ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PLANAR ELLIPTIC SYSTEMS WITH STRONG COMPETITION

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ABSTRACT. We study a class of planar nonlinear elliptic systems with competition which includes the Hartree-Fock type approximation for a system of Bose-Einstein condensates in multiple hyperfine states as derived by Esry et al. [14]. We study the limit behaviour of solutions in the case where the repulsive interaction tends to infinity and phase separation is expected. In particular, we prove the continuity of the limit shape and derive limit equations satisfied within its nodal sets. By this we complement recent work of Chang et al. [8] where additional assumptions had to be made.

1. INTRODUCTION

In this paper we are concerned with the following class of parameter-dependent systems of elliptic equations with k components:

$$\begin{cases} -\Delta u_i = f_i(u_i)u_i - \sum_{\substack{j=1\\ j\neq i}}^{k} \alpha_{ij} f_{ij}(u_j) u_i & \text{in } \Omega, \\ u_1, \dots, u_k > 0 & \text{in } \Omega, \\ u_1 = \cdots = u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $f_i, f_{ij} : [0,\infty) \to \mathbb{R}$ are continuous functions and $\alpha_{ij} > 0$ are parameters for $i, j = 1, \dots, k$, $j \neq i$. We make the following assumptions on the coupling functions:

(A1) $f_{ij}(t) > 0$ for $t > 0$ and $i \neq j$.

(A2) There exists $\tau > 0$ such that $\lim\limits_{t\to 0} \frac{f_{ij}(t)}{t^\tau} = 0$ for $i \neq j$.

We point out two special cases of system (1.1). The case $f_{ij}(t) = t$ for $i \neq j$ corresponds to a Lotka-Volterra type system modelling the interaction between biological species in population ecology. In particular, this case has been considered by Dancer and Du [11], Conti, Terracini and Verzini [9] and Caffarelli and Lin [6]. Another case where the right hand side $f_i(u_i)u_i - \sum_{\substack{j=1\\ j\neq i}}^{k} \alpha_{ij} f_{ij}(u_j) u_i$ is replaced by $A(x)\Pi_{i=1}^k u_i^{a_i}$

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arises in combustion theory and has been considered recently by Caffarelli and Roquejoffre in [7]. Our interest in this problem arose from the following elliptic system:

$$
\begin{cases} -\Delta u_i + \lambda_i u_i = \sum_{j=1}^{n} \beta_{ij} u_j^2 u_i & \text{in } \Omega, \\ u_1, \dots, u_k > 0 & \text{in } \Omega, \\ u_1 = \dots = u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}
$$

Here $\beta_{ij} < 0$ for $i \neq j$ and $\beta_{ii}, \lambda_i \in \mathbb{R}$, $i = 1, \dots, k$. This is a special case of (1.1) with $f_{ij}(t) = t^2$, $\alpha_{ij} = -\beta_{ij}$ for $i \neq j$ and $f_i(t) = \beta_{ii} t^2 - \lambda_i t$ for $i = 1, \dots, k$. System (1.2) arises in the Hartree-Fock-theory of a mixture of Bose-Einstein condensates in multiple hyperfine states where the interaction between the different states is repulsive, see [14]. We note that (1.2) has a variational structure if $\beta_{ij} = \beta_{ji}$ for all i, j; solutions of (1.2) can be found as critical points of the energy functional $\Phi : H^1(\Omega, \mathbb{R}^k) \to \mathbb{R}$ given by

$$
\Phi(u) = \sum_{i=1}^{k} \int_\Omega \Big[\frac{1}{2}(|\nabla u_i|^2 + \lambda_i u_i^2 - \frac{1}{4}\beta_{ii} u_i^4\Big] dx - \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{k} \beta_{ij} \int_\Omega u_i^2 u_j^2 \, dx.
$$

Existence and multiplicity of critical points of Φ has been obtained under different assumptions on the parameters λ_i and β_{ij}, see e.g. [3–5, 8, 12, 19, 20, 23, 25, 26]. The variational principles yielding these critical points imply uniform energy bounds independent of the coupling coefficients β_{ij}, $i \neq j$. It is natural to try to understand the asymptotic profile of these solutions in the "strong repulsion limit" $\beta_{ij} \to -\infty$, which corresponds to $\alpha_{ij} \to \infty$ in (1.1). It is easy to see that uniform Φ-bounds for a sequence of solutions of (1.2) – corresponding to bounded diagonal parameters α_i, β_{ii} and unbounded β_{ij} – yield uniform $H^1(\Omega)$-bounds, and these in turn yield uniform $L^\infty(\Omega)$-bounds by standard elliptic regularity. It is expected that components with bounded L^∞-norm tend to separate in different regions of the underlying domain Ω, a phenomenon physicists describe as "*phase separation*" in the context of (1.2), see e.g. [8, 16, 17, 24]. However, from a rigorous mathematical point of view, the nature of this limit and the spatial separation is not well understood so far. The following is our main result concerning (1.1).

Theorem 1.1. *Let $(A_1), (A2)$ be satisfied and let $\alpha_{ij}^n > 0$, $n \in \mathbb{N}$, $i \neq j$ be such that $\alpha_{ij}^n \to \infty$ as $n \to \infty$ and*

$$
\max_{i \neq j} \alpha_{ij}^n \le C \min_{i \neq j} \alpha_{ij}^n \qquad \text{for some } C > 0 \text{ and all } n.
$$

Moreover, for every n let $u_n = (u_{1,n}, \dots, u_{k,n}) \in C^2(\overline{\Omega}, \mathbb{R}^k)$ be a solution of (1.1) corresponding to $\alpha_{ij} = \alpha_{ij}^n$ such that the sequence $(u_n)_n$ is bounded in $L^\infty(\Omega, \mathbb{R}^k)$. Then:

(a) The sequence $(u_n)_n$ is uniformly equicontinuous. Hence there exists $u = (u_1, \dots, u_k) \in C(\overline{\Omega}, \mathbb{R}^k)$ such that, for a subsequence,

$$
u_{1,i} \to u_i \qquad \text{uniformly on } \Omega \text{ for } i = 1, \dots, k. \tag{1.3}
$$

(b) If $u = (u_1, \dots, u_k) \in C(\overline{\Omega}, \mathbb{R}^k)$ *satisfies* (1.3) *and*

$$N_i := \{x \in \Omega \,:\, u_i > 0\} \qquad \text{for } i = 1, \dots, k,$$

then the sets N_i *are open and disjoint. Moreover, if* f_i *is Hölder continuous, then* $u_i|_{N_i} \in C^2(N_i)$ *is a classical solution of the equation*

$$-\Delta u_i = f_i(u_i)u_i \qquad \text{in } N_i. \tag{1.4}$$

Remark 1.1. (i) We will prove properties (a) and (b) for a more general sequence of vector-valued functions $u_n \in L^\infty(\Omega, \mathbb{R}^k)$ satisfying a nonlinear system of differential inequalities, see Theorem 3.1 below. Moreover, our proof carries over to the case of x-dependent functions $f_i = f_i(x, u)$ which are continuous on $\Omega \times [0, \infty)$. In some cases, one is also led to study functions $f_i = f_i^n$ depending on n, see e.g. [8]. Then one has to assume that $f_i^n \circ u_{i,n}$ is bounded in $L^\infty(\Omega)$ independently on n, so that a subsequence of these functions has a weak*-limit in $L^\infty(\Omega)$ which then appears in the right hand side of (1.4) in place of $f_i(u_i)$. We omit these straightforward extensions to keep the presentation short.

(ii) Any uniform $L^\infty(\Omega)$-bound for solutions of (1.1) yields a uniform $H_0^1(\Omega)$-bound, since

$$\int_\Omega |\nabla u_i|^2 \le \int_\Omega f_i(u_i)u_i^2 \, dx$$

for every i. On the other hand, if we assume in addition that $f_i(t) \le C_i(1 + t^\lambda)$ for some $\lambda > 0$, then, by Sobolev embeddings and classical subsolution estimates, a uniform $H_0^1(\Omega)$-bound also yields a uniform $L^\infty(\Omega)$-bound.

(iii) In the special case of system (1.2), Theorem 1.1 improves [8, Theorem 1.1] of Chang et al.. In [8], the authors consider a sequence of solutions of (1.2) satisfying the assumptions of Theorem 1.1, but they could only prove *weak* convergence in $H_0^1(\Omega, \mathbb{R}^k)$, and the limit equations (1.4) were only derived under the *assumption* that the sets N_i are *open*.

(iv) In the case $f_{ij}(t) = t$, Conti-Terracini-Verzini [9] proved uniform Hölder bounds for solutions of (1.1). Their method works in arbitrary dimension but relies crucially on the specific form of the coupling. In fact, via a blow up argument rescaling Hölder quotients, they are led to study vector-valued functions $(u_1, \dots, u_k)$ defined on $\mathbb{R}^N$ with the following property:

$$u_i \text{ is subharmonic and } u_i - \sum_{j \ne i} \frac{\alpha_{ij}}{\alpha_{ji}} u_j \text{ is superharmonic for } i = 1, \dots, n \tag{1.5}$$

(as was pointed out to us [10], this property is assumed in [9, Proposition 7.2]). Then they conclude via interesting new monotonicity theorems. However, due to the fact that the coupling terms in (1.2) are non-symmetric (even when the β_{ij} are symmetric), it is unclear whether (1.5) extends to limiting functions arising from (1.2). Moreover, any rescaling of Hölder or uniform gradient norms seems unsuitable for the general system (1.1) when no homogeneity is assumed.

The proof of Theorem 1.1 is based on a rescaling of the form

$$u_n \mapsto v_n := u_n(x_n + A_n r_n(\cdot))$$

with suitably chosen $x_n \in \Omega$, $A_n \in O(2)$ and $r_n \to 0$ as $n \to \infty$. Extending v_n trivially to all of $\mathbb{R}^2$, we may pass to a subsequence such that the weak*-limit v of v_n in $L^\infty(\mathbb{R}^2)$ exists and is a subharmonic function. Liouville's theorem (see Section 2) implies that v is almost everywhere constant. By a careful analysis of the values taken by v_n on circles, we then come to a contradiction. In particular, here we use properties of the spherical cap-symmetrization of v_n. The assumption that $N = 2$, i.e. the domain is planar, enters at three points. First, when we use Liouville's theorem. Second, when we look for suitable exponents in Morrey's lemma to get local oscillation estimates, see the proof of Lemma 3.2 below. And last, we use the fact that we have a well defined trace of H^1-function on line segments, which also requires $N = 2$. Whether Theorem 1.1 carries over to the case $N \geq 3$ is an interesting question which is open even in the special case of (1.2).

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2. Preliminaries

Here we recall some facts on subharmonic functions and the (spherical) cap-symmetrization. A function $u \in L^1_{loc}(\mathbb{R}^2)$ is called (weakly) subharmonic if, for every $R > 0$, one of the following equivalent properties are satisfied.

(S1) For almost every $x \in \mathbb{R}^2$,

$$u(x) \leq \frac{1}{|B_R|} \int_{B_R(x)} u(y)\, dy \qquad \text{for every } R > 0.$$

(S2) For every nonnegative $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} u \Delta \varphi \geq 0.$$

For the equivalence of these properties, see e.g. [18, Theorem 9.3]. As stated in [18, Theorem 9.3], every function $u \in L^1_{loc}(\mathbb{R}^2)$ satisfying $(S1)$ or $(S2)$ has an *upper semicontinuous* representative $\tilde{u} : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ such that $\tilde{u}(x) = u(x)$ for a.e. $x \in \mathbb{R}^2$ and (S1) holds for every $x \in \mathbb{R}^2$ with $\tilde{u}(x) < \infty$. In the standard literature on potential theory (see e.g. [13]), these properties are part of the definition of a subharmonic function. We recall the following classical result, see e.g. [13].

Theorem 2.1. *(weak version of Liouville's Theorem)*
If $u \in L^\infty(\mathbb{R}^2)$ is subharmonic, then there exists $c \in \mathbb{R}$ such that $u \equiv c$ almost everywhere on $\mathbb{R}^2$.

Next we recall facts about the cap-symmetrization. Let $e \subset \mathbb{R}^2$ denote a fixed unit vector, and let $S(r) := \{x \in \mathbb{R}^2 \ : \ |x| = r\}$ for $r > 0$. For a Borel set $A \subset S(r)$ we define the cap-symmetrization A^* of A as the closed geodesic ball in $S(r)$ centered at

re and having the same surface measure as A. For a function $u \in C(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$, the cap-symmetrization $u^* : \mathbb{R}^2 \to \mathbb{R}$ of u is the function defined by the relations

$$\{x \in S(r) \,:\, u^*(x) > d\} = \{x \in S(r) \,:\, u(x) > d\}^* \qquad \text{for } r > 0,\ d \in \mathbb{R}.$$

It is well known that $u^* \in C(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$, and that

$$\|u^*\|_{L^p(B_R(0))} = \|u\|_{L^p(B_R(0))} \qquad \text{for } R > 0,\ p \geq 1, \text{ and} \tag{2.1}$$

$$\|u^*\|_{H^1(B_R(0))} \leq \|u\|_{H^1(B_R(0))} \qquad \text{for } R > 0. \tag{2.2}$$

We note that u^* is axially symmetric with respect to the axis $\mathbb{R}e$, and it is decreasing in the polar angle $\theta = \arccos\left(\frac{x}{|x|} \cdot e\right)$ from this axis. This in particular implies that, for every $r > 0$,

$$u^*(re) = \max_{x \in S(r)} u(x) \qquad \text{and} \qquad u^*(-re) = \min_{x \in S(r)} u(x). \tag{2.3}$$

It is easy to see that the map $u \mapsto u^*$ is continuous with respect to local L^p-norms. More precisely, if $u, v \in C(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$, then

$$\|u^* - v^*\|_{L^p(B_R(0))} \leq \|u - v\|_{L^p(B_R(0))} \text{ for } R > 0,\ p \geq 1. \tag{2.4}$$

The map $u \mapsto u^*$ is not continuous with respect to local H^1-norms, see e.g. [2]. However, it is weak-to-weak continuous in the following sense.

Lemma 2.1. *If $u, u_n \in C(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$, $n \in \mathbb{N}$ are functions such that $u_n \rightharpoonup u$ weakly in $H^1(B_R(0))$ for some $R > 0$, then also $u_n^* \rightharpoonup u^*$ weakly in $H^1(B_R(0))$.*

Proof. By assumption and compactness of the embedding $H^1(B_R(0)) \hookrightarrow L^1(B_R(0))$, we infer that

$$u_n \to u \qquad \text{strongly in } L^1(B_R(0)),$$

so that

$$u_n^* \to u^* \qquad \text{strongly in } L^1(B_R(0)) \tag{2.5}$$

by (2.4). Suppose by contradiction that there exists $\varphi \in H^1(B_R(0))$ such that, after passing to a subsequence,

$$\liminf_{n \to \infty} \langle u_n^* - u^*, \varphi \rangle > 0, \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $H^1(B_R(0))$. By (2.2), the sequence $(u_n^*)_n$ is bounded, so we may pass again to a subsequence such that $u_n^* \rightharpoonup w$ in $H^1(B_R(0))$. By compactness, $u_n^* \to w$ strongly in $L^1(B_R(0))$ and therefore $w = u^*$ by (2.5). This however contradicts (2.6). $\square$

3. Proof of the main theorem

As announced in the introduction, we prove a more general version of Theorem 1.1. Consider the following nonlinear system of differential inequalities for $u = (u_1, \dots, u_k)$:

$$\begin{cases} -\alpha\, g_i(u^i)u_i \le -\,\Delta u_i - f_i(u_i)u_i \le -\beta\, g_i(u^i)u_i & \text{in } \Omega, \\ u_1, \dots, u_k > 0 & \text{in } \Omega, \\ u_1 = \dots = u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $f_1, \dots, f_k : \mathbb{R}_+ \to \mathbb{R}$ are continuous (where, as usual, $\mathbb{R}_+ = [0, \infty)$) and $\alpha > \beta > 0$ are parameters. Moreover, $u^i = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_k) \in \mathbb{R}_+^{k-1}$, and for the functions $g_i \in C(\mathbb{R}_+^{k-1}, \mathbb{R})$ we make the following assumptions:

(B1) $g_i(t_1, \dots, t_{k-1}) > 0$ if $\max\{t_1, \dots, t_{k-1}\} > 0$.
(B2) There exists $\tau > 0$ such that, for all $i = 1, \dots, k$,

$$\frac{g_i(t_1, \dots, t_{k-1})}{(\max\{t_1, \dots, t_{k-1}\})^\tau} \to 0 \qquad \text{as} \quad \max\{t_1, \dots, t_{k-1}\} \to 0.$$

Note that every solution $u = (u_1, \dots, u_k)$ of (1.1) satisfies (3.1) with

$$\alpha = \max_{i \ne j} \alpha_{ij}, \qquad \beta = \min_{i \ne j} \alpha_{ij} > 0$$

and g_i given by

$$g_i(u^i) = \sum_{j \ne i} f_{ij}(u_j).$$

Now Theorem 1.1 is a direct consequence of the following result.

Theorem 3.1. *Let $(B1), (B2)$ be satisfied and let $\alpha_n > \beta_n > 0$, $n \in \mathbb{N}$ be such that $\alpha_n, \beta_n \to \infty$ as $n \to \infty$ and*

$$\alpha_n \le C\beta_n \qquad \textit{for some } C > 0 \textit{ and all } n.$$

Moreover, for every n let $u_n = (u_{1,n}, \dots, u_{k,n}) \in C^2(\overline{\Omega}, \mathbb{R}^k)$ be a solution of (3.1) corresponding to $\alpha = \alpha_n$, $\beta = \beta_n$ such that the sequence $(u_n)_n$ is bounded in $L^\infty(\Omega, \mathbb{R}^k)$. Then (a) and (b) of Theorem 1.1 hold.

The remainder of this section is devoted to the proof of this theorem. So consider a sequence $u_n = (u_{1,n}, \dots, u_{k,n}) \in C^2(\overline{\Omega}, \mathbb{R}^k)$ satisfying the assumptions. We put

$$U_0 = \sup_{\substack{i=1,\dots,k \\ n \in \mathbb{N}}} \|u_{i,n}\|_{L^\infty(\Omega)}, \quad F_0 = \sup_{\substack{i=1,\dots,k \\ n \in \mathbb{N}}} \|f_i(u_{i,n})\|_{L^\infty(\Omega)} \quad \text{and } G_0 = \sup_{\substack{i=1,\dots,k \\ n \in \mathbb{N}}} \|g_i(u^{i,n})\|_{L^\infty(\Omega)}.$$

Multiplying the equations in (3.1) for u_n with $u_{i,n}$ and integrating over Ω, we infer that

$$\int_\Omega |\nabla u_{i,n}|^2\, dx \le \int_\Omega f_i(u_{i,n}) u_{i,n}^2\, dx \le |\Omega| F_0 U_0^2 \qquad \text{for all } i, n. \tag{3.2}$$

It is convenient to extend u_n by zero on $\mathbb{R}^2\backslash\Omega$. Then $u_n \in L^\infty(\mathbb{R}^2,\mathbb{R}^k)\cap W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^k)$, and in distributional sense it satisfies the differential inequalities

$$-\Delta u_{i,n} \le f_i(u_{i,n})u_{i,n} - \beta_n g_i(u_n^i)u_{i,n} \qquad \text{in } \mathbb{R}^2 \text{ for } i = 1,\dots,k. \tag{3.3}$$

We first provide some crucial estimates. Here we use some ideas from Chang-Lin-Lin-Lin [8] and Conti-Terracini-Verzini [9].

Lemma 3.1. *There is $C_0 > 0$ such that $\|\nabla u_{i,n}\|_{L^\infty(\mathbb{R}^2)} \le C_0\sqrt{\beta_n}$ for $i = 1,\dots,k$.*

This is a generalization of [8, Lemma 2.2]. In [8] the estimate was proved for solutions of (1.2) but only for points in Ω with a fixed lower bound on the distance to $\partial\Omega$.

Lemma 3.2. *If $(x_n)_n \subset \Omega$ is a sequence such that $\varepsilon := \inf_{n\in\mathbb{N}} u_{i,n}(x_n) > 0$ for some i, then $u_{j,n}(x_n) = O(\beta_n^{-\eta})$ for $j \ne i$ and every $\eta > 0$.*

This lemma can be seen as an improvement of [8, Prop. 2.1 and 2.2] since we consider the general system (3.1) and do not assume a lower bound on $\mathrm{dist}(x_n, \partial\Omega)$.

Proof of Lemma 3.1. If the statement was false, we may pass to a subsequence such that, for some $i \in \{1,\dots,k\}$, there exists points $x_n \in \Omega$, $n \in \mathbb{N}$ such that

$$a_n := |\nabla u_{i,n}(x_n)| \ge n^2\sqrt{\beta_n}. \tag{3.4}$$

We consider $b_n := \frac{a_n}{n}$, the rescaled domains

$$\Omega_n := \{x \in \mathbb{R}^2 \ : \ x_n + \frac{x}{b_n} \in \Omega\}$$

and the rescaled functions

$$w_n : \mathbb{R}^2 \to \mathbb{R}, \qquad w_n(x) = u_{i,n}(x_n + \frac{x}{b_n})$$

for $i = 1,\dots,k$, $n \in \mathbb{N}$. Then $0 \in \Omega_n$, $\|w_n\|_{L^\infty(\mathbb{R}^2)} \le U_0$ and

$$|\nabla w_n(0)| = n. \tag{3.5}$$

Moreover, w_n is a solution of the rescaled problem

$$\begin{cases} -\frac{\alpha_n}{b_n^2}\, G_0 w_n \le -\Delta w_n + \frac{f_i(w_n)}{b_n^2} \le -\frac{\beta_n}{b_n^2}\, G_0 w_n & \text{in } \Omega_n, \\ w_n > 0 \quad \text{in } \Omega_n, \qquad w_n = 0 \quad \text{on } \mathbb{R}^2 \setminus \Omega_n. \end{cases} \tag{3.6}$$

Since $\beta_n \le \alpha_n \le C\beta_n$ and $b_n \ge n\sqrt{\beta_n}$ for all n, (3.6) implies that

$$-\Delta w_n = o(1) \quad \text{in } \Omega_n \quad \text{for } i = 1,\dots,k. \tag{3.7}$$

where $o(1) \to 0$ in the L^∞-norm. For a subsequence, we may now distinguish the following two cases.

Case 1: $B_r(0) \subset \Omega_n$ for some $r > 0$ and all $n \in \mathbb{N}$. In this case, standard elliptic regularity using equation (3.7) in $B_r(0)$ implies that $|\nabla w_n(0)|$ is uniformly bounded. This contradicts (3.5).

Case 2: $r_n := \text{dist}(0, \partial\Omega_n) \to 0$. Let $r > 0$ be fixed. Since $\partial\Omega$ is smooth, there are C^2-diffeomorphisms $\psi_n : B_r(0) \to \psi_n(B_r(0))$ for n large which straighten the boundary portions $\partial\Omega_n \cap B_r(0)$. More precisely, the maps ψ_n can be chosen such that $\psi_n(0) = (0, r_n)$, $\psi_n(\partial\Omega_n \cap B_r(0)) \subset \mathbb{R} \times \{0\}$ and that ψ_n converges to the inclusion $B_r(0) \hookrightarrow \mathbb{R}^2$ as $n \to \infty$ with respect to the $C^2(B_r(0))$-norm. It it then easy to see that there exists $s > 0$ such that

$$B_s^+ := \{x \in \mathbb{R}^2 \ : x_2 > 0, \ |x| \le s\} \subset \psi_n(B_r(0) \cap \Omega_n)$$

and

$$H_s := \{x \in \mathbb{R}^2 \ : x_2 = 0, \ |x_1| \le s\} \subset \psi_n(B_r(0) \cap \partial\Omega_n)$$

for n large enough. Now the function $z_n : B_s^+ \to \mathbb{R}$ defined by $z_n(x) = w_n(\psi_n^{-1} x)$ satisfies

$$L_n z_n = o(1) \quad \text{in } B_s^+, \qquad z_n = 0 \quad \text{on } H_s,$$

where L_n is a second order differential operator whose coefficients are uniformly bounded as $n \to \infty$ (for details, see e.g. [15, Proof of Lemma 6.5]). Since also $\|z_n\|_{L^\infty(B_R(0))}$ is uniformly bounded, elliptic estimates near flat boundary portions yield that $|\nabla z_n(0, r_n)|$ remains bounded and therefore $|\nabla w_n(0)|$ remains bounded as $n \to \infty$. Again this contradicts (3.5). The proof is finished □

Proof of Lemma 3.2. Without loss, we may assume that $i = 1$. By assumption (B1),

$$g_* := \inf_{j=2,\dots,n} \ \inf_{\frac{\varepsilon}{2} \le \max\{t_1,\dots,t_{k-1}\} \le U_0} g_j(t_1, \dots, t_{k-1}) > 0. \tag{3.8}$$

Let $\eta > 0$ be given and fix

$$\eta_1 > \max\{\frac{8\eta}{\sqrt{g_*}}, 1\} \qquad \text{and} \qquad 0 < \rho < \min\{\frac{1}{2}, \frac{1}{2\exp(U_0^2)}\}. \tag{3.9}$$

For every n, we consider the function

$$h_n : (0, \infty) \to \mathbb{R}, \qquad h_n(r) = \frac{1}{2\pi r} \int_{\partial B_r(x_n)} u_{1,n}^2 \, ds,$$

and we put

$$s_n := \eta_1 \beta_n^{-1/2} \log \beta_n, \qquad t_n := \beta_n^{-\rho}.$$

By definition of h_n and U_0,

$$0 \le h_n(r) \le U_0^2 \qquad \text{for every } n \in \mathbb{N}, \ r > 0. \tag{3.10}$$

For n large we have $s_n < t_n$, and we claim the following:

$$\text{there exists } \xi_n \in (s_n, t_n) \text{ such that } h_n'(\xi_n) \le -\frac{1}{\xi_n \log \xi_n}. \tag{3.11}$$

We prove (3.11) by contradiction. If $h_n'(r) > -\frac{1}{r\log r}$ for every $r \in (s_n, t_n)$, then

$$U_0^2 \geq h_n(t_n) - h_n(s_n) > \int_{s_n}^{t_n} (-\frac{1}{\tau \log \tau})\, d\tau = \log(\frac{\log s_n}{\log t_n})$$
$$= \log(\frac{\log(\eta_1 \log \beta_n) - \frac{1}{2}\log \beta_n}{-\rho \log \beta_n}) \;\to\; \log\frac{1}{2\rho} \qquad \text{as } n \to \infty.$$

This contradicts the choice of ρ, see (3.9). Hence (3.11) holds for large n, and we conclude that

$$\int_{\partial B_{\xi_n}(x_n)} u_{1,n} \frac{\partial u_{1,n}}{\partial \nu}\, ds = \pi \xi_n h_n'(\xi_n) \leq -\frac{C_1}{\log \xi_n}. \tag{3.12}$$

Here and in the following, $C_1, C_2, \dots$ denote positive constants. Combining (3.12) with (3.3), we obtain the estimate

$$\int_{B_{s_n}(x_n)} |\nabla u_{1,n}|^2\, dx \leq \int_{B_{\xi_n}(x_n)} |\nabla u_{1,n}|^2\, dx \leq \int_{\partial B_{\xi_n}(0)} u_{1,n} \frac{\partial u_{1,n}}{\partial \nu}\, ds + U_0 F_0 \pi \xi_n^2$$
$$\leq -\frac{C_1}{\log \xi_n} + U_0 F_0 \pi \xi_n^2 \leq -\frac{C_2}{\log \xi_n} \leq \frac{C_3}{\log \beta_n}.$$

We fix $2 < p < 3$ and put $\gamma = 1 - \frac{2}{p}$. Then Morrey's Lemma (see e.g. [15, Theorem 7.17]) and Lemma 3.1 imply

$$\text{osc}_{B_{s_n}(x_n)} u_{1,n} \leq C_4 s_n^\gamma \left(\int_{B_{s_n}(x_n)} |\nabla u_{1,n}|^p\, dx \right)^{1/p} \leq C_5 s_n^\gamma (\sqrt{\beta_n})^{\frac{p-2}{p}} \left(\int_{B_{s_n}(x_n)} |\nabla u_{1,n}|^2\, dx \right)^{1/p}$$
$$\leq C_5 s_n^\gamma \beta_n^{\frac{1}{2} - \frac{1}{p}} \left(\frac{C_3}{\log \beta_n} \right)^{1/p} \leq C_6 \eta_1^\gamma (\log \beta_n)^{\gamma - \frac{1}{p}}$$
$$= C_7 (\log \beta_n)^{1 - \frac{3}{p}} \to 0 \qquad \text{as } n \to \infty.$$

Hence $u_{1,n} \geq \frac{\varepsilon}{2}$ in $B_{s_n}(x_n)$ for n large. By (3.3) and the definition of g_* we conclude that, for $j = 2, \dots, k$ and large n,

$$-\Delta u_{j,n} \leq [F_0 - \beta_n g_j(u_n^j)] u_{j,n} \leq [F_0 - \beta_n g_*] u_{j,n} \leq -\frac{\beta_n g_*}{2} u_{j,n} \qquad \text{in } B_{s_n}(x_n),$$

while $u_{j,n} \leq U_0$ on $\partial B_{s_n}(x_n)$. Hence the subsolution estimate given in [9, Lemma 4.4] yields that

$$u_{j,n}(x_n) \leq C_8 U_0 e^{-\frac{s_n}{4}\sqrt{\frac{\beta_n g_*}{2}}} = C_9 e^{-\frac{\eta_1 \sqrt{g_*}}{4\sqrt{2}} \log \beta_n} \leq C_9 e^{-\eta \log \beta_n} = C_9 \beta_n^{-\eta}$$

□

Now we have all the tools to complete the proof of Theorem 3.1.
We first consider part (b). Let $i \in \{1, \dots, k\}$ be fixed. Since u_i is continuous, the set N_i is open. Let $K \subset N_i$ be a compact set and fix $\eta > \frac{1}{\tau}$, where τ is given by assumption (B2). Then Lemma 3.2 implies that

$$u_{j,n} = O(\beta_n^{-\eta}) \qquad \text{uniformly on } K \text{ for } j \neq i. \tag{3.13}$$

and therefore $g_i(u_n^i) = O(\beta_n^{-\tau\eta})$ uniformly on K by assumption $(B2)$. Hence

$$\beta_n\, g_i(u_n^i)u_{i,n} \le \beta_n g_i(u_n^i)L_0 = O(\beta_n^{1-\tau\eta}) \to 0 \qquad \text{uniformly on } K. \tag{3.14}$$

Since $\alpha_n \le C\beta_n$, we also have

$$\alpha_n\, g_i(u_n^i)u_{i,n} \to 0 \qquad \text{uniformly on } K. \tag{3.15}$$

Since (3.13) holds for arbitrary compact subsets $K \subset N_i$, we infer that $u_j = 0$ on N_i for $j \neq i$, i.e., $N_j \cap N_i = \varnothing$. Moreover, passing to the limit in (3.1) and using (3.14), (3.15), we conclude that $u_i|_{N_i} \in C^2(N_i)$ is a distributional solution of $-\Delta u_i = f_i(u_i)u_i$ in N_i. If f_i is Hölder continuous, standard elliptic regularity yields that $u_i|_{N_i} \in C^2(N_i)$ solves the latter equation in classical sense.

In the remainder of this section we complete the proof of Theorem 1.1(a). We suppose by contradiction that the sequence $(u_{1,n}, \dots, u_{k,n})_n$ is not uniformly equicontinuous. Then there exists $\delta > 0$, $i \in \{1, \dots, k\}$ and a subsequence – denoted as before – such that

$$\inf\Big\{|x-y| \;:\; x, y \in \Omega, |u_{i,n}(x) - u_{i,n}(y)| \ge 2\delta\Big\} \to 0 \qquad \text{as } n \to \infty.$$

Since all functions $u_{i,n}$ are nonnegative, there exists, for every n, points $x_n, y_n \in \Omega$ such that

$$r_n := |x_n - y_n| \to 0 \qquad \text{as } n \to \infty \tag{3.16}$$

and

$$d_n := u_{i,n}(y_n) \ge \delta, \qquad u_{i,n}(x_n) \ge d_n + \delta \qquad \text{for all } n \in \mathbb{N}. \tag{3.17}$$

By adjusting the choice of $x_n, y_n \in \Omega$ and $i \in \{1, \dots, k\}$ and passing to a subsequence we may further assume that

$$|u_{j,n}(x) - u_{j,n}(y)| \le \delta \quad \begin{cases} \text{for } j = 1, \dots, k \text{ and every } x, y \in \Omega \text{ with} \\ u_{j,n}(x), u_{j,n}(y) \ge \delta \text{ and } |x-y| \le r_n. \end{cases} \tag{3.18}$$

Without loss of generality, we may assume that $i = 1$. We denote $e_1 = (1, 0) \in \mathbb{R}^2$ and choose $A_n \in O(2)$, $n \in \mathbb{N}$ such that $A_n e_1 = r_n^{-1}(y_n - x_n)$. We consider the rescaled domains

$$\Omega_n := \{x \in \mathbb{R}^2 \;:\; x_n + r_n A_n x \in \Omega\}$$

and the rescaled functions

$$v_{i,n} : \mathbb{R}^2 \to \mathbb{R}, \qquad v_{i,n}(x) = u_{i,n}(x_n + r_n A_n x)$$

for $i = 1, \dots, k$, $n \in \mathbb{N}$ (recall that we have extended the functions $u_{i,n}$ trivially to all of $\mathbb{R}^2$). Then

$$v_{1,n}(e_1) = d_n \ge \delta, \qquad \text{and} \qquad v_{1,n}(0) \ge v_{1,n}(e_1) + \delta \qquad \text{for all } n. \tag{3.19}$$

Moreover, $v_n = (v_{1,n}, \dots, v_{k,n})$ is a solution of the rescaled problem

$$\begin{cases} -CM_n g_i(v_n^i)v_{i,n} \le -\Delta v_{i,n} - l_{i,n}v_{i,n} \le -M_n g_i(v_n^i)v_{i,n} & \text{in } \Omega_n, \\ v_{1,n}, \dots, v_{k,n} > 0 & \text{in } \Omega_n, \\ v_{1,n} = \dots = v_{k,n} = 0 & \text{in } \mathbb{R}^2 \setminus \Omega_n, \end{cases} \tag{3.20}$$

where

$$M_n = r_n^2 \beta_n, \qquad v_n^i = (v_{1,n}\,,\, \dots\,,\, v_{i-1,n}\,,\, v_{i+1,n}\,,\, \dots\,,\, v_{k,n})$$

and

$$l_{i,n} := r_n^2 f_i(v_{i,n}) \to 0 \qquad \text{in } L^\infty(\mathbb{R}^2) \text{ as } n \to \infty.$$

In particular, by Kato's inequality, $v_{1,n} \in H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is a distributional solution of the differential inequality

$$-\Delta v_{1,n} \le l_{1,n} v_{1,n} - M_n g_1(v_n^1) v_{1,n} \le l_{1,n} v_{1,n} \qquad \text{in } \mathbb{R}^2. \tag{3.21}$$

By a standard argument analyzing the asymptotic behaviour of $\text{dist}(x_n, \Omega)$, we find that $\Omega_n \to \Omega_\infty$ in the sense that $\Omega_n \cap K \to \Omega_\infty \cap K$ in Hausdorff distance for every compact set $K \subset \mathbb{R}^2$, where either $\Omega_\infty = \mathbb{R}^2$ or $\Omega_\infty = \mathcal{H}$, a halfspace. In both cases, (3.18) implies that $\text{dist}(0, \partial\Omega_n) \ge 1$ and therefore $\text{dist}(0, \partial\Omega_\infty) \ge 1$. For a subsequence, we may assume that $v_{1,n} \rightharpoonup v \in L^\infty(\mathbb{R}^2)$ in the weak*-topology. Passing to the distribution limit in (3.21), we see that v satisfies (S2) and therefore is a subharmonic function on $\mathbb{R}^2$. By Theorem 2.1, there is $c \in \mathbb{R}$ such that $v \equiv c$ almost everywhere in $\mathbb{R}^2$. Passing again to a subsequence, we may distinguish two cases.

Case 1: M_n remains bounded.

Then the right hand side of (3.20) remains uniformly bounded in $L^\infty(\Omega_n)$, so elliptic regularity implies that

$$v_{1,n} \to c \text{ uniformly on compact subsets of } \Omega_\infty. \tag{3.22}$$

If $\Omega_\infty = \mathbb{R}^2$, we obtain a contradiction, since in this case (3.22) yields $0 = \lim_n [v_{1,n}(0) - v_{1,n}(e_1)] \ge \delta$.

If $\Omega_\infty = \mathcal{H}$, then $v \equiv 0$ on $\mathbb{R}^2 \setminus \overline{\mathcal{H}}$ and therefore $c = 0$. Consequently, $\lim_n v_{1,n}(0) = 0$ by (3.22) since 0 is in the interior of $\mathcal{H}$. This contradicts (3.19).

In the remainder of this section, we will consider

Case 2: $M_n \to \infty$ as $n \to \infty$.

From Lemma 3.2 we directly deduce the following.

Lemma 3.3. *If $v_{1,n}(x_n)_n \ge \varepsilon$ for a sequence $(x_n) \subset \mathbb{R}^2$ and some $\varepsilon > 0$, then $v_{j,n}(x_n) = O(\beta_n^{-\eta})$ for $j = 2, \dots, k$ and every $\eta > 0$.*

Next we pass to subsequence such that one of the following two cases occurs.

Case 2.1: $\min\limits_{\partial B_R(0)} v_{1,n} < d_n$ for $2 \le R \le 3$.

Case 2.2: There exist a sequence of radii $R_n \in [2, 3]$ such that $v_{1,n} \ge d_n$ on $\partial B_{R_n}(0)$.

First we consider **Case 2.1**. As noted in (3.21),

$$-\Delta v_{1,n} \le l_{1,n} v_{1,n} \quad \text{in } \mathbb{R}^2 \qquad \text{and} \qquad l_{1,n} v_{1,n} \to 0 \text{ in } L^\infty(\mathbb{R}^2). \tag{3.23}$$

Using (3.19) and a standard inequality for subsolutions of Poisson's equation (see e.g. [15, page 71]), we obtain

$$d_n + \delta \le v_{1,n}(0) \le \frac{1}{|B_1(0)|} \int_{B_1(0)} v_{1,n}\, dx + o(1). \tag{3.24}$$

Recall that $v_{1,n} \to c \in L^\infty(\mathbb{R}^2)$ in the weak*-topology, where c is a constant. Moreover, from (3.2) and our rescaling we infer that $v_{1,n}$ is bounded in $H^1(B_3(0))$, hence

$$v_{1,n} \rightharpoonup c \quad \text{weakly in } H^1(B_3(0)) \qquad \text{and} \qquad v_{1,n} \to c \quad \text{in } L^1_{loc}(B_3(0)).$$

Passing to a subsequence, we may assume that also $d := \lim\limits_{n\to\infty} d_n$ exists. Then (3.24) yields

$$c = \lim_{n\to\infty} \frac{1}{|B_1(0)|} \int_{B_1(0)} v_{1,n}\, dx \ge \delta + d. \tag{3.25}$$

Now let $w_n := v_{1,n}^* \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$ denote the cap-symmetrization of $v_{1,n}$ with respect to the unit vector e_1, see Section 2. By Lemma 2.1,

$$w_n \rightharpoonup c \qquad \text{weakly in } H^1(B_3(0)). \tag{3.26}$$

Consider the line segment $\Gamma := \{-se_1 \;:\; 2 \le s \le 3\} \subset \mathbb{R}^3$. By definition of Case 2.1 and (2.3),

$$w_n < d_n \text{ on } \Gamma \text{ for every } n, \tag{3.27}$$

so that

$$\limsup \int_\Gamma w_n\, ds \le d \le c - \delta \tag{3.28}$$

by (3.25). Let $T : H^1(B_3(0)) \to L^1(\Gamma)$ denote the usual trace map on Γ satisfying $Tu = u|_\Gamma$ for every $u \in H^1(B_3(0)) \cap C(B_3(0))$, see e.g. [1]. It is well known that T is a compact operator, so it follows from (3.26) that $Tw_n \to Tc = c$ strongly in $L_1(\Gamma)$. This contradicts (3.28).

Finally, we consider **Case 2.2**. We put $B_n := B_{R_n}(0)$, and $S_n := \partial B_n$ for $n \in \mathbb{N}$, so that $\min_{S_n} v_{1,n} \ge d_n \ge \delta$ for all n. We fix $\eta > \frac{1}{\tau}$, where τ is given by assumption (B2), then Lemma 3.3 implies

$$\max_{j=2,\dots,k} \max_{S_n} v_{j,n} = O(\beta_n^{-\eta}) \qquad \text{as } n \to \infty. \tag{3.29}$$

Moreover, $-\Delta v_{j,n} - l_{j,n} v_{j,n} \le 0$ in $B_n \subset \Omega_n$, so by the standard subsolution estimate (see [15, Theorem 3.7]),

$$\max_{B_n} v_{j,n} \le \max_{S_n} v_{j,n} + C_{10} \max_{B_n} l_{j,n} v_{j,n}.$$

Here the constant $C_{10} > 0$ does not depend on R_n since $2 \le R_n \le 3$. Recalling that $l_{j,n} \to 0$ in $L^\infty(\mathbb{R}^2)$, we conclude that

$$\max_{B_n} v_{j,n} \le C_{11} \sup_{S_n} v_{j,n} \qquad \text{for large } n.$$

Hence (3.29) implies that

$$\max_{j=2,\dots,k}\ \max_{B_n} v_{j,n} = O(\beta_n^{-\eta}) \qquad \text{as } n\to\infty \tag{3.30}$$

and therefore $\max\limits_{B_n} g_1(v_n^1) = O(\beta_n^{-\tau\eta})$ by assumption $(B2)$. Consequently,

$$M_n\ \max_{B_n} g_1(v_n^1)v_{1,n} \le \beta_n\Big(\max_{B_n} g_1(v_n^1)\Big)L_0 = O(\beta_n^{1-\tau\eta}) \to 0,$$

and thus by (3.20) we have

$$-\Delta v_{1,n} = k_n \quad \text{in } B_n\subset\Omega_n, \qquad \text{where } \|k_n\|_{L^\infty(B_n)}\to 0. \tag{3.31}$$

In the following, let G_n denote the Green function for the Dirichlet Laplacian on B_n given by

$$G(x,y) = \frac{1}{2\pi}\begin{cases} \ln|x-y| - \ln\Big|\dfrac{x|y|}{R_n} - \dfrac{R_n y}{|y|}\Big|, & y\neq 0\\ \ln|x| - \ln R_n, & y=0.\end{cases}$$

Moreover, let K_n denote the corresponding Poisson kernel, i.e,

$$K_n(x,y) = \frac{\partial}{\partial\nu_y}G(x,y) = \frac{R_n^2-|x|^2}{2\pi R_n|x-y|^2} \qquad \text{for } x\in B_n,\ y\in S_n.$$

Recalling (3.19) and (3.31) we find

$$\begin{aligned} d_n = v_{1,n}(e_1) &= \int_{B_n} G_n(e_1,y)k_n(y)\,dy + \int_{S_n} K_n(e_1,y)v_{1,n}(y)\,dy\\ &= o(1) + \int_{S_n} K_n(e_1,y)v_{1,n}(y)\,ds_y,\end{aligned}$$

so that

$$\int_{S_n} K_n(e_1,y)[v_{1,n}(y)-d_n]\,ds_y = \int_{S_n} K_n(e_1,y)v_{1,n}(y)\,ds_y\ - d_n \to 0 \quad \text{as } n\to\infty. \tag{3.32}$$

Since $2\le R_n\le 3$ for all n, we have

$$K_n(e_1,y) = \frac{R_n^2-1}{2\pi R_n|e_1-y|^2} \ge \frac{R_n^2-1}{2\pi R_n(R_n+1)^2} \ge \frac{3}{16}\,\frac{1}{2\pi R_n} = \frac{3}{16}K_n(0,y) \qquad \text{for } y\in S_n.$$

Since also $v_{1,n}\ge d_n$ on S_n by assumption, we have by (3.19), (3.31) and (3.32)

$$\begin{aligned}\delta \le v_{1,n}(0) - d_n &= \int_{B_n} G_n(0,y)k_n(y)\,dy + \int_{S_n} K_n(0,y)[v_{1,n}(y)-d_n]\,ds_y\\ &\le o(1) + \frac{16}{3}\int_{S_n} K_n(e_1,y)[v_{1,n}(y)-d_n]\,ds_y = o(1),\end{aligned}$$

a contradiction for n large.

We conclude that neither Case 2.1 nor Case 2.2 can occur, so the proof is finished.

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