equations with vorticity concentrated around traveling circles with thin section are called vortex rings

Axisymmetric Euler:

$$u(r,z,t) = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z$$
$$\omega(r,z,t) = \omega^r(r,z,t)e_r + \omega^\theta(r,z,t)e_\theta + \omega^z(r,z,t)e_z$$
$$e_r = \frac{1}{r}(x,y,0)^T,\; e_\theta = \frac{1}{r}(-y,x,0)^T,\; e_z = (0,0,1)^T$$

the 3D Euler becomes

$$\begin{cases} |\log\varepsilon| u_{1,t} + u^r u_{1,r} + u^z u_{1,z} = 2u_1\psi_{1,z} \\ |\log\varepsilon| W_t + u^r W_r + u^z W_z = (u_1^2)_z \\ -[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\psi_1 = W \end{cases}$$

where

$$u_1 = \frac{u^\theta}{r}, \quad W = \frac{\omega^\theta}{r}, \quad \psi_1 = \frac{\psi^\theta}{r},$$
$$u^r = -r\psi_{1,z}, \quad u^z = 2\psi_1 + r\psi_{1,r}$$

No-swirl case, i.e., $u_1 = 0$. For $x = (r, z)$, $r > 0$

$$|\log \varepsilon| r\, W_t + \nabla^\perp(r^2\, \Psi) \cdot \nabla W = 0, \quad \frac{\partial \Psi}{\partial r} = 0 \quad \text{in } r = 0,$$

$$-\Delta_5 \Psi = W, \quad \Delta_5 := \partial^2_{rr} + \frac{3}{r}\partial_r + \partial^2_{zz}$$

Helmholtz (1858), Fraenkel (1970-1972):

Exact traveling ring solutions $W(r, z, t) = w(r, z - \alpha t)$ solve

$$\nabla^\perp \left[r^2 (\psi - \frac{\alpha}{2} |\log \epsilon|) \right] \cdot \nabla w = 0, \quad -\Delta_5 \psi = w$$

Take $w = f(r^2(\psi - \frac{\alpha}{2}|\log \epsilon|))$, then

$$-\Delta_5 \psi = f(r^2(\psi - \frac{\alpha}{2}|\log \epsilon|)) = w$$

It is expected for the vorticity $r\, w_\varepsilon \rightharpoonup 8\pi\delta_{P_0}$. Take $P_0 = (r_0, 0)$, $r_0 > 0$

Let me explain the construction of Fraenkel's ring solution by Gluing Methods.

The Green's function for $\Delta_5 := \partial^2_{rr} + \frac{3}{r}\partial_r + \partial^2_{zz}$

$$-\Delta_5 G(x, P_0) = 8\pi\delta_{P_0}, \quad \frac{\partial G}{\partial r} = 0, \qquad \text{in } r = 0$$
$$G(x, P_0) \to 0 \quad \text{as } |x| \to \infty$$

for $x = (r, z)$. Locally around P_0, $G(\cdot, P_0)$ has the expansion

$$G(x, P_0) = \log \frac{1}{|x - P_0|^4}\left(1 - \frac{3}{2r_0}(r - r_0) + O(|x - P_0|^2)\right) + O(|x - P_0|^2),$$

Let

$$r_0\,\psi_\varepsilon^0 = \log \frac{1}{\varepsilon^2 + |x - P_0|^4}\left(1 - \frac{3}{2r_0}(r - r_0)\right).$$

Take $f(s) = \varepsilon^{2-\alpha r_0} e^{\frac{s}{r_0}}$, $w_\varepsilon^0 = f(r^2(\psi - \frac{\alpha}{2}|\log\epsilon|))$. So

$$r_0\,w_\varepsilon^0(x) \sim \frac{1}{\varepsilon^2} U(\frac{x - P_0}{\varepsilon}), \quad U(y) = \frac{1}{\pi(1 + |y|^2)^2}$$

U Kaufmann-Scully vortex.

Fraenkel's vortex ring $(\psi_\varepsilon, w_\varepsilon) \sim (\psi_\varepsilon^0, w_\varepsilon^0)$ near P_0

In the expanded variables $y = \frac{x - P_0}{\varepsilon}$ the error

$$e_0 := \varepsilon^4 \left[\Delta_5 \psi_\varepsilon^0 + f(r^2(\psi_\varepsilon^0 - \frac{\alpha}{2} |\log \epsilon|)) \right]$$

$$= \varepsilon y_1 U [\frac{3}{2} + \frac{1}{2r_0} (\Gamma_0 - 4(2 - \alpha r_0) \log \varepsilon)] + O(\frac{\varepsilon^2}{1 + |y|^2})$$

with $U = e^{\Gamma_0}$. To improve the error, solve

$$L(\psi) := \Delta \psi + U \psi = e_0$$

Γ_0 solves the **Liouville equation** $\Delta \Gamma_0 + e^{\Gamma_0} = 0$ in $\mathbb{R}^2$, and $L(Z_j) = 0$, $j = 0, 1, 2$

$$Z_0 = 2 + \nabla \Gamma_0 \cdot y, \quad Z_j(y) = \partial_{y_j} \Gamma_0(y), \, j = 1, 2.$$

Thus solvability condition is required to solve $L(\psi) = e_0$ which at main order is

$$\int_{\mathbb{R}^2} y_1 U[\frac{3}{2} + \frac{1}{2r_0}(\Gamma_0 - 4(2 - \alpha r_0)\log\varepsilon)]Z_1(y)\,dy = 0.$$

This gives

$$\alpha = \frac{2}{r_0} + \frac{\beta_\varepsilon(r_0)}{|\log\varepsilon|}, \quad \beta_\varepsilon = O(1) \quad \text{as} \quad \varepsilon \to 0$$

Fraenkel (1970-1972)

Existence of a single vortex-ring solution via constrained variational method:Arnold (1964), Fraenkel-Berger (1974), Benjaman (1976), Friedman-Turkington (1981),Burton (1983), Ambrosetti-Struwe (1989), Benedetto-Caglioti-Marchioro (2000)

Traveling clustered vortex rings

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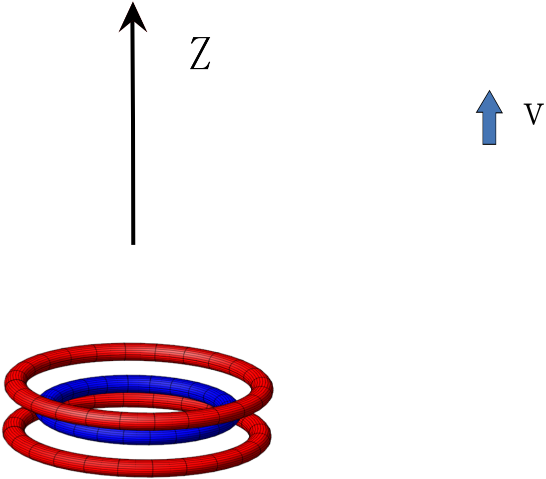
- (Ao-Liu-Wei 2021) Traveling Wave Solutions with multiple vortex rings

$$\omega_\varepsilon(r,z) \to \sum_{j=1}^{m+n} k_j \delta_{(r_j,z_j)}$$

where (r_j, z_j) are roots of generalized Adler-Moser Polynomials.

$$k_j > 0, j = 1, ..., m; k_j < 0, j = m+1, ..., m+n$$


Z
V

Interacting Vortex Rings

- Let $(r_0, 0)$ be the Fraenkel's ring
- If we write

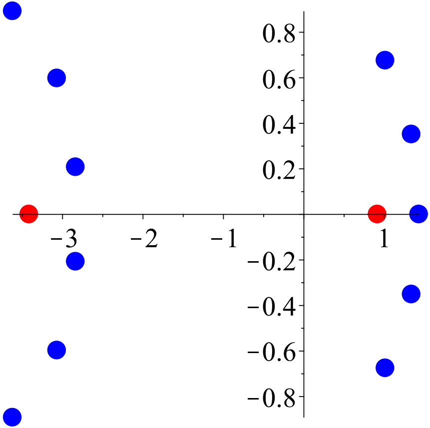
$$(r_i, z_i) = (r_0, 0) + \frac{(\hat{r}_i, \hat{z}_i)}{|\ln \varepsilon|}.$$

Then $(\hat{r}_i, \hat{z}_i)$ will be perturbation from solution of the following system:

$$\begin{cases} \sum_{j=1, j\neq k}^{m} \frac{\gamma_j}{\mathbf{a}_k - \mathbf{a}_j} - \sum_{j=1}^{n} \frac{\beta_j}{\mathbf{a}_k - \mathbf{b}_j} = \sigma_k, \text{ for } k = 1, ..., m, \\ -\sum_{j=1, j\neq k}^{n} \frac{\beta_j}{\mathbf{b}_k - \mathbf{b}_j} + \sum_{j=1}^{m} \frac{\gamma_j}{\mathbf{b}_k - \mathbf{a}_j} = -\rho_k, \text{ for } k = 1, ..., n. \end{cases}$$

where m, n corresponds to the number of positive and negative circulation, γ_j, β_j corresponds to the absolute value of the circulation, σ_j, ρ_j are some constants related to the radius and travelling speed of the ring.

$m = 2, n = 11$



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The same reduced problem has been derived when we study the multi vortex ring solution for the 3-dimensional Gross-Pitaevskii equation in (Ao-Huang-Liu-Wei 2020) when all the degree of the standard vortex are equal to $+1$ or -1. It has been shown that the existence and non-degeneracy of symmetric $(\mathbf{a}_j, \mathbf{b}_\ell)$ are related to some generalized Adler-Moser polynomials.

Traveling Wave Solutions to Euler Equations with Swirl

In the case with swirl, we have the Long-Squire equation (or more generally the Grad-Shafranov equation)

$$-\Delta_5\psi = F\left(r^2\psi - \frac{\alpha}{2}\left|\ln\varepsilon\right| r^2\right) + \frac{G\left(r^2\psi - \frac{\alpha}{2}\left|\ln\varepsilon\right| r^2\right)}{r^2} = W.$$

The same method as that of the non-swirl case can be applied to swirl, with the choice that $F = G = e^s$ near the vortex rings(also using a cutoff function to make them zero away from the vortex ring).

Result: for ε sufficiently small, there is a solution with two vortex rings to the Euler equation with swirl.

Summary on vortex ring solutions

- Fraenkel's ring solution

$$\omega_\varepsilon(r, z) \to \kappa\delta_{(r_0,0)}$$

- (Ao-Liu-Wei 2021) Multiple vortex rings

$$\omega_\varepsilon(r, z) \to \sum_{j=1}^{m+n} k_j\delta_{(r_j,z_j)}$$

$$(r_j, z_j) = (r_0, 0) + O(\frac{1}{|\ln \varepsilon|})$$

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$$(r_j, z_j) = (r_0, 0) + O(\frac{1}{|\ln\varepsilon|})$$

- What happens when

$$|(r_j, z_j) - (r_0, 0)| >> O(\frac{1}{|\ln\varepsilon|})?$$

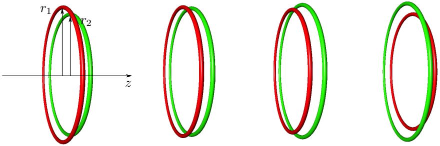
Nearly Parallel Vortex-Rings: Leap-frogging Phenomena

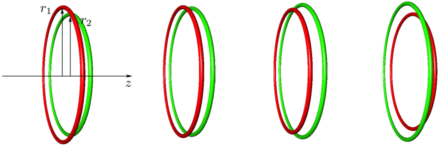
When two vortex-rings interact, Helmholtz predicts the following:

Helmholtz 1858: We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.

$$\text{speed} \sim \frac{1}{\text{radius}}$$

The motion described by Helmholtz, is often termed leapfrogging in the fluid mechanics community. Even though it has been widely studied since Helmholtz, as far as we know it has not been mathematically justified in the context of the Euler equation, even in the axi-symmetric case without swirl. As a matter of fact, the interaction leading to the leapfrogging motion is somehow borderline in strength compared to the stability of isolated vortex rings.





Aim: mathematically justify the leap-frogging dynamics for 3D axisymmetric Euler flow without swirls.

$$S(w,\psi) := |\log\varepsilon|\, r\, w_t + \nabla^{\perp}\left[r^2 \left(\psi - \frac{\alpha}{2}|\log\epsilon|\right)\right]\cdot\nabla w = 0,$$
$$-\Delta_5\psi = w, \quad r>0, z\in\mathbb{R}, \quad t\in[0,T)$$

where $\Delta_5 = \partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2$

Formal Derivation of Leap-Frogging

For $x = (r, z)$, take two **Fraenkel solutions**

$$w_\varepsilon^0(x,t) = \sum_{j=1}^{2} \frac{1}{r_j \varepsilon_j^2} U(\frac{x - P_j}{\varepsilon_j}), \quad P_j = (r_j, z_j)$$

$$\psi_\varepsilon^0(x,t) = \sum_{j=1}^{2} \frac{1}{r_j} \log \frac{1}{(\varepsilon_j^2 + |x - P_j|^2)^2} \left[1 - \frac{3}{2r_j}(r - r_j)\right]$$

Here $\varepsilon_j = \varepsilon_j(t)$, $P_j = P_j(t)$. Choose

$$r_j(t)\, \epsilon_j^2(t) = r_0\, \epsilon^2, \quad \alpha = \frac{2}{r_0}$$

To describe the dynamics around P_1, we expand variable

$$y = \frac{x - P_1}{\varepsilon_1}, \quad y = \rho e^{i\theta}.$$

To describe the error we use the notation

$$E_k = E_{k1}(\rho, t) \cos k\theta + E_{k2}(\rho, t) \sin k\theta,$$

with

$$E_{k1}(\rho, t), E_{k2}(\rho, t) = O(1) \quad \text{as } \varepsilon \to 0$$

At $x = P_1 + \epsilon_1 y$, we compute the error

$$\varepsilon_1^4\, S(w_\varepsilon^0, \psi_\varepsilon^0) = \varepsilon_1^4 \left[|\log \varepsilon|\, r\, \partial_t w_\varepsilon^0 + \nabla^\perp \left[r^2 \left(\psi_\varepsilon^0 - \frac{\alpha}{2} |\log \epsilon| \right) \right] \cdot \nabla w_\varepsilon^0 \right]$$

$$= \varepsilon_1 \left(-|\log \varepsilon| \partial_t P_1 + 4 \sum_{\ell \neq j} \frac{(P_1 - P_2)^\perp}{|P_1 - P_2|^2} + \frac{2}{r_1} |\log \varepsilon|\, \mathbf{e_1} \right) \cdot \nabla U$$

$$+ \frac{\varepsilon^2 |\log \varepsilon|}{1 + \rho^4} E_2.$$

An analogous computation around P_2.

The error term of size ε is cancelled if, for $j = 1, 2$

$$\partial_t P_j = \frac{2}{r_j}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{|\log \epsilon|}\sum_{i\neq j}\frac{(P_j - P_i)^{\perp}}{|P_j - P_i|^2} + O(\frac{1}{|\log \epsilon|})$$

Fraenkel's (1972) single-ring traveling:

$$r_1 \partial_t P_1 = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad r = r_0, \quad z = z_0 + \frac{2}{r_0}\, t$$

Choose

$$P_j = \left(r_0,\ z_0 + \frac{2}{r_0}\, t\right) + Q_j, \quad Q_j = \frac{1}{\sqrt{|\log \epsilon|}} b_i(t)$$

Here $b_i(t) = (r(b_i(t)), z(b_i(t)))$ satisfies the following
Leapfrogging dynamics

$$(LeapFrog) \quad \begin{cases} \dot{b}_i(t) = \sum_{j \neq i} \frac{(b_i - b_j)^\perp}{\|b_i - b_j\|^2} - \frac{2r(b_i)}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ b_i(0) = b_i^0 \end{cases}$$

Then the error's size becomes ε-smaller

$$\varepsilon_1^4 S(w_\varepsilon^0, \psi_\varepsilon^0) = \frac{\varepsilon^2 |\log \varepsilon|}{1 + \rho^4} E_2$$

Theorem [Dávila, del Pino, Musso, Wei, 2022]
Let $(b_1(t), \ldots, b_N(t))$ be a colisionless solution of System *(LeapFrog)*

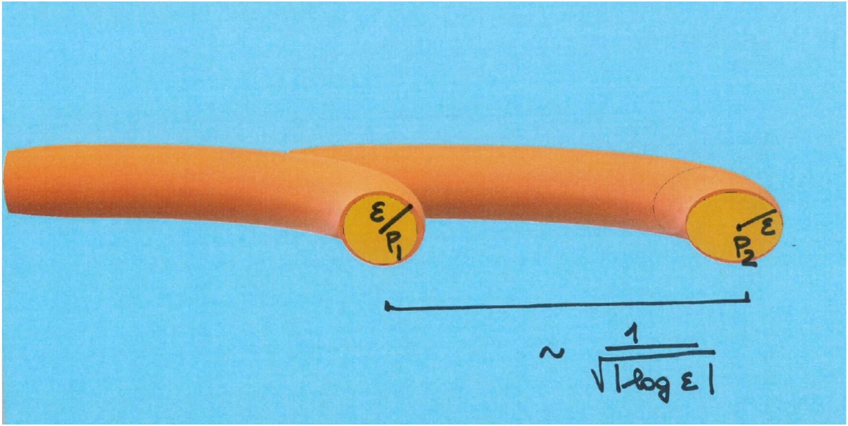
$$P_i = \left(r_0 + \frac{r(b_i(t))}{\sqrt{|\log \epsilon|}}, z_0 + \frac{2t}{r_0} + \frac{z(b_i(t))}{\sqrt{|\log \epsilon|}} \right) \quad \text{in} \quad (0, T)$$

Then there exists a solution ω_ε of 3D axisymmetric Euler flow (without swirl) of the form

$$w_\varepsilon(x,t) \sim \sum_{j=1}^{N} \frac{1}{r_j \varepsilon_j^2} U\left(\frac{(r,z) - P_j}{\varepsilon_j} \right)$$

$$\psi_\varepsilon(x,t) \sim \sum_{j=1}^{N} \frac{1}{r_j} \log \frac{1}{(\varepsilon_j^2 + |x - P_j|^2)^2} [1 - \frac{3}{2r_j}(r - r_j)]$$

Multiple-scalings



Ingredients in the construction

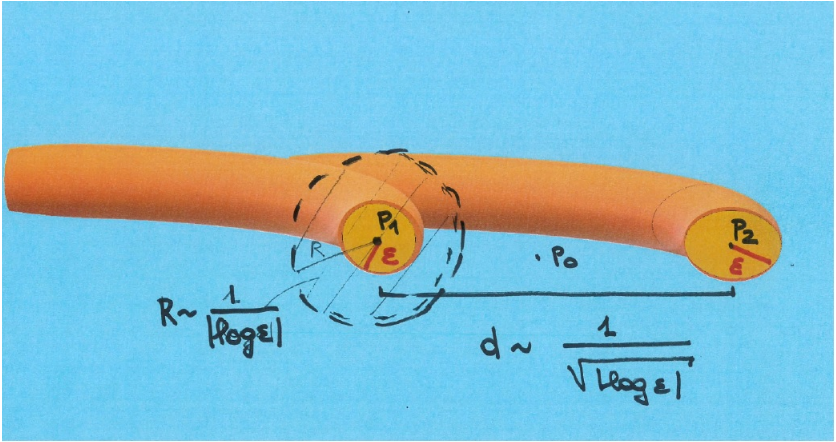
- Improvement of the approximation in powers of ε: $(w_\varepsilon^*, \psi_\varepsilon^*)$
- Setting up the problem as a coupled system of inner problems near the singularities and and an outer problem more regular (the inner-outer gluing scheme)
  Inner-outer gluing scheme:
  elliptic (del Pino-Kowalczyk-Wei (2011))—counterexample to De Giorgi's Conjecture
  parabolic (Davila-del Pino-Wei (2020))—singularities of Harmonic Map Flows
- New ingredients of gluing for Euler:
  - The inner problem is highly degenerate;
  - The outer problem is transport equation–lack of regularity.

Sketch of the proof. We want to solve the equation $S(\omega, \psi) = 0$, where

$$S(\omega, \psi) := |\log \varepsilon| r \partial_t \omega + \nabla^{\perp} \left[r^2 (\psi - \frac{\alpha}{2} |\log \epsilon|) \right] \cdot \nabla \omega = 0,$$
$$-\Delta_5 \psi = \omega.$$

Introduce cut-off functions

$$\eta_j(x, t) = \eta(|\log \varepsilon||x - P_j|), \quad \eta(s) = \begin{cases} 1 & s \leq 1 \\ 0 & s \geq 2 \end{cases}$$



P1
R
ε
P2
ε
· P0
R~ 1/|log ε|
d ~ 1/√|log ε|

The inner-outer gluing scheme

$$\psi(x,t) = \psi^*_\varepsilon + \sum_{j=1}^{2} \frac{\eta_j}{r_j} \psi_j\left(\frac{x-P_j}{\varepsilon_j}, t\right) + \psi^{out}(x,t)$$

$$\omega(x,t) = w^*_\varepsilon + \sum_{j=1}^{2} \frac{\eta_j}{r_j \varepsilon_j^2} \phi_j\left(\frac{x-P_j}{\varepsilon_j}, t\right) + \phi^{out}(x,t)$$

where $-\Delta_5 \psi = \omega$, $\phi_j = -\Delta_{5,j} \psi_j$

$$\Delta_{5,j}\psi_j := -[\Delta_y \psi_j + \frac{3\varepsilon_j}{r_j + \varepsilon_j y_1} \partial_1 \psi_j], \quad y = \frac{x - P_j}{\varepsilon_j}.$$

The problem becomes

$$S(\omega,\psi) = 0 \quad \text{if} \quad \begin{cases} E_j^{in}[\phi_j, \psi_j, \psi^{out}, P] = 0, \quad j = 1, 2, \\ E^{out}[\phi^{out}, \psi^{out}, \phi_j, \psi_j, P] = 0 \end{cases}$$

For $\tilde{\phi}(x,t) = \frac{1}{r_j \varepsilon_j^2} \phi(\frac{x-P_j}{\varepsilon_j}, t)$, $y = \frac{x-P_j}{\varepsilon_j}$

$$\varepsilon_j^4 |\log \varepsilon| r \partial_t \tilde{\phi} = |\log \varepsilon| (1 + \frac{\varepsilon_j}{r_j} y_1) \left[\varepsilon_j^2 \partial_t \phi - \underbrace{(\frac{\partial_t r_j}{r_j} + \frac{2\partial_t \varepsilon_j}{\varepsilon_j})}_{=0} \varepsilon_j^2 \phi \right]$$

$$- \varepsilon_j |\log \varepsilon| \nabla \phi \cdot \partial_t P_j - \varepsilon_j |\log \varepsilon| \underbrace{\left(\partial_t \varepsilon_j \nabla \phi \cdot y + \frac{\varepsilon_j}{r_j} y_1 \partial_t P_j \cdot \phi \right)}_{= \varepsilon_j \frac{\partial_t r_j}{2 r_j} (y_1 \partial_1 \phi - y_2 \partial_2 \phi)}$$

$$- \varepsilon_j^2 |\log \varepsilon| \partial_t \varepsilon_j r_j^{-1} y_1 \nabla \phi \cdot y$$

$$= \varepsilon_j^2 |\log \varepsilon| \partial_t \phi + \varepsilon_j |\log \varepsilon| \nabla \phi \cdot \partial_t P_j + \varepsilon_j^2 |\log \varepsilon| B_0(\phi)$$

If ϕ is a radial function, $B_0(\phi)$ has Fourier modes 2, or 1 with an extra ε_j.

A simplified version of E_j^{in}

$$E_j^{in}(y,t) := \varepsilon_j^2|\log\varepsilon|\partial_t\phi_j + \varepsilon_j|\log\varepsilon|\nabla\phi\cdot\partial_t P_j + \varepsilon_j^2|\log\varepsilon|B_0(\phi_j)$$
$$+ \nabla^\perp\left(\Gamma_0 + \frac{\varepsilon_j}{2r_j}y_1\Gamma_0\right)\cdot\nabla\phi_j + \nabla^\perp(\psi_j + \frac{2\varepsilon_j}{r_1}y_1\psi_j)\cdot\nabla U$$
$$+ \nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j + r_j\psi^{out})\right)\cdot\nabla\phi_j$$
$$+ \eta_j e^*_{final}, \qquad y\in B(0,\varepsilon^{-1}|\log\varepsilon|^{-1}),\ t\in[0,T)$$

where $y = \frac{x-P_j}{\varepsilon_j}$, and e^*_{final} is the final error

$$e^*_{final} = \varepsilon_j^4 S(w^*_\varepsilon,\psi^*_\varepsilon)(\varepsilon_j y + P_j)$$

A simplified version of E^{out}

$$
\begin{aligned}
E^{out}(x,t) := & \, |\log\varepsilon|\, r\, \phi_t^{out} + \nabla_x^{\perp}(r^2(\Psi^0 - \alpha|\log\varepsilon|)) \cdot \nabla_x \phi^{out} \\
& + \sum_{j=1}^{2} \left[r\,|\log\varepsilon|\,\partial_t \bar\eta_{j1} + \nabla_x^{\perp}(r^2(\Psi^0 - \alpha|\log\varepsilon|))\nabla\bar\eta_{1j} \right] \frac{\phi_j}{\varepsilon_j^2 r_j} \\
& + \left[\sum_{j=1}^{2} (\eta_{2j} - \eta_{1j}) \nabla_x^{\perp}(r^2(\frac{\psi_j}{r_j} + \psi^{out})) + \frac{r^2\psi_j}{r_j}\nabla_x^{\perp}\eta_{2j} \right] \cdot \nabla_x W^0 \\
& + (1 - \sum_{j=1}^{2} \eta_{j1}) S(w_\varepsilon^*, \psi_\varepsilon^*) = 0 \quad r > 0,\ z \in \mathbb{R},\ t \in [0, T)
\end{aligned}
$$

To decouple the inner and outer problems, we need the inner functions ϕ_j to decay as ρ becomes large

For the **inner problem** we solve in $\mathbb{R}^2$

$$\varepsilon^2 |\log \varepsilon| \phi_t - \nabla^\perp \Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$
$$-\Delta\psi = \phi \quad \text{in } \mathbb{R}^2 \times [0, T]$$

A central ingredient is an L^2-a priori estimate:
Lemma: A priori estimates If ϕ is a solution and satisfies certain orthogonality conditions, then the following estimate holds

$$\boxed{\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)} \le C\varepsilon^{-2}|\log \varepsilon| \sup_{t\in[0,T]} \|E(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}}$$

The **inner problem**

$$\varepsilon^2|\log\varepsilon|\phi_t - \nabla^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + B(\phi) + Q(\phi) + e^*_{final} = 0,$$
$$\phi(y,0) = 0, \quad -\Delta_5\psi = \phi \quad \text{in } \mathbb{R}^2 \times [0,T]$$

The whole construction works if we get to an approximation $(w^*_\varepsilon, \psi^*_\varepsilon)$ with a final error

$$e^*_{final} = \frac{\varepsilon^5|\log\varepsilon|^\beta}{1+\rho^3}, \quad \beta > 0$$

How do we improve the approximation? Recall

$$w_\varepsilon^0(x,t) = \sum_{j=1}^{2} \frac{1}{\varepsilon_j^2 r_j} U\Big(\frac{x-P_j}{\varepsilon_j}\Big), \quad P_j = (r_j, z_j)$$

$$\psi_\varepsilon^0(x,t) = \sum_{j=1}^{2} \frac{1}{r_j} \log \frac{1}{(\varepsilon_j^2 + |x-P_j|^2)^2} \left[1 - \frac{3}{2r_j}(r-r_j)\right]$$

The points

$$P_i = \left(r_0, \; z_0 + \frac{2t}{r_0}\right) + \frac{1}{\sqrt{|\log \epsilon|}} b_i(t) + a_j(t)$$

with $|a_j(t)|_{L^\infty(0,T)} \le \varepsilon^2$. So: $|P_1 - P_2| \sim \frac{1}{\sqrt{|\log \epsilon|}}$.

Improvement of the error near P_j

The inner equation:

$$\varepsilon_j^2 |\log \varepsilon| \partial_t \phi_j + \varepsilon_j |\log \varepsilon| \partial_t P_j \cdot \nabla \phi_j + \varepsilon_j^2 |\log \varepsilon| B_0(\phi_j)$$
$$+ \nabla^{\perp} \left(\Gamma_0 + \frac{\varepsilon_j}{2r_j} y_1 \Gamma_0 \right) \cdot \nabla \phi_j + \nabla^{\perp} (\psi_j + \frac{2\varepsilon_j}{r_1} y_1 \psi_j) \cdot \nabla U$$
$$+ \nabla^{\perp} \left((1 + \frac{\varepsilon_j}{r_j} y_1)^2 (\psi_j + r_j \psi^{out}) \right) \cdot \nabla \phi_j + e_0 \sim 0$$

where e_0 is the initial error

$$e_0 = \frac{\varepsilon^2 |\log \varepsilon|}{1 + \rho^4} E_2, \quad \rho = |y|, \quad y = \frac{x - P_j}{\varepsilon_j}.$$

We obtain the first reduction of the error solving the elliptic equation

$$\nabla^{\perp}\Gamma_0 \cdot \nabla\phi + \nabla^{\perp}\psi \cdot \nabla U + e_0 = 0, \quad -\Delta\psi = \phi.$$

Since $U = e^{\Gamma_0}$, the problem becomes

$$-\nabla_y^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + U\psi) + e_0 = 0$$

In polar coordinates $y = \rho e^{i\theta}$ we see that

$$\mathcal{L}[\psi] := \nabla_y^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + U\psi) = \frac{-4}{1+\rho^2}\frac{\partial}{\partial\theta}[\Delta_y\psi + U\psi]$$

The operator $\mathcal{L}$ (for Liouville) is highly degenerate:

- All radial functions are in its kernel
- Kernel of

$$\Delta_y \psi + U\psi = 0, \quad \psi \in < 2 + \nabla\Gamma_0 \cdot y, \partial_{y_1}\Gamma_0, \partial_{y_2}\Gamma_0 >$$

Since

$$e_0 = \frac{\varepsilon^2 |\log \varepsilon|}{1+\rho^4} E_2$$

we can solve $\mathcal{L}[\psi_1, \phi_1] + e_0 = 0$ and the new error

$$e_1 = \frac{\varepsilon^3 |\log \varepsilon|^2}{1+\rho^3} E_1$$

We obtain the second reduction of the error solving the elliptic equation

$$\mathcal{L}[\psi_2, \phi_2] + e_1 = 0, \quad e_1 = \frac{\varepsilon^3}{1+\rho^3} E_1$$

which can be done with an adjustment of the points $a_j(t)$. We get a new error

$$e_2 = \frac{\varepsilon^4}{1+\rho^2} E_0$$

We **cannot** use $\mathcal{L}$ to improve $\frac{\varepsilon^4}{1+\rho^2} E_0$. We solve the ODE

$$\varepsilon_j^2 |\log \varepsilon| \partial_t \phi_3 + e_2 = 0.$$

The solution $\phi_3 = \frac{\varepsilon^2 |\log \varepsilon|}{1+\rho^2}$ is a function of 0-Fourier mode. The new error

$$e_3 = \frac{\varepsilon^3}{1+\rho^3} \sin \theta$$

We obtain the further reduction of the error solving with

$$\mathcal{L}[\psi_4, \phi_4] + \frac{\varepsilon^3}{1+\rho^3} \sin\theta = 0$$

which can be done with a further adjustment of the points . This time we get a new error of the form

$$e_4 = \frac{\varepsilon^4}{1+\rho^2} E_2$$

We use again Liouville $\mathcal{L}[\psi_5, \phi_5] + \frac{\varepsilon^4}{1+\rho^2} = 0$ to get

$$e_5 = \frac{\varepsilon^5}{1+\rho} E_{i\geq 0}$$

How do we get decay?

We solve with the transport equation

$$\mathcal{T}(\phi_6) := |\log\varepsilon|\,\varepsilon_j^2\partial_t\phi_6 + |\log\varepsilon|\varepsilon_j^2 B_0(\phi_6) + \nabla^\perp\left(\Gamma + \frac{\varepsilon_j}{2r_j}\,y_1\,\Gamma\right)\cdot\nabla\phi_6 = e_5 = \frac{\varepsilon^5}{1+\rho}E_{i\geq 0}$$

Even if we have no control on the Fourier mode of ϕ_6, the new error

$$e_6 = \frac{\varepsilon^3}{1+\rho^5}E_1$$

This is like $e_1 = \frac{\varepsilon^3}{1+\rho^3}E_1$, but with faster decay.

We start the process again

.

Scheme for the inner approximation

$$e_0 = \frac{\varepsilon^2}{1+\rho^4} E_2 \underset{\mathcal{L}}{\Rightarrow} e_1 = \frac{\varepsilon^3}{1+\rho^3} E_1 \underset{\mathcal{L}\&a_j}{\Rightarrow} e_2 = \frac{\varepsilon^4}{1+\rho^2} E_0$$

$$\underset{ODE}{\Rightarrow} e_3 = \frac{\varepsilon^3}{1+\rho^3} \sin\theta \underset{\mathcal{L}\&a_j}{\Rightarrow} e_4 = \frac{\varepsilon^4}{1+\rho^2} E_2 \underset{\mathcal{L}}{\Rightarrow} e_5 = \frac{\varepsilon^5}{1+\rho} E_0$$

$$\underset{\mathcal{T}}{\Rightarrow} e_6 = \frac{\varepsilon^3}{1+\rho^5} E_1 \underset{\mathcal{L}\&a_j}{\Rightarrow} e_7 = \frac{\varepsilon^4}{1+\rho^4} E_0 \underset{ODE}{\Rightarrow} e_8 = \frac{\varepsilon^3}{1+\rho^5} \sin\theta$$

$$\underset{\mathcal{L}\&a_j}{\Rightarrow} e_9 = \frac{\varepsilon^4}{1+\rho^4} E_2 \underset{\mathcal{L}}{\Rightarrow} e_{10} = \frac{\varepsilon^5}{1+\rho^3} E_0$$

Lemma: A priori estimates: If ϕ solves in $\mathbb{R}^2 \times [0.T]$

$$\varepsilon^2 \phi_t + \nabla^\perp \Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$

and satisfies the orthogonality conditions

$$\int_{B(0,\delta\varepsilon^{-1})} y_i \phi(y,t) dy = 0, \quad i = 1, 2,$$

$$\int_{\mathbb{R}^2} \phi(y,t) dy = \int_{\mathbb{R}^2} \phi(y,t) \frac{1 - 2|y|^2}{1 + |y|^2} U(y) dy = 0,$$

then the following estimate holds

$$\|\phi(\cdot,t) U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon^{-2} |\log \varepsilon| \sup_{t \in [0,T]} \|E(\cdot,t) U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}$$

Proof.

$$\varepsilon^2\phi_t + \nabla^\perp\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$

We use the test function $g = \frac{\phi}{U} - (-\Delta)_{\mathbb{R}^2}\phi$, so

$$Ug = -(\Delta\psi + f'(\Gamma_0)\psi).$$

and

$$\varepsilon^2\partial_t\int_{\mathbb{R}^2}\phi g = \int_{\mathbb{R}^2} U^{-1}\nabla^\perp\Gamma_0\nabla(U^2g^2) + 2\int_{\mathbb{R}^2} Eg$$

The second integral is zero for

$$\int_{\mathbb{R}^2} U^{-1}\nabla^\perp\Gamma_0\nabla(U^2g^2) = -\int_{\mathbb{R}^2}\nabla\cdot(U^{-1}\nabla^\perp\Gamma_0)\nabla(U^2g^2)$$

and since Γ_0 and U are radial,

$$\nabla\cdot(U^{-1}\nabla^\perp\Gamma_0) = 0.$$

Thus

$$\varepsilon^2 \partial_t \int_{\mathbb{R}^2} \phi g = 2 \int_{\mathbb{R}^2} Eg \le C \| E U^{-\frac{1}{2}} \|_{L_2} \| \| g U^{\frac{1}{2}} \|_{L^2}$$

and integrating,

$$\varepsilon^2 \int_{\mathbb{R}^2} \phi g(\cdot, t) \le \max_{t \in (0,T)} C \| E(\cdot, t) U^{-\frac{1}{2}} \|_{L_2} \| \| g(\cdot, t) U^{\frac{1}{2}} \|_{L^2}.$$

Under the orthogonality conditions assumed on ϕ we can prove the following Poincare inequality:

$$(*) \qquad \frac{\gamma}{|\log \varepsilon|} \int_{\mathbb{R}^2} \phi^2 U^{-1} \le \int_{\mathbb{R}^2} \phi g$$

while we always have

$$\int_{\mathbb{R}^2} g^2 U \le C \int_{\mathbb{R}^2} \phi^2 U^{-1}.$$

From these inequalities the desired estimate follows.

To prove the Poincare inequality

$$\frac{\gamma}{|\log \varepsilon|} \int_{\mathbb{R}^2} \phi^2 U^{-1} \leq \int_{\mathbb{R}^2} \phi g$$

we set $\tilde{\phi} = U^{-1}\phi$. Using stereographic projection we see that

$$\int_{S^2} \tilde{\phi}^2 = \int_{\mathbb{R}^2} \phi^2 U^{-1}, \quad \int_{S^2} \tilde{\phi} = \int_{\mathbb{R}^2} \phi = 0.$$

Besides

$$\int_{\mathbb{R}^2} \phi g = \int_{S^2} \tilde{\phi}(\tilde{\phi} - 2(-\Delta_{S^2})^{-1}\tilde{\phi}).$$

Expanding $\tilde{\phi}$ in the orthonormal basis in $L^2(S^2)$ of spherical harmonics we get

$$\tilde{\phi} = \sum_{j=0}^{\infty} \tilde{\phi}_j e_j(z) = \sum_{j=0}^{3} \tilde{\phi}_j e_j + \tilde{\phi}^{\perp},$$

where $-\Delta_{S^2} e_j = \lambda_j e_j$.

Here $\lambda_0 = 0$ and e_0 is constant, while $\lambda_1 = \lambda_2 = \lambda_3 = 2$, with $e_j(z) = z_j$. Thus $\tilde{\phi}_0 = 0$ and also $\tilde{\phi}_3 = 0$ because of our orthogonality condition:

$$\int_{\mathbb{R}^2} \phi(y,t)dy = \int_{\mathbb{R}^2} \phi(y,t) \frac{1 - 2|y|^2}{1 + |y|^2} U(y)dy = 0,$$

$$\int_{\mathbb{R}^2} \phi g = \sum_{j=4}^{\infty} \left(1 - \frac{2}{\lambda_j}\right) \tilde{\phi}_j^2 \sim \|\tilde{\phi}^\perp\|^2_{L^2(S^2)}$$

We also have, $j = 2, 3$

$$0 = \int_{B_R} \phi y_j = c\tilde{\phi}_j + O(\|\tilde{\phi}^\perp\|_{L^2(S^2)}) |\log R|^{\frac{1}{2}}$$

with $R = \delta\varepsilon^{-1}$ which gives

$$\tilde{\phi}_j = O(\|\tilde{\phi}^\perp\|_{L^2(S^2)}) |\log \varepsilon|^{\frac{1}{2}}.$$

From here it follows that

$$\int_{\mathbb{R}^2} \phi g \geq \gamma |\log \varepsilon|^{-1} \int_{S^2} \tilde{\phi}^2$$

as we wanted.

Remarks

1. Klein-Majda-Damodaran (1995) formally derived the LeapFrogging dynamics.

2. Jerrard-Smets (2018): gave the first mathematical justification of leapfrogging in three-dimensional Gross-Pitaeskii equation

$$iu_t - \Delta u = \frac{1}{\varepsilon^2}(1 - |u|^2)u \quad \text{in} \quad \mathbb{R}^3$$

$u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^2$

3. The gluing approach we developed will be useful for the Vortex Filament Conjecture.

Related results: Gallay-Smets (2018) (spectrum analysis for the vortex line filament)

Approximate Binormal Conjecture

Davila-del Pino-Musso-Wei 2022: For each fixed integer N, there exists a solution to 3D Euler flow

$$\omega^\epsilon = F_1 \gamma_s + F_2 n_1 + F_3 n_2 + \text{smooth}$$

such that 3D Euler flow can be solved up to $O(\epsilon^N)$.

Main Difficulty: Lack of L^2- estimates for the spectral problem of inviscid columnar vortices. Gallay-Smets 2018: showed (numerically) all eigenvalues are on the imaginary axis. This is the first step. But the L^{-1} bound depends e^t.

Thanks for your attention