ON THE UNIQUENESS OF SOLUTIONS OF A NONLOCAL ELLIPTIC SYSTEM

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Abstract. We consider the following elliptic system with fractional Laplacian

$$-(-\Delta)^s u = uv^2, \quad -(-\Delta)^s v = vu^2, \quad u, v > 0 \text{ on } \mathbb{R}^n,$$

where $s \in (0,1)$ and $(-\Delta)^s$ is the s-Lapalcian. We first prove that all positive solutions must have polynomial bound. Then we use the Almgren monotonicity formula to perform a blown-down analysis. Finally we use the method of moving planes to prove the uniqueness of the one dimensional profile, up to translation and scaling.

1. Introduction and main results

In this paper we prove the uniqueness of positive solutions (u, v), up to scaling and translations, of the following nonlocal elliptic system

$$-(-\Delta)^s u = uv^2, -(-\Delta)^s v = vu^2, \; u, v > 0 \text{ in } \mathbb{R}^1 \tag{1.1}$$

where $(-\Delta)^s$ is the s-Laplacian with $0 < s < 1$.

When $s = 1$, problem (1.1) arises as limiting equation in the study of phase separations in Bose-Einstein system and also in the Lotka-Volterra competition systems. More precisely, we consider the classical two-component Lotka-Volterra competition systems

$$\begin{cases} -\Delta u + \beta_1 u^3 + \beta v^2 u = \lambda_1 u \;\text{ in }\; \Omega, \\ -\Delta v + \beta_2 v^3 + \beta u^2 v = \lambda_2 v \;\text{ in }\; \Omega, \\ u > 0, \quad v > 0 \;\text{ in }\; \Omega, \\ u = 0, \quad v = 0 \;\text{ on }\; \partial\Omega\,, \\ \int_\Omega u^2 = N_1, \quad \int_\Omega v^2 = N_2\,, \end{cases} \tag{1.2}$$

where $\beta_1, \beta_2, \beta > 0$ and Ω is a bounded smooth domain in $\mathbb{R}^n$. Solutions of (1.2) can be regarded as critical points of the energy functional

$$E_\beta(u, v) = \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) + \frac{\beta_1}{2} u^4 + \frac{\beta_2}{2} v^4 + \frac{\beta}{2} u^2 v^2\,, \tag{1.3}$$

on the space $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with constraints

$$\int_\Omega u^2 dx = N_1, \qquad \int_\Omega v^2 dx = N_2. \tag{1.4}$$

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Of particular interest is the asymptotic behavior of a family of bounded energy solutions (u_β, v_β) in the case of *strong competition*, i.e., when $\beta \to +\infty$, which produces spatial segregation in the limiting profiles. After suitable scaling and blowing up process (see Berestycki-Lin-Wei-Zhao [2] and Noris-Tavares-Terracini-Verzini [16]), we arrive at the following nonlinear elliptic system

$$\Delta u = uv^2\,, \quad \Delta v = vu^2\,, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^n\,. \tag{1.5}$$

Recently there have been intense studies on the elliptic system (1.5). In [2, 3] the relationship between system (1.5) and the celebrated Allen-Cahn equation is emphasized. A De Giorgi's-type and a Gibbons'-type conjecture for the solutions of (1.5) are formulated. Now we recall the following results for the system (1.5).

(1) When $n = 1$, it has been proved that the one-dimensional profile must have linear growth, and it is reflectionally symmetric, i.e., there exists x_0 such that $u(x - x_0) = v(x_0 - x)$, and is unique, up to translation and scaling. Furthermore this solution is nondegenerate and stable. See Berestycki-Terracini-Wang-Wei [3] and Berestycki-Lin-Wei-Zhao [2].
(2) When $n \geq 2$, all sublinear growth solutions are trivial (Noris-Tavares-Terracini-Verzini [16]). Furthermore, Almgren's and Alt-Caffarelli-Friedman monotonicity formulas are derived (Noris-Tavares-Terracini-Verzini [16]).
(3) When $n = 2$, the monotonic solution, i.e. (u, v) satisfies

$$\frac{\partial u}{\partial x_n} > 0, \qquad \frac{\partial v}{\partial x_n} < 0, \tag{1.6}$$

must be one-dimensional (Berestycki-Lin-Wei-Zhao [2]), provided that (u, v) has the following linear growth

$$u(x) + v(x) \leq C(1 + |x|). \tag{1.7}$$

Same conclusion holds if we consider stable solutions (Berestycki-Terracini-Wang-Wei [3]). It has also been proved by Farina [11] that the conditions (1.6)-(1.9) can be reduced to

$$\frac{\partial u}{\partial x_n} > 0 \tag{1.8}$$

and

$$u(x) + v(x) \leq C(1 + |x|)^d, \text{ for some positive integer } d. \tag{1.9}$$

The Gibbon's conjecture has also been solved under the polynomial growth condition (1.9) (Farina-Soave [12]).

(4) In $\mathbb{R}^2$, for each positive integer d there are solutions to (1.5) with polynomial growth of degree d (Berestycki-Terracini-Wang-Wei [3]). Moreover there are solutions in $\mathbb{R}^2$ which are periodic in one direction and have exponential growth in another direction (Soave-Zilio [19]).

(5) In two recent papers of the first author [25, 26], it is proved that any solution of (1.5) with linear growth is one dimensional, for any $n \geq 2$.

In this paper, we will generalize part of (1) and (2) to the fractional case.

In [22]-[23], Terracini, Verzini and Zillo initiated the study of competition-diffusion nonlinear systems involving *fractional Lapalcian* of the form

$$\begin{cases} (-\Delta)^s u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^2, \quad i = 1, ..., k \\ u_i \in H^s(\mathbb{R}^n) \end{cases} \tag{1.10}$$

where $n \geq 1$, $a_{ij} = a_{ji}$, β is positive and large, and the fractional Lapalcian $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{pv} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy.$$

Here $c_{n,s}$ is a constant depending only on n and s.

It is well known that fractional diffusion arises when the Gaussian statistics of the classical Brownian motion is replaced by a different one, allowing for the Lévy jumps (or flights). The operator $(-\Delta)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion process (Applebaum [1]). This operator arises in several areas such as physics, biology and finance. In particular in population dynamics while the standard Laplacian seems well suited to describe the diffusion of predators in presence of an abundant prey, when the prey is sparse observations suggest that fractional Laplacians give a more accurate model (Humphries [9]). Mathematically (1.10) is a more challenging problem because the operator is of the nonlocal nature.

In [22, 23, 24], Terracini et. al. derived the corresponding Almgren's and Alt-Caffarelli-Friedman's monotonicity formula and proved that the bounded energy solutions have uniform Hölder regularity with small Hölder exponent $\alpha = \alpha(N, s)$. As in the standard diffusion case, a key result to prove is to show that there are no entire solutions to the blown-up limit system

$$-(-\Delta)^s u = uv^2, \quad -(-\Delta)^s v = vu^2, \quad u, v > 0 \text{ in } \mathbb{R}^n, \tag{1.11}$$

with small Hölder continuous exponent.

In this paper, we study some basic qualitative behavior of solutions to (1.11), including (cp. the results (1) and (4) in the classical Laplacian case)

(a) are all one-dimensional solutions unique, up to translation and scaling?
(b) do all solutions have polynomial bounds?

We shall answer both questions affirmatively. To state our results, we consider the Caffarelli-Silvestre extension of (1.11). Letting $a := 1 - 2s \in (-1, 1)$, as in [7], we introduce the elliptic operator

$$L_a v := \mathrm{div}\left(y^a \nabla v\right),$$

for functions defined on the upper half plane $\mathbb{R}^{n+1}_+$. For simplicity of notations, define

$$\partial^a_y v := \lim_{y \to 0^+} y^a \frac{\partial v}{\partial y}.$$

The problem (1.11) is equivalent to the following extension problem

$$\begin{cases} L_a u = L_a v = 0, \text{ in } \mathbb{R}^{n+1}_+, \\ \partial^a_y u = uv^2, \quad \partial^a_y v = vu^2 \text{ on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{1.12}$$

Indeed, solutions of this extension problem, when restricted to $\partial\mathbb{R}^{n+1}_+$, can be seen as solutions of (1.11) in the viscosity sense.

Note that the problem (1.12) is invariant under the scaling $(u(z), v(z)) \mapsto (\lambda^s u(\lambda z), \lambda^s v(\lambda z)$ and translations in $\mathbb{R}^n$ directions. It is also invariant under the involution $(u, v) \mapsto (v, u)$.

Our first main result is

Theorem 1.1. *When $n = 1$ and $s \in (1/4, 1)$, the positive solution (u, v) of (1.12) is unique up to a scaling and translation in the x-direction. In particular, there exists a constant T such that*

$$u(x, y) = v(2T - x, y), \quad \text{in } \mathbb{R}^2_+.$$

It turns out that many trivial facts in the classical Laplacian case (cp. [2]) become serious problems in the fractional setting. Hence the proof is quite involved and basically splits into three steps.

(1) With the help of Almgren monotonicity formula, we perform a blowing down analysis for solutions of (1.11). Then we classify the blowing down limits. This gives the first order expansion of (u, v) at infinity.
(2) By establishing some decay estimates, we then use the Fourier mode analysis to get the next order expansion of (u, v) at infinity. It is in this step we need the technical assumption $s > 1/4$.
(3) With the above refined asymptotics of (u, v) at infinity, we can use a refinement of the moving plane method used in [3] to finish the proof of Theorem 1.1.

In the first step, we also need the following result.

Theorem 1.2. *When $n \geq 1, s \in (0, 1)$, the positive solution (u, v) of (1.12) must have at most polynomial growth: there exists $d > 0$ such that*

$$u(x, y) + v(x, y) \leq C(1 + |x| + |y|)^d. \tag{1.13}$$

Compared to the classical Laplacian case (e.g. solutions with exponential growth as constructed in Soave-Zilio [19]), this is quite surprising.

Let us put our results in broader context. The uniqueness for fractional nonlinear elliptic equations is a very challenging problem. The only results known in this direction are due to Frank-Lenzmann [13] and Frank-Lenzmann-Silvestre [14], in which they proved the nondegeneracy and uniqueness of radial ground states for the following fractional nonlinear Schrödinger equation

$$-(-\Delta)^s Q - Q + Q^p = 0, \ Q > 0, \ Q \in H^s(\mathbb{R}^n). \tag{1.14}$$

Our proof of Theorem 1.1 is completely different from theirs: we make use of the method of moving planes (as in [3]) to prove uniqueness. To apply the method of moving plane, we have to know precise asymptotics of the solutions up to high orders. This is achieved by blown-down analysis and Fourier mode expansions. (The condition that $s > \frac{1}{4}$ seems to be technical only.) In dealing with nonlocal equations some "trivial" facts can become quite nontrivial. For example, one of "trivial" question is whether or not one dimensional profile has linear growth. (When $s = 1$ this is a trivial consequence of Hamiltonian identity. See [2].) To prove this for the fractional Laplacian case we employ Yau's gradient estimates. A surprising result is that this also gives the polynomial bound for all solutions (Theorem 1.2). This is in sharp contrast with $s = 1$ case since there are exponential growth solutions ([19]).

The rest of the paper is organized as follows: In Section 2 we prove Yau's estimates for positive $s-$subharmonic functions from which we prove Theorem 1.2. Sections 3 and 4 contain the Almgren's monotonicity formula and the blown-down process to $s-$harmonic functions. We prove Theorem 1.1 in Sections 5-8: we first classify the blown-down limit when $n = 1$ (Section 5). Then we prove the growth bound and decay estimates (Section 6). In order to apply the method of moving planes we need to obtain refined asymptotics (Section 7). Finally we apply the method of moving planes to prove the uniqueness result. We list some basic facts about L_a-subharmonic functions in the appendix.

Throughout this paper, we take the following notations. $z = (x, y)$ denotes a point in $\mathbb{R}^{n+1}_+$ where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}_+$. In polar coordinates, $y = r\sin\theta$ where $\theta \in [0, \pi]$. When $n = 1$, we also use the notation $z = x + iy = (r\cos\theta, r\sin\theta)$. The half ball $B^+_r(z_0) = B_r(z_0) \cap \mathbb{R}^{n+1}_+$, the positive part of its boundary $\partial^+ B^+_r(z_0) = \partial B_r(z_0) \cap \mathbb{R}^{n+1}_+$ and the flat part $\partial^0 B^+_r(z_0) = \partial B^+_r(z_0) \setminus \partial^+ B^+_r$. Moreover, if the center of ball is the origin 0, it will be omitted. We use C, c and M to denote various constants, which may be different from lines to lines, and φ and ψ to denote functions, which could be different in different sections.

2. Gradient estimate for positive L_a-harmonic functions and Proof of Theorem 1.2

In this section we prove the following Yau's type gradient estimate for positive L_a-harmonic functions and use it in combination with an observation due to Markovic [15], to give a polynomial bound for solutions of (1.12). Regarding Yau's estimates for harmonic functions on manifolds, we refer to the book by Schoen-Yau [17].

Theorem 2.1. *Let u be a positive L_a-harmonic function in $\mathbb{R}^{n+1}_+$. There exists a constant $C(n)$ such that*

$$\frac{|\nabla u(x,y)|}{u(x,y)} \le \frac{C(n)}{y}, \quad \text{in } \mathbb{R}^{n+1}_+.$$

Proof. Let $v := \log u$, which satisfies

$$-\Delta v = |\nabla v|^2 + a y^{-1}\frac{\partial v}{\partial y}. \tag{2.1}$$

By a direct calculation we have

$$\frac{1}{2}\Delta|\nabla v|^2 = |\nabla^2 v|^2 - \nabla|\nabla v|^2\cdot\nabla v - \frac{a}{2y}\frac{\partial}{\partial y}|\nabla v|^2 + \frac{a}{y^2}\Big|\frac{\partial v}{\partial y}\Big|^2. \tag{2.2}$$

For any $z_0 = (x_0, y_0) \in \mathbb{R}^{n+1}_+$, let $R = y_0/3$. Take a nonnegative function $\eta \in C^\infty_0(B_{2R}(z_0))$ and let $w := |\nabla v|^2\eta$. Since w vanishes on $\partial B_{2R}(z_0)$, it attains its maximum at an interior point, say z_1.

At z_1,

$$0 = \nabla w = \eta\nabla|\nabla v|^2 + |\nabla v|^2\nabla\eta, \tag{2.3}$$

$$0 \ge \Delta w = \eta\Delta|\nabla v|^2 + 2\nabla|\nabla v|^2\cdot\nabla\eta + |\nabla v|^2\Delta\eta. \tag{2.4}$$

Substituting (2.2) and (2.3) into (2.4) leads to

$$\begin{aligned} 0 &\geq 2|\nabla^2 v|^2\eta + 2|\nabla v|^2\nabla v\cdot\nabla\eta + ay^{-1}|\nabla v|^2\frac{\partial\eta}{\partial y} \\ &\quad +2ay^{-2}\left(\frac{\partial v}{\partial y}\right)^2\eta - 2|\nabla v|^2\eta^{-1}|\nabla\eta|^2 + |\nabla v|^2\Delta\eta. \end{aligned}$$

By the Cauchy inequality and (2.1),

$$\begin{aligned} |\nabla^2 v|^2 &\geq \frac{1}{n+1}(\Delta v)^2 \\ &= \frac{1}{n+1}\left(|\nabla v|^4 + 2ay^{-1}|\nabla v|^2\frac{\partial v}{\partial y} + \frac{a^2}{y^2}\Big|\frac{\partial v}{\partial y}\Big|^2\right). \end{aligned}$$

Combining these two inequalities gives

$$\begin{aligned} 0 &\geq \frac{2}{n+1}|\nabla v|^4\eta + \frac{4a}{(n+1)y}|\nabla v|^2\frac{\partial v}{\partial y}\eta + \frac{2a^2}{(n+1)y^2}\Big|\frac{\partial v}{\partial y}\Big|^2\eta \\ &\quad +2|\nabla v|^2\nabla v\cdot\nabla\eta + ay^{-1}|\nabla v|^2\frac{\partial\eta}{\partial y} + 2ay^{-2}\left(\frac{\partial v}{\partial y}\right)^2\eta \\ &\quad -2|\nabla v|^2\eta^{-1}|\nabla\eta|^2 + |\nabla v|^2\Delta\eta. \end{aligned}$$

Now take an $\varphi \in C_0^\infty(B_{2R}(z_0))$, satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_R(z_0)$ and $|\nabla\varphi|^2 + |\Delta\varphi| \leq 100R^{-2}$. Choose an $m \geq 3$ and substitute $\eta = \varphi^{2m}$ into the above inequality, which results in

$$\begin{aligned} |\nabla v|^4\varphi^{2m} &\leq C(n)y^{-1}|\nabla v|^3\varphi^{2m} + C(n)y^{-2}|\nabla v|^2\varphi^{2m} \\ &\quad +C(n,m)\varphi^{2m-1}|\nabla v|^3|\nabla\varphi| + C(n,m)\varphi^{2m-1}y^{-1}|\nabla v|^2|\nabla\varphi| \\ &\quad +C(n,m)\varphi^{2m-2}|\nabla v|^2|\nabla\varphi|^2 + C(n,m)\varphi^{2m-1}|\Delta\varphi||\nabla v|^2. \end{aligned}$$

Applying the Young inequality to the right hand side, we obtain

$$\begin{aligned} |\nabla v|^4\varphi^{2m} &\leq \frac{1}{2}|\nabla v|^4\varphi^{2m} \\ &\quad +C(n,m)\left(y^{-4}\varphi^{2m} + \varphi^{2m-4}|\nabla\varphi|^4 + \varphi^{2m-2}y^{-2}|\nabla\varphi|^2 + \varphi^{2m-2}|\Delta\varphi|^2\right). \end{aligned}$$

By our assumption on φ, and because $y^{-1} \leq 4R^{-1}$ in $B_{2R}(z_0)$, this gives

$$|\nabla v(z_0)|^4 \leq |\nabla v(z_1)|^4\varphi(z_1)^{2m} \leq C(n,m)R^{-4},$$

which clearly implies the bound on $u^{-1}|\nabla u|$. □

A direct consequence of this gradient estimate is a Harnack inequality for positive L_a-harmonic functions.

Corollary 2.2. *Let u be a positive L_a-harmonic function in $\mathbb{R}^{n+1}_+$. There exists a constant $C(n)$ such that, for any $(x,y) \in \mathbb{R}^{n+1}_+$,*

$$\sup_{B_{y/2}(x,y)} u \leq C(n) \inf_{B_{y/2}(x,y)} u.$$

Iterating this Harnack inequality using chains of balls gives an exponential growth bound on u. However, we can get a more precise estimate using the hyperbolic geometry (as in [15]).

Now we come to the proof of Theorem 1.2. In fact, we have the following polynomial bound for positive s-subharmonic function on $\mathbb{R}^n$.

Theorem 2.3. *Let $u \in C(\overline{\mathbb{R}^{n+1}})$ be a solution of the problem*

$$\begin{cases} L_a u = 0, \ \ \textit{in } \mathbb{R}^{n+1}_+, \\ u \geq 0, \ \ \textit{on } \overline{\mathbb{R}^{n+1}_+}, \\ \partial^a_y u \geq 0, \quad \textit{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

There exists a constant C depending only on the dimension n and a such that,

$$u(x,y) \leq Cu(0,1)\left(1+|x|^2+y^2\right)^C \quad \textit{on } \overline{\mathbb{R}^{n+1}}.$$

Proof. **Step 1. Estimates in $\{y \geq 1/2\}$**

As in [15], for any two different points $z_i = (x_i, y_i) \in \mathbb{R}^{n+1}_+$ and a C^1 curve $\gamma(t) = (\gamma_1(t), \cdots, \gamma_{n+1}(t)) \subset \mathbb{R}^{n+1}_+, t \in [0,1]$ connecting them,

$$\begin{aligned} \log\frac{u(z_2)}{u(z_1)} &= \int_0^1 \nabla \log u(\gamma(t)) \cdot \frac{d\gamma(t)}{dt} dt \\ &\leq \int_0^1 \Big|\nabla \log u(\gamma(t))\Big|\Big|\frac{d\gamma(t)}{dt}\Big| dt \\ &\leq C\int_0^1 \frac{\left|\frac{d\gamma(t)}{dt}\right|}{\gamma_{n+1}(t)} dt \qquad \text{(by Theorem 2.1)} \\ &\leq C\text{Length}_H(\gamma). \end{aligned}$$

Here $\text{Length}_H(\gamma)$ is the length of γ with respect to the hyperbolic metric on $\mathbb{R}^{n+1}_+$,

$$ds^2 := \frac{dx^2+dy^2}{y^2}.$$

In particular, we can take γ to be the geodesic between z_1 and z_2. This gives

$$\log\frac{u(z_2)}{u(z_1)} \leq C\text{dist}_H(z_1,z_2).$$

However, we know the distance function dist_H has the form

$$\text{dist}_H(z_1,z_2) = \text{arccosh}\left(1+\frac{|x_1-x_2|^2+(y_1-y_2)^2}{2y_1y_2}\right).$$

This then implies that

$$\frac{u(z_2)}{u(z_1)} \leq \left(1+\frac{|x_1-x_2|^2+(y_1-y_2)^2}{2y_1y_2}\right)^C. \tag{2.5}$$

In particular, for any $(x,y)\in\mathbb{R}^{n+1}_+$,

$$u(x,y)\leq u(0,1)\left(1+\frac{|x|^2+(y-1)^2}{2y}\right)^C.$$

Hence, in $\{y\geq 1/2\}$,

$$u(x,y)\leq C\left(|x|^2+y^2+1\right)^C. \tag{2.6}$$

Step 2. Estimates in $\{0\leq y\leq 1/2\}$.

The estimates in Step 1 does not give any information near the boundary $\partial\mathbb{R}^{n+1}_+$. To get a growth bound in the part where y is small, we have to use the boundary condition on $\partial\mathbb{R}^{n+1}_+$. This is possible by using the Poisson representation formula.

For every $t\in(0,1/2)$, let $P^t(x,y)$ be the Poisson kernel of the elliptic operator $\Delta+a(y+t)^{-1}\partial_y$ on $\mathbb{R}^{n+1}_+$. Note that when $t=0$, this is the usual Poisson kernel for the operator L_a. By [7, Section 2.4], modulo a constant,

$$P^0(x,y)=\frac{y^{2s}}{\left(|x|^2+y^2\right)^{\frac{n+2s}{2}}}.$$

From the uniqueness of the Poisson kernel we deduce the following product rule: for $y>t$,

$$P^0(x,y+t)=\int_{\mathbb{R}^n}P^t(x-\xi,y)P^0(\xi,t)d\xi. \tag{2.7}$$

Denote the Fourier transform of $P^t(x,y)$ in x by $\hat{P}^t(\zeta,y)$. $\hat{P}^0(\zeta,y)$ has the form (modulo a constant) $\Phi(y|\zeta|)$, where

$$\Phi(|\zeta|)=d_{n,s}\int_{\mathbb{R}^n}\left(1+|x|^2\right)^{-\frac{n+2s}{2}}e^{-\sqrt{-1}x\cdot\zeta}dx.$$

Here $d_{n,s}$ is a normalization constant.

Since $\hat{P}^0$ satisfies

$$-|\zeta|^2\hat{P}^0(\zeta,y)+\frac{\partial^2}{\partial y^2}\hat{P}^0(\zeta,y)+\frac{a}{y}\frac{\partial}{\partial y}\hat{P}^0(\zeta,y)=0,$$

Φ satisfies

$$\Phi''(t)+at^{-1}\Phi'(t)-\Phi(t)=0,\quad\text{in }(0,+\infty).$$

By definition and the Lebesgue-Riemann lemma, $\Phi(0)=1$ and $\lim_{t\to+\infty}\Phi(t)=0$, where the decay rate is exponential (by the equation for Φ). Then by a maximum principle argument, we know $\Phi(t)>0$ and $\Phi(t)$ is decreasing in t.

By (2.7),

$$\hat{P}^t(\zeta,y)=\frac{\hat{P}^0(\zeta,y+t)}{\hat{P}^0(\zeta,t)}=\frac{\Phi^0((y+t)|\zeta|)}{\Phi(t|\zeta|)}.$$

Hence there exists a constant C depending only on n and a so that for all $t\in[0,1/2]$,

$$P^t(0,1-t)=\int_{\mathbb{R}^n}\frac{\Phi(|\zeta|)}{\Phi(t|\zeta|)}d\zeta\geq\int_{\mathbb{R}^n}\Phi(|\zeta|)d\zeta=P^0(0,1)\geq\frac{1}{C}. \tag{2.8}$$

Since P^t is a positive solution of

$$\Delta P^t + \frac{a}{y+t}\frac{\partial P^t}{\partial y} = 0, \quad \text{in } \mathbb{R}^{n+1}_+,$$

the gradient estimate Theorem 2.1 holds for P^t with the same constant $C(n)$. Then similar to (2.5), we get

$$P^t(x, 1-t) \geq P^t(0, 1-t)\left(1 + \frac{|x|^2}{2(1-t)^2}\right)^{-C}. \tag{2.9}$$

By the Poisson representation,

$$u(0,1) \geq \int_{\mathbb{R}^n} P^t(-x, 1-t)u(x,t)dx. \tag{2.10}$$

In fact, for any $R > 0$, consider the boundary value $u(x,t)\chi_{\{|x|<R\}}$, and let w^r be the solution of

$$\begin{cases} L_a w^r = 0, \text{ in } B_r^+, \\ w^r = u(x,t)\chi_{\{|x|<R\}}, \quad \text{on } \partial^0 B_r^+, \\ w^r = 0, \quad \text{on } \partial^+ B_r^+. \end{cases}$$

Such w^r exists and is unique. By the maximum principle, as $r \to +\infty$, they are uniformly bounded and increase to

$$\int_{\{|x|<R\}} P^t(x-\zeta, y)u(\zeta, t)d\zeta.$$

Here we have used the fact that there is a unique bounded L_a-harmonic function in $\mathbb{R}^{n+1}_+$ with boundary value $u(x,t)\chi_{\{|x|<R\}}$.

By the comparison principle, for each $r > 0$, $w^r \leq u$. Thus we have

$$u(0,1) \geq \int_{\{|x|<R\}} P^t(-x, 1-t)u(x,t)dx.$$

Then let $R \to +\infty$ we get (2.10).

Substituting (2.8) and (2.9) into (2.10), we see for any $t \in (0, 1/2)$,

$$\int_{\mathbb{R}^n} \frac{u(x,t)}{(|x|^2+1)^C}dx \leq C(n,s)u(0,1).$$

Integrating t in $[0, 1/2]$ gives

$$\int_0^{1/2}\int_{\mathbb{R}^n} \frac{u(x,y)}{(|x|^2+1)^C}dxdy \leq Cu(0,1). \tag{2.11}$$

Next we divide the proof into two cases.

Case 1. First assume $s \leq 1/2$, hence $a \geq 0$ and y^a is bounded in $\{0 \leq y \leq 1/2\}$. For any $x_0 \in \mathbb{R}^n$ with $|x_0| > 2$, by the co-area formula, we find an $r \in (1,2)$ so that

$$\int_{\partial^+ B_r^+(x_0,0)} y^a u \quad \leq \quad \int_{B_2^+(x_0,0)\setminus B_1^+(x_0,0)} y^a u(x,y)dxdy$$

$$\begin{aligned}
&\leq C\left(1+|x_0|^2\right)^C \int_{\left(B_2^+(x_0,0)\setminus B_1^+(x_0,0)\right)\cap\{0<y<1/2\}} \frac{u(x,y)}{\left(1+|x|^2\right)^C}dxdy \\
&\quad+\int_{\left(B_2^+(x_0,0)\setminus B_1^+(x_0,0)\right)\cap\{y>1/2\}} u(x,y)dxdy \\
&\leq Cu(0,1)\left(1+|x_0|^2\right)^C,
\end{aligned} \tag{2.12}$$

thanks to (2.6) and (2.11).

Case 2. If $s>1/2$, $a<0$ and y^a is unbounded in $\{0\leq y\leq 1/2\}$. Hence the above argument does not work. Instead, we take two positive constants p,q so that

$$1<p<\frac{1}{2s-1},\quad \frac{1}{p}+\frac{1}{q}=1.$$

By noting that

$$\begin{cases}
L_a u^q\geq 0, \text{ in } \mathbb{R}^{n+1}_+,\\
u^q\geq 0, \text{ on } \overline{\mathbb{R}^{n+1}_+},\\
\partial_y^a u^q\geq 0, \text{ on } \partial\mathbb{R}^{n+1}_+,
\end{cases}$$

we can still apply the argument leading to (2.11) to deduce that

$$\int_0^{1/2}\int_{\mathbb{R}^n} \frac{u(x,y)^q}{\left(|x|^2+1\right)^C}dxdy\leq Cu(0,1)^q. \tag{2.13}$$

Then in (2.12), we use the Hölder inequality to get

$$\begin{aligned}
&\int_{\partial^+B_r^+(x_0,0)} y^a u \\
\leq &\int_{B_2^+(x_0,0)\setminus B_1^+(x_0,0)} y^a u(x,y)dxdy \\
\leq &\left(\int_{\left(B_2^+(x_0,0)\setminus B_1^+(x_0,0)\right)\cap\{0<y<1/2\}} y^{pa}dxdy\right)^{\frac{1}{p}}\left(\int_{\left(B_2^+(x_0,0)\setminus B_1^+(x_0,0)\right)\cap\{0<y<1/2\}} u(x,y)^q dxdy\right)^{\frac{1}{q}} \\
&+\int_{\left(B_2^+(x_0,0)\setminus B_1^+(x_0,0)\right)\cap\{y>1/2\}} u(x,y)dxdy \\
\leq &C\left(1+|x_0|^2\right)^C\int_{\left(B_2^+(x_0,0)\setminus B_1^+(x_0,0)\right)\cap\{0<y<1/2\}} \frac{u(x,y)^q}{\left(1+|x|^2\right)^C}dxdy \\
&+\int_{\left(B_2^+(x_0,0)\setminus B_1^+(x_0,0)\right)\cap\{y>1/2\}} u(x,y)dxdy \\
\leq &Cu(0,1)\left(1+|x_0|^2\right)^C,
\end{aligned} \tag{2.14}$$

After extending u evenly to $B_r(x_0,0)$, u becomes a positive L_a-subharmonic function, thanks to its boundary condition on $\partial^0\mathbb{R}^{n+1}_+$. With the help of (2.12) or (2.14), Lemma A.2

implies that

$$\sup_{B_{1/2}(x_0,0)} u \le C(n,a)\int_{\partial B_r(x_0,0)} y^a u \le Cu(0,1)\left(1+|x_0|^2\right)^C.$$

Together with (2.6), we get a polynomial bound for u as claimed. □

3. Almgren monotonicity formula

In this section we present the Almgren monotonicity formula for solutions of (1.12) and some of its consequences. In the next section these will be used in the blow down analysis. Throughout this section, (u,v) denotes a solution of (1.12).

We first state a Pohozaev identity for the application below.

Lemma 3.1. *For any $x\in\mathbb{R}^n$ and $r>0$,*

$$\begin{aligned}
&(n-1+a)\int_{B_r^+(x,0)} y^a\left(|\nabla u|^2+|\nabla v|^2\right)\\
=\ & r\int_{\partial^+ B_r^+(x,0)} y^a\left(|\nabla u|^2+|\nabla v|^2\right)-2y^a\left(\left|\frac{\partial u}{\partial r}\right|^2+\left|\frac{\partial v}{\partial r}\right|^2\right)\\
&+r\int_{S_r^n(x,0)} u^2v^2-n\int_{\partial^0 B_r^+(x,0)} u^2v^2.
\end{aligned}$$

Here $S_r^n(x,0)$ is the sphere with radius r and center x in $\mathbb{R}^n$.

Proof. This can be proved by multiplying the equation (1.12) by $z\cdot\nabla u$ (respectively, $z\cdot\nabla v$) and integrating by parts on B_r^+, cf. [7, Lemma 6.2] and the Pohozaev identity in [22]. □

Let

$$E(r):=\frac{1}{r^{n-1+a}}\int_{B_r^+} y^a\left(|\nabla u|^2+|\nabla v|^2\right)+\frac{1}{r^{n-1+a}}\int_{\partial^0 B_r^+} u^2v^2,$$

$$H(r):=\frac{1}{r^{n+a}}\int_{\partial^+ B_r^+} y^a\left(u^2+v^2\right),$$

and $N(r):=E(r)/H(r)$.

We have the following (cf. [22] for the 1/2-Lapalcian case and [23, Proposition 6] for general s-Laplacian case).

Proposition 3.2 (Almgren monotonicity formula). *$N(r)$ is non-decreasing in $r>0$.*

Proof. Direct calculation using the equation (1.12) shows that

$$\begin{aligned}
H'(r) &= \frac{2}{r^{n+a}}\int_{\partial^+ B_r^+} y^a\left(u\frac{\partial u}{\partial r}+v\frac{\partial v}{\partial r}\right)\\
&= \frac{2}{r^{n+1}}\int_{B_r^+} y^a\left(|\nabla u|^2+|\nabla v|^2\right)+\frac{4}{r^{n+a}}\int_{\partial^0 B_r^+} u^2v^2 \qquad (3.1)\\
&= \frac{2E(r)}{r}+\frac{2}{r^{n+a}}\int_{\partial^0 B_r^+} u^2v^2.
\end{aligned}$$

Using Lemma 3.1, we have

$$E'(r)=\frac{1}{r^{n-1+a}}\int_{\partial^+B_r^+}y^a\left(\left|\frac{\partial u}{\partial r}\right|^2+\left|\frac{\partial v}{\partial r}\right|^2\right)+\frac{1-a}{r^{n+a}}\int_{\partial^0B_r^+}u^2v^2. \tag{3.2}$$

Combining these two, we obtain

$$\begin{aligned}\frac{1}{2}\frac{N'(r)}{N(r)}\ &\geq\ \frac{\int_{\partial^+B_r^+}y^a\left(\left|\frac{\partial u}{\partial r}\right|^2+\left|\frac{\partial v}{\partial r}\right|^2\right)}{\int_{\partial^+B_r^+}y^a\left(u\frac{\partial u}{\partial r}+v\frac{\partial v}{\partial r}\right)}-\frac{\int_{\partial^+B_r^+}y^a\left(u\frac{\partial u}{\partial r}+v\frac{\partial v}{\partial r}\right)}{\int_{\partial^+B_r^+}y^a\left(u^2+v^2\right)}\\ &\quad+\frac{1-a}{N(r)}\frac{\int_{\partial^0B_r^+}u^2v^2}{\int_{\partial^+B_r^+}y^a\left(u^2+v^2\right)},\end{aligned} \tag{3.3}$$

which is nonnegative. □

Note that (3.1) also implies that

$$\frac{d}{dr}\log H(r)=\frac{2N(r)}{r}+\frac{2\int_{\partial^0B_r^+}u^2v^2}{\int_{\partial^+B_r^+}y^a\left(u^2+v^2\right)}\geq\frac{2N(r)}{r}. \tag{3.4}$$

Combining this with Proposition 3.2 we have

Proposition 3.3. *Let (u,v) be a solution of* (1.12). *If $N(R)\geq d$, then for $r>R$, $r^{-2d}H(r)$ is nondecreasing in r.*

The following result states a doubling property of (u,v).

Proposition 3.4. *Let (u,v) be a solution of* (1.12) *on B_R^+. If $N(R)\leq d$, then for every $0<r_1\leq r_2\leq R$*

$$\frac{H(r_2)}{H(r_1)}\leq e^{\frac{d}{1-a}}\frac{r_2^{2d}}{r_1^{2d}} \tag{3.5}$$

Proof. This is similar to the proof of [3, Proposition 5.2]. Since for all $r\in(0,R]$, $N(r)\leq d$, by (3.3) and (3.4) we have

$$\begin{aligned}\frac{d}{dr}\log H(r)\ &\leq\ \frac{2d}{r}+\frac{2\int_{\partial^0B_r^+}u^2v^2}{\int_{\partial^+B_r^+}y^a\left(u^2+v^2\right)}\\ &\leq\ \frac{2d}{r}+\frac{1}{1-a}N'(r).\end{aligned}$$

Integrating this from r_1 to r_2, since $N(r_1)\geq0$ and $N(r_2)\leq d$, we get (3.5). □

Proposition 3.5. *Let (u,v) be a solution of* (1.12) *on $\mathbb{R}^{n+1}_+$. For any $d>0$, the following two conditions are equivalent:*

(1) **(Polynomial growth)** *There exists a positive constant C such that*

$$u(x,y)+v(x,y)\leq C\left(1+|x|^2+y^2\right)^{\frac{d}{2}}. \tag{3.6}$$

(2) **(Upper bound on $N(R)$)** *For any $R>0$, $N(R)\leq d$.*

Proof. Since the even extension of u and v to $\mathbb{R}^{n+1}$ are L_a-subharmonic, (2) $\Rightarrow$ (1) is a direct consequence of Proposition 3.4 and Lemma A.2.

On the other hand, if we have (3.6), but there exists some $R_0 > 0$ such that $N(R_0) \geq d+\delta$, where $\delta > 0$. By Proposition 3.3, for all $R > R_0$,

$$\sup_{\partial^+ B_R^+} \left(u^2 + v^2\right) \geq H(R) \geq \frac{H(R_0)}{R_0^{2d+2\delta}} R^{2d+2\delta},$$

which clearly contradicts (3.6). In other words, for any $R > 0$, we must have $N(R) \leq d$. □

4. Blow down analysis

In this section we perform the blow down analysis for solutions to (1.12). This gives the asymptotic behavior of these solutions at infinity.

Let (u, v) be a solution of (1.12). By Theorem 2.3 and Proposition 3.5, there exists a constant $d > 0$ so that

$$\lim_{R \to +\infty} N(R) := d < +\infty.$$

The existence of this limit is guaranteed by the Almgren monotonicity formula (Proposition 3.2). Note that for any $R < +\infty$, $N(R) \leq d$.

For $R \to +\infty$, define

$$u_R(z) := L_R^{-1} u(Rz), \qquad v_R(z) := L_R^{-1} v(Rz),$$

where L_R is chosen so that

$$\int_{\partial^+ B_1^+} y^a \left(u_R^2 + v_R^2\right) = 1. \tag{4.1}$$

(u_R, v_R) satisfies

$$\begin{cases} L_a u_R = L_a v_R = 0, \ \text{in } \mathbb{R}^{n+1}_+, \\ \partial^a_y u_R = \kappa_R u_R v_R^2, \ \ \partial^a_y v_R = \kappa_R v_R u_R^2 \ \ \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{4.2}$$

where $\kappa_R = L_R^2 R^{1-a}$.

By (4.1),

$$L_R^2 = R^{-n} \int_{\partial^+ B_R^+} y^a \left(u^2 + v^2\right).$$

By the Liouville theorem [23, Propostion 12] (Note that only a growth bound, not the global Hölder bound, is needed to deduce this Liouville theorem), for some $\alpha > 0$ small, there exists a constant C_α such that

$$L(R) \geq C_\alpha R^\alpha. \tag{4.3}$$

Thus $\kappa_R \to +\infty$ as $R \to +\infty$.

With the bound on $N(R)$ in hand, we can use Proposition 3.4 to deduce that, for any $r > 1$,

$$r^{-n-a} \int_{\partial^+ B_r^+} y^a \left(u_R^2 + v_R^2\right) \leq r^{2d}.$$

Since $\partial_y^a u_R \geq 0$ on $\partial^+\mathbb{R}^{n+1}_+$, its even extension to $\mathbb{R}^{n+1}$ is L_a-subharmonic. Thus by Lemma A.2 we can get a uniform bound from the above integral bound,

$$\sup_{B_r^+}(u_R + v_R) \leq Cr^d, \ \forall r > 1.$$

Then by the uniform Hölder estimate in [23], for some $\alpha \in (0, s)$, (u_R, v_R) are uniformly bounded in $C^\alpha_{loc}(\overline{\mathbb{R}^{n+1}_+})$.

Because $N(r; u_R, v_R) = N(Rr; u, v) \leq d$,

$$\int_{B_r^+} y^a \left(|\nabla u_R|^2 + |\nabla v_R|^2\right) + \int_{\partial^0 B_r^+} \kappa_R u_R^2 v_R^2 \leq dr^{n-1+a+2d}, \ \forall r > 1. \tag{4.4}$$

After passing to a subsequence of R, we can assume that (u_R, v_R) converges to (u_∞, v_∞) weakly in $H^{1,a}_{loc}(\mathbb{R}^{n+1}_+)$, and uniformly in $C^\alpha_{loc}(\overline{\mathbb{R}^{n+1}_+})$.

Then for any $r > 1$,

$$\begin{aligned}\int_{\partial^0 B_r^+} u_\infty^2 v_\infty^2 &= \lim_{R\to+\infty} \int_{\partial^0 B_r^+} u_R^2 v_R^2 \\ &\leq \lim_{R\to+\infty} \kappa_R^{-1} dr^{n-1+a+2d} = 0.\end{aligned}$$

Thus $u_\infty v_\infty \equiv 0$ on $\partial\mathbb{R}^{n+1}_+$.

Lemma 4.1. *(u_R, v_R) converges strongly to (u_∞, v_∞) in $H^{1,a}_{loc}(\mathbb{R}^{n+1}_+)$. $\kappa_R u_R^2 v_R^2$ converges to 0 in $L^1_{loc}(\partial\mathbb{R}^2_+)$.*

For a proof see [23, Lemma 4.5] (and the corresponding results in [22] for the 1/2-Lapalcian case).

Corollary 4.2. *For any $r > 0$,*

$$N(r; u_\infty, v_\infty) := \frac{r\int_{B_r^+} y^a \left(|\nabla u_\infty|^2 + |\nabla v_\infty|^2\right)}{\int_{\partial^+ B_r^+} y^a \left(u_\infty^2 + v_\infty^2\right)} \equiv d.$$

Proof. For any fixed $r > 0$, by Lemma 4.1,

$$\int_{B_r^+} y^a \left(|\nabla u_\infty|^2 + |\nabla v_\infty|^2\right) = \lim_{R\to+\infty} \int_{B_r^+} y^a \left(|\nabla u_R|^2 + |\nabla v_R|^2\right) + \int_{\partial^0 B_r^+} \kappa_R u_R^2 v_R^2.$$

By the uniform convergence of u_R and v_R, we also have

$$\int_{\partial^+ B_r^+} y^a \left(u_\infty^2 + v_\infty^2\right) = \lim_{R\to+\infty} \int_{\partial^+ B_r^+} y^a \left(u_R^2 + v_R^2\right).$$

Thus

$$N(r; u_\infty, v_\infty) = \lim_{R\to+\infty} N(r; u_R, v_R) = \lim_{R\to+\infty} N(Rr; u, v) = d. \qquad \square$$

For any $\eta \in C_0^\infty(\mathbb{R}^{n+1})$ nonnegative and even in y, multiplying the equation of u_R by η and integrating by parts, we obtain

$$\int_{\partial\mathbb{R}^{n+1}_+} \eta \partial_y^a u_R dx = \int_{\partial\mathbb{R}^{n+1}_+} \eta \kappa_R u_R v_R^2 dx = \int_{\mathbb{R}^{n+1}_+} u_R L_a \eta, \tag{4.5}$$

which is uniformly bounded as $R \to +\infty$. Hence we can assume that (up to a subsequence) $\partial_y^a u_R dx = \kappa_R u_R v_R^2 dx$ converges to a positive Radon measure μ. On the other hand, passing to the limit in (4.5) gives $\mu = \partial_y^a u_\infty dx$. Here $\partial_y^a u_\infty \geq 0$ on $\partial\mathbb{R}^{n+1}_+$ in the weak sense, that is, $\partial_y^a u_\infty dx$ is a positive Radon measure on $\partial\mathbb{R}^{n+1}_+$.

Lemma 4.3. *The limit (u_∞, v_∞) satisfies*

$$\begin{cases} L_a u_\infty = L_a v_\infty = 0, \ \text{in } \mathbb{R}^{n+1}_+, \\ u_\infty \partial_y^a u_\infty = v_\infty \partial_y^a v_\infty = 0 \quad \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{4.6}$$

Here the second equation in (4.6) is equivalent to the statement that the support of $\partial_y^a u_\infty dx$ belongs to $\{u_\infty = 0\}$.

Proof. The first equation can be directly obtained by passing to the limit in $L_a u_R = L_a v_R = 0$ and using the uniform convergence of (u_R, v_R).

To prove the second one, take an arbitrary point $z_0 = (x_0, 0) \in \{u_\infty > 0\}$. Since u_∞ is continuous, we can find an $r_0 > 0$ and $\delta_0 > 0$ such that $u_\infty \geq 2\delta_0$ in $B^+_{r_0}(z_0)$. By the segregated condition, $v_\infty(z_0) = 0$. Thus by decreasing r_0 if necessary, we can assume that

$$v_\infty \leq \delta_0 \ \text{ in } \overline{B^+_{r_0}(z_0)}.$$

Then by the uniform convergence of u_R and v_R, for all R large,

$$u_R \geq \delta_0, \quad v_R \leq 2\delta_0 \ \text{ in } \overline{B^+_{r_0}(z_0)}.$$

Thus

$$\partial_y^a v_R \geq \kappa_R \delta_0^2 v_R \ \text{ on } \partial^0 B^+_{r_0}(z_0).$$

By applying Lemma A.3, we obtain

$$\sup_{\partial^0 B^+_{r_0/2}(z_0)} v_R \leq C(r_0, \delta_0)\kappa_R^{-1}.$$

Then $\partial_y^a u_R = \kappa_R u_R v_R^2$ is uniformly bounded in $C^\beta(\partial^0 B^+_{r_0}(z_0))$ for some $\beta > 0$.

Let $w_R = y^a \frac{\partial u_R}{\partial y}$. It can be directly checked that w_R satisfies (see [7, Section 2.3])

$$\operatorname{div}\left(y^{-a}\nabla w_R\right) = 0.$$

By (4.4),

$$\int_{B^+_{r_0/2}(z_0)} y^{-a} w_R^2 \leq \int_{B^+_{r_0/2}(z_0)} y^a |\nabla u_R|^2$$

are uniformly bounded. Then by the boundary Hölder estimate ([18]), w_R are uniformly bounded in $C^\beta(\overline{B^+_{r_0/2}(z_0)})$. Because $w_R \geq 0$ on $\partial^0 B^+_{r_0}(z_0)$ and $w_R u_R \to 0$ in $L^1(\partial^0 B^+_{r_0}(z_0))$, by letting $R \to +\infty$ and using the uniform Hölder continuity of u_R and w_R, we get

$$\partial_y^a u_\infty = 0 \quad \text{on } \partial^0 B^+_{r_0/2}(z_0).$$

In the blow down procedure, we have shown that $u_\infty \in C^\alpha_{loc}(\overline{\mathbb{R}^{n+1}_+})$ for some $\alpha > 0$. Hence u_∞ is continuous on $\partial\mathbb{R}^{n+1}_+$. The above argument also shows that $y^a \partial_y u_\infty$ is continuous up to $\{u_\infty > 0\} \cap \partial\mathbb{R}^{n+1}_+$ and $\partial_y^a u_\infty = 0$ on $\{u_\infty > 0\} \cap \partial\mathbb{R}^{n+1}_+$. This completes the proof. □

Integrating by parts using (4.6), we get

$$\int_{\partial^+ B_r^+} y^a u_\infty \frac{\partial u_\infty}{\partial r} = \int_{B_r^+} y^a |\nabla u_\infty|^2, \quad \int_{\partial^+ B_r^+} y^a v_\infty \frac{\partial v_\infty}{\partial r} = \int_{B_r^+} y^a |\nabla v_\infty|^2, \tag{4.7}$$

for any ball B_r^+.

Let

$$E_\infty(r) := r^{1-n-a} \int_{B_r^+} y^a \left(|\nabla u_\infty|^2 + |\nabla v_\infty|^2 \right),$$

$$H_\infty(r) := r^{-n-a} \int_{\partial^+ B_r^+} y^a \left(u_\infty^2 + v_\infty^2 \right),$$

and $N_\infty(r) := E_\infty(r)/H_\infty(r)$.

By (4.7) and calculating as in (3.1), we still have

$$\frac{d}{dr} \log H_\infty(r) = \frac{2N_\infty(r)}{r}. \tag{4.8}$$

Since $N_\infty(r) \equiv d$, integrating this and by noting the normalization condition (4.1), which passes to the limit, gives

$$H_\infty(r) \equiv r^{2d}. \tag{4.9}$$

The following lemma is essentially [23, Propostion 6].

Lemma 4.4. *For any $r \in (0, +\infty)$, $H_\infty(r) > 0$ and $E_\infty(r) > 0$. Moreover,*

$$\frac{1}{2} \frac{N_\infty'(r)}{N_\infty(r)} \geq \frac{\int_{\partial^+ B_r^+} y^a \left(\left| \frac{\partial u_\infty}{\partial r} \right|^2 + \left| \frac{\partial v_\infty}{\partial r} \right|^2 \right)}{\int_{\partial^+ B_r^+} y^a \left(u_\infty \frac{\partial u_\infty}{\partial r} + v_\infty \frac{\partial v_\infty}{\partial r} \right)} - \frac{\int_{\partial^+ B_r^+} y^a \left(u_\infty \frac{\partial u_\infty}{\partial r} + v_\infty \frac{\partial v_\infty}{\partial r} \right)}{\int_{\partial^+ B_r^+} y^a \left(u_\infty^2 + v_\infty^2 \right)} \geq 0, \tag{4.10}$$

in the distributional sense.

Proof. The Pohozaev identity for (u_R, v_R) reads as

$$\begin{aligned} & (n-1+a) \int_{B_r^+} y^a \left(|\nabla u_R|^2 + |\nabla v_R|^2 \right) \\ = \; & r \int_{\partial^+ B_r^+} y^a \left(|\nabla u_R|^2 + |\nabla v_R|^2 \right) - 2y^a \left(\left| \frac{\partial u_R}{\partial r} \right|^2 + \left| \frac{\partial v_R}{\partial r} \right|^2 \right) \\ & + r \int_{S_r^n} \kappa_R u_R^2 v_R^2 - n \int_{\partial^0 B_r^+} \kappa_R u_R^2 v_R^2. \end{aligned}$$

By Lemma 4.1, for all but countable $r \in (0, +\infty)$, we can pass to the limit in the above identity, which gives

$$\begin{aligned} & (n-1+a) \int_{B_r^+} y^a \left(|\nabla u_\infty|^2 + |\nabla v_\infty|^2 \right) \\ = \; & r \int_{\partial^+ B_r^+} y^a \left(|\nabla u_\infty|^2 + |\nabla v_\infty|^2 \right) - 2y^a \left(\left| \frac{\partial u_\infty}{\partial r} \right|^2 + \left| \frac{\partial v_\infty}{\partial r} \right|^2 \right). \end{aligned} \tag{4.11}$$

The following calculation is similar to the proof of Proposition 3.2. $\square$

Lemma 4.5. *For any $\lambda > 0$,*

$$u_\infty(\lambda z) = \lambda^d u_\infty(z), \quad v_\infty(\lambda z) = \lambda^d v_\infty(z).$$

Proof. By Corollary 4.2, $N_\infty(r) \equiv d$. Then by the previous lemma, for a.a. $r > 0$,

$$\frac{\int_{\partial^+ B_r^+} y^a \left(\left|\frac{\partial u_\infty}{\partial r}\right|^2 + \left|\frac{\partial v_\infty}{\partial r}\right|^2 \right)}{\int_{\partial^+ B_r^+} y^a \left(u_\infty \frac{\partial u_\infty}{\partial r} + v_\infty \frac{\partial v_\infty}{\partial r} \right)} - \frac{\int_{\partial^+ B_r^+} y^a \left(u_\infty \frac{\partial u_\infty}{\partial r} + v_\infty \frac{\partial v_\infty}{\partial r} \right)}{\int_{\partial^+ B_r^+} y^a \left(u_\infty^2 + v_\infty^2 \right)} = 0.$$

By the characterization of the equality case in the Cauchy inequality, there exists a $\lambda(r) > 0$, such that

$$\frac{\partial u_\infty}{\partial r} = \lambda(r) u_\infty, \quad \frac{\partial v_\infty}{\partial r} = \lambda(r) v_\infty \quad \text{on } \partial^+ B_r^+.$$

Integrating this in r, we then get two functions $g(r)$ defined on $(0, +\infty)$ and $(\varphi(\theta), \psi(\theta))$ defined on $\partial^+ B_1^+$, such that

$$u_\infty(r, \theta) = g(r)\varphi(\theta), \quad v_\infty(r, \theta) = g(r)\psi(\theta).$$

By (4.9), we must have $g(r) \equiv r^d$. □

Remark 4.6. *By definition, we always have $d > 0$. In the standard Laplacian case, we can show that d must be a positive integer (see* [3]*). However, we do not know if such a quantization phenomena holds for this problem. For related studies see* [22, 23]*.*

5. Classification of the blow down limit in dimension 2

From now on assume $n = 1$. In the previous section we proved that the blow down limit

$$u_\infty(r, \theta) = r^d \varphi(\theta), \quad v_\infty(r, \theta) = r^d \psi(\theta),$$

where the two functions φ and ψ are defined on $[0, \pi]$. In this section we determine the explicit form of φ and ψ.

5.1. **Classification.**

By denoting

$$L_\theta^a \varphi = \varphi_{\theta\theta} + a \cot\theta \varphi_\theta,$$

the equation for (φ, ψ) reads as

$$\begin{cases} L_\theta^a \varphi + d(d+a)\varphi = L_\theta^a \psi + d(d+a)\psi = 0, \text{ in } (0, \pi), \\ \varphi \partial_\theta^a \varphi = \psi \partial_\theta^a \psi = 0, \text{ at } \{0, \pi\}, \\ \varphi(0)\psi(0) = \varphi(\pi)\psi(\pi) = 0. \end{cases} \tag{5.1}$$

Here $\partial_\theta^a \phi(0) = \lim_{\theta \to 0} (\sin\theta)^a \varphi_\theta(\theta)$, and we have a similar definition at π.

There are two cases.

Case 1. $\varphi(0) \neq 0, \varphi(\pi) \neq 0$.

By this assumption, we have

$$\partial_\theta^a \varphi(0) = \partial_\theta^a \varphi(\pi) = 0. \tag{5.2}$$

Using the equation for φ, we know φ is continuous on $[0, \pi]$. Thus by our assumption $\varphi > 0$ on $[0, \pi]$.

Multiplying the equation of φ by $\varphi(\theta)^{-1}(\sin\theta)^a$ and integrating by parts on $[0,\pi]$, with the help of (5.2) we arrive at

$$-d\,(d+a)\int_0^\pi (\sin\theta)^a\,d\theta = \int_0^\pi (\sin\theta)^a\,\frac{\varphi'(\theta)^2}{\varphi(\theta)^2}d\theta \geq 0. \tag{5.3}$$

On the other hand, multiplying the equation of φ by $\varphi(\theta)(\sin\theta)^a$ and integrating by parts on $[0,\pi]$, we have

$$d\,(d+a) = \frac{\int_0^\pi (\sin\theta)^a\,\varphi'(\theta)^2 d\theta}{\int_0^\pi (\sin\theta)^a\,\varphi(\theta)^2 d\theta} \geq 0. \tag{5.4}$$

Combining (5.3) and (5.4), we see

$$d(d+a) = 0.$$

Since $d > 0$, we must have $d = -a = 2s-1$. Note that this is only possible when $s > 1/2$. This then implies that φ is a constant.

In this case we must have $\psi(0) = \psi(\pi) = 0$. We claim that $\psi \equiv 0$. In fact, since v_∞ is homogeneous of degree $d = 2s-1$ and L_a-harmonic in $\mathbb{R}^2_+$, by [23, Proposition 7], $v_\infty \equiv 0$ in $\mathbb{R}^2_+$.

We conclude that in this case φ is a constant and $\psi \equiv 0$. Note that this is possible only if $s > \frac{1}{2}$.

In the subsection below, we shall prove that this is impossible.

Case 2. $\varphi(0) \neq 0, \varphi(\pi) = 0$ or $\psi(0) = 0, \psi(\pi) \neq 0$.

By this assumption, $\partial^a_\theta \varphi(0) = 0$. Hence we can extend φ to an even function in $[-\pi,\pi]$. It satisfies

$$\begin{cases} L^a_\theta \varphi + d(d+a)\varphi = 0, \text{ in } (-\pi,\pi), \\ \varphi > 0, \text{ in } (-\pi,\pi), \\ \varphi(-\pi) = \varphi(\pi) = 0. \end{cases}$$

In other words, φ is the first eigenfunction of L^a_θ in $H^1_0((-\pi,\pi))$. Then it can be directly checked that, up to the multiplication of a positive constant, $\varphi(\theta) = (\cos\frac{\theta}{2})^{2s}$. Similarly, $\psi(\theta) = (\sin\frac{\theta}{2})^{2s}$. Moreover, $d = s$ in this case.

By Corollary 4.2, either $\lim_{R\to+\infty} N(R) = s$ or $\lim_{R\to+\infty} N(R) = 2s-1$ (when $s > 1/2$).

5.2. **Self-segregation.** Here we exclude the possibility that the blow down limit $(\varphi,\psi) = (1,0)$ when $s > 1/2$.

Assume the blow down limit $(\varphi,\psi) = (1,0)$. First we claim that

Lemma 5.1. *There exists a constant $c > 0$ such that*

$$u \geq c \quad on\ \partial\mathbb{R}^2_+.$$

Proof. Assume that we have a sequence R_i such that $u(R_i,0) \to 0$. Then necessarily $R_i \to \infty$. Let (u_{R_i}, v_{R_i}) be the blow down sequence defined as before. Then (u_{R_i}, v_{R_i}) converges to $(r^{2s-1}, 0)$ (modulo a normalization constant) in $C_{loc}(\overline{\mathbb{R}^2_+})$. However, by our assumption, because $L(R_i) \to +\infty$ (see (4.3)),

$$u_{R_i}(1,0) = \frac{u(R_i,0)}{L(R_i)} \to 0,$$

which is a contradiction. □

By the bound on $N(R)$ and Proposition 3.4, there exists a constant C such that

$$\int_{B_r^+} y^a \left(u^2+v^2\right) \leq C r^{2+2(2s-1)}, \quad \forall\, r>1.$$

For each $r>1$, let

$$\tilde{u}(z)=u(rz), \quad \tilde{v}(z)=v(rz).$$

Then $\tilde{v}$ satisfies

$$\begin{cases} L_a \tilde{v}=0, \ \text{in } B_1^+, \\ \partial_y^a \tilde{v} = r^{1-a}\tilde{u}^2\tilde{v} \geq c r^{1-a}\tilde{v}, \quad \text{on } \partial^0 B_1^+. \end{cases}$$

Here we have used the previous lemma which says $\tilde{u}\geq c$ on $\partial^0 B_1^+$.

Applying Lemma A.3, we obtain

$$\sup_{\partial^0 B_{1/2}^+} \tilde{v} \leq C r^{-1}.$$

Letting $r\to+\infty$, we see $v\equiv 0$ on $\partial\mathbb{R}^2_+$.

Now since the growth bound of v is controlled by r^{2s-1}, applying [23, Proposition 7], we get $v\equiv 0$ in $\mathbb{R}^2_+$.

The equation for u becomes

$$\begin{cases} L_a u=0, \ \text{in } \mathbb{R}^2_+, \\ \partial_y^a u = 0, \quad \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

Because the growth bound of u is controlled by r^{2s-1}, applying [23, Corollary 2], u is a constant. This is a contradiction with the condition on $N(R)$.

5.3. Combining the results in the previous two subsections, we have proved that the blow down limit must be

$$u_\infty = \alpha_+ r^s (\cos\frac{\theta}{2})^{2s}, \quad v_\infty=\alpha_- r^s(\sin\frac{\theta}{2})^{2s}, \tag{5.5}$$

for two suitable positive constants α_+ and α_-.

Here we note that, the blow down limit cannot be

$$u_\infty = \beta_+ r^s (\sin\frac{\theta}{2})^{2s}, \quad v_\infty=\beta_- r^s(\cos\frac{\theta}{2})^{2s}. \tag{5.6}$$

In other words, only one of the above two limits is possible and the blow down limit must be unique (the constant a_+ and a_- will be shown to be independent of the choice of subsequences $R_i\to+\infty$ in the next section). For example, if both these two arise as the blow down limit (from different subsequence of $R\to+\infty$), then we can find a sequence of $R_i\to+\infty$ satisfying $u(R_i,0)=v(R_i,0)$. Using these R_i to define the blow down sequence, we get a blow down limit (u_∞,v_∞) satisfying $u_\infty(1,0)=v_\infty(1,0)$. This is a contradiction with the two forms given above.

Lemma 5.2. *$\alpha_+=\alpha_-$.*

This can be proved by the Pohozaev identity for (u_∞, v_∞), (4.11), where we replace the ball B_r^+ by $\overline{B_r^+(t,0)}$ and let t vary (cf. [21]). We can also use the stationary condition: for any $X \in C_0^\infty(\overline{\mathbb{R}^2_+}, \mathbb{R}^2)$ satisfying $X^2 = 0$ on $\partial\mathbb{R}^2_+$, we have

$$\int_{\mathbb{R}^2_+} y^a \left(|\nabla u_\infty|^2 + |\nabla v_\infty|^2\right) \mathrm{div}X - 2y^a DX(\nabla u_\infty, \nabla u_\infty)$$
$$-2y^a DX(\nabla v_\infty, \nabla v_\infty) + a y^{a-1} X^2 \left(|\nabla u_\infty|^2 + |\nabla v_\infty|^2\right) = 0.$$

We have proved that the blow down limit (u_∞, v_∞) satisfies

$$\int_{\partial^+ B_1^+} y^a u_\infty^2 = \int_{\partial^+ B_1^+} y^a v_\infty^2.$$

By the analysis in Section 4, for any $R_i \to +\infty$, the blowing down sequences (u_{R_i}, v_{R_i}) satisfy

$$\lim_{R_i\to+\infty} \int_{\partial^+ B_1^+} y^a u_{R_i}^2 = \int_{\partial^+ B_1^+} y^a u_\infty^2, \quad \lim_{R_i\to+\infty} \int_{\partial^+ B_1^+} y^a v_{R_i}^2 = \int_{\partial^+ B_1^+} y^a v_\infty^2.$$

By a compactness argument, we get a constant C so that for all $R \geq 1$,

$$\frac{1}{C} \leq \frac{\int_{\partial^+ B_R^+} y^a u^2}{\int_{\partial^+ B_R^+} y^a v^2} \leq C. \tag{5.7}$$

6. Growth bound

In this section we prove various growth bound and decay estimates for u and v.

Proposition 6.1 (**Upper bound**). *There exists a constant C so that*

$$u(z) + v(z) \leq C\left(1 + |z|\right)^s.$$

Proof. Because for any r, $N(r) \leq s$. Proposition 3.4 implies that

$$H(r) \leq H(1) r^{2s}, \quad \forall r > 1.$$

Then because the even extension of u to $\mathbb{R}^2$ is L_a-subharmonic, by Lemma A.2 we get

$$\sup_{B_{r/2}} u \leq C H(r)^{1/2} \leq C H(1)^{1/2} r^s.$$ □

Because for any $R > 0$, $N(R) \leq s$, the bound on $H(r)$ also gives

Corollary 6.2. *For any $R > 1$,*

$$\int_{B_R^+} y^a \left(|\nabla u|^2 + |\nabla v|^2\right) + \int_{\partial^0 B_R^+} u^2 v^2 \leq CR.$$

Next we give a lower bound for the growth of u and v.

Proposition 6.3 (**Lower bound**). *There exists a constant c such that*

$$\int_{\partial^+ B_r^+} y^a u^2 \geq c r^2, \quad \int_{\partial^+ B_r^+} y^a v^2 \geq c r^2, \quad \forall\ r > 1. \tag{6.1}$$

We first present two lemmas needed in the proof of this proposition.

Lemma 6.4. *For any $K > 0$, there exists an $R(K)$ such that $\{Kx > y > 0\} \cap B^c_{R(K)} \subset \{u > v\}$ and $\{-Kx > y > 0\} \cap B^c_{R(K)} \subset \{u < v\}$.*

Proof. This is because, there exists a $\delta(K) > 0$ so that for any $R \geq R(K)$,

$$\sup_{\overline{B_1^+}} \left|u^R - \alpha r^s \left(\cos\frac{\theta}{2}\right)^{2s}\right| + \left|v^R - \alpha r^s \left(\sin\frac{\theta}{2}\right)^{2s}\right| \leq \delta(K),$$

and

$$\alpha r^s \left(\cos\frac{\theta}{2}\right)^{2s} \geq \alpha r^s \left(\sin\frac{\theta}{2}\right)^{2s} + \delta(K), \quad \text{in } \{Kx > y > 0\} \cap \left(\overline{B_1^+} \setminus B_{1/2}^+\right).$$

These two imply that $u^R > v^R$ in $\overline{B_1^+} \setminus B_{1/2}^+$. By noting that this holds for any $R \geq R(K)$, we complete the proof. □

Lemma 6.5. *As $x \to +\infty$, $u(x,0) \to +\infty$ and $v(x,0) \to 0$. As $x \to -\infty$, $v(x,0) \to +\infty$ and $u(x,0) \to 0$.*

Proof. For any $\lambda > 0$ large, let

$$u^\lambda(x,y) := \lambda^{-s} u(\lambda x, \lambda y), \quad v^\lambda(x,y) := \lambda^{-s} v(\lambda x, \lambda y).$$

By the previous lemma and Proposition 6.1,

$$u^\lambda \geq v^\lambda, \quad v^\lambda \leq C \quad \text{on } \overline{B_{1/2}^+(1,0)}.$$

v^λ satisfies

$$\begin{cases} L_a v^\lambda = 0, \text{ in } B_{1/2}^+(1,0), \\ \partial_y^a v^\lambda = \lambda^{4s} \left(u^\lambda\right)^2 v^\lambda \geq \lambda^{4s} \left(v^\lambda\right)^3, \text{ on } \partial^0 B_{1/2}^+(1,0). \end{cases}$$

Then $(v^\lambda - \lambda^{-\frac{4s}{3}})^+$ satisfies

$$\begin{cases} L_a (v^\lambda - \lambda^{-\frac{4s}{3}})^+ \geq 0, \text{ in } B_{1/2}^+(1,0), \\ \partial_y^a (v^\lambda - \lambda^{-\frac{4s}{3}})^+ \geq \lambda^{\frac{4s}{3}} (v^\lambda - \lambda^{-\frac{4s}{3}})^+, \text{ on } \partial^0 B_{1/2}^+(1,0). \end{cases}$$

By Lemma A.3 we get

$$\sup_{\partial^0 B_{1/4}^+(1,0)} v^\lambda \leq \sup_{\partial^0 B_{1/4}^+(1,0)} (v^\lambda - \lambda^{-\frac{4s}{3}})^+ + \lambda^{-\frac{4s}{3}} \leq C\lambda^{-\frac{4s}{3}}.$$

Rescaling back we get $v(\lambda, 0) \leq C\lambda^{-s/3}$ for all λ large.

Next assume that there exists $\lambda_i \to +\infty$, $u(\lambda_i, 0) \leq M$ for some $M > 0$. Then by defining the blow down sequence $(u^{\lambda_i}, v^{\lambda_i})$ as before, following the proof of Lemma 5.1 we can get a contradiction. Indeed, the blow down analysis gives $u^{\lambda_i}(1,0) \to \alpha$ for some constant $\alpha > 0$, while our assumption and (4.3) implies that

$$u^{\lambda_i}(1,0) \leq CM\lambda_i^{-s} \to 0.$$

This is a contradiction. □

Now we can prove Proposition 6.3.

Proof of Proposition 6.3. By the previous lemma, there exists a constant M^* such that $w_1 := (u-M^*)_+$ and $w_2 := (v-M^*)_+$ have disjoint supports on $\partial\mathbb{R}^2_+$. Both of these two functions are nonnegative, continuous and L_a-subharmonic. By assuming $M^* > \max\{u(0,0), v(0,0)\}$, $w_1(0,0) = w_2(0,0) = 0$. Moreover, they satisfy

$$\begin{cases} w_i L_a w_i = 0, \text{ in } \mathbb{R}^2_+, \\ \partial^a_y w_i \geq 0 \text{ on } \partial\mathbb{R}^2_+. \end{cases}$$

This then implies that for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2_+} y^a \nabla w_i \cdot \nabla (w_i \phi) = -\int_{\partial\mathbb{R}^2_+} w_i \partial^a_y w_i \phi \leq 0. \tag{6.2}$$

Then by [23, Proposition 4] (note that here the dimension $n=1$ and hence in that proposition the exponent $\nu^{ACF} = s$),

$$J(r) := r^{-4s} \left(\int_{B_r^+} y^a \frac{|\nabla w_1|^2}{|z|^a} \right) \left(\int_{B_r^+} y^a \frac{|\nabla w_2|^2}{|z|^a} \right)$$

is non-decreasing in $r > 0$. This then implies the existence of a constant c so that

$$\left(\int_{B_r^+} y^a \frac{|\nabla w_1|^2}{|z|^a} \right) \left(\int_{B_r^+} y^a \frac{|\nabla w_2|^2}{|z|^a} \right) \geq c r^{4s}, \quad \forall\, r > R^*. \tag{6.3}$$

Here we choose R^* large so that w_1 and w_2 are not constant in $B^+_{R^*}$, which implies

$$\int_{B^+_{R^*}} y^a \frac{|\nabla w_1|^2}{|z|^a} \geq c, \quad \int_{B^+_{R^*}} y^a \frac{|\nabla w_2|^2}{|z|^a} \geq c,$$

where $c > 0$ is a constant depending on the solution (u, v) and R^*.

Take an $\eta \in C_0^\infty(B_2)$ such that $\eta \equiv 1$ in B_1. For any $r > 1$, let $\eta^r(z) = \eta(r^{-1}z)$. Substituting $\phi = (\eta^r)^2 |z|^{-a}$ into (6.2) and integrating by parts gives (cf. the derivation of [23, Eq. (4)])

$$\int_{B_r^+} y^a \frac{|\nabla w_i|^2}{|z|^a} \leq C r^{-2-a} \int_{B^+_{2r} \setminus B^+_r} y^a w_i^2. \tag{6.4}$$

Substituting this into (6.3) leads to

$$\int_{B^+_{2r}} y^a \left(u^2 + v^2\right) \geq \int_{B^+_{2r}} y^a \left(w_1^2 + w_2^2\right) \geq c r^{2+a+2s}.$$

Because u^2 and v^2 are L_a-subharmonic, by the mean value inequality, this can be transformed to

$$\int_{\partial^+ B_r^+} y^a \left(u^2 + v^2\right) \geq c r^2, \quad \forall r > 2.$$

Then by noticing (5.7), we finish the proof. □

Remark 6.6. *With Proposition 6.1 and Proposition 6.3 in hand, in the blow down analysis we can choose*

$$u^R(z) := R^{-s} u(Rz), \quad v^R(z) := R^{-s} v(Rz).$$

By the blow down analysis, for any $R_i \to +\infty$, there exists a subsequence of R_i (still denoted by R_i) such that

$$u^{R_i} \to br^s\left(\cos\frac{\theta}{2}\right)^{2s}, \quad v^{R_i} \to br^s\left(\sin\frac{\theta}{2}\right)^{2s},$$

in $C(\overline{B_1^+}) \cap H^{1,a}(B_1^+)$, for some constant $b > 0$.

We claim that b is independent of the sequence R_i, thus the blow down limit is unique. By (6.4) *and Proposition 6.1,*

$$\lim_{R\to+\infty} J(R) < +\infty, \tag{6.5}$$

where the limit exists because $J(R)$ is non-decreasing.

For each R, let $w_1^R = (u_R - M^* R^{-s})_+ = R^{-s} w_1(Rz)$ and $w_2^R = (v_R - M^* R^{-s})_+ = R^{-s} w_2(Rz)$. Then a rescaling gives

$$J(R_i) = \left(\int_{B_1^+} y^a \frac{|\nabla w_1^{R_i}|^2}{|z|^a}\right)\left(\int_{B_1^+} y^a \frac{|\nabla w_2^{R_i}|^2}{|z|^a}\right).$$

For any $\delta > 0$ small, by (6.4),

$$\lim_{i\to+\infty} \int_{B_\delta^+} y^a \frac{|\nabla w_1^{R_i}|^2}{|z|^a} \le C \lim_{i\to+\infty}\left(\sup_{B_{2\delta}^+} w_1^{R_i}\right)^2 = O(\delta^{2s}),$$

because $w_1^{R_i}$ converges uniformly to $br^s\left(\cos\frac{\theta}{2}\right)^{2s}$. Using this estimate and the strong convergence of $w_1^{R_i}$ in $H^{1,a}(B_1^+)$, we obtain

$$\begin{aligned}\lim_{i\to+\infty} \int_{B_1^+} y^a \frac{|\nabla w_1^{R_i}|^2}{|z|^a} &= \lim_{i\to+\infty} \int_{B_1^+\setminus B_\delta^+} y^a \frac{|\nabla w_1^{R_i}|^2}{|z|^a} + \lim_{i\to+\infty} \int_{B_\delta^+} y^a \frac{|\nabla w_1^{R_i}|^2}{|z|^a} \\ &= b^2 \int_{B_1^+\setminus B_\delta^+} y^a \frac{|\nabla r^s\left(\cos\frac{\theta}{2}\right)^{2s}|^2}{|z|^a} + O(\delta^{2s}).\end{aligned}$$

After applying (6.4) *to $br^s\left(\cos\frac{\theta}{2}\right)^{2s}$ and letting $\delta \to 0$, this gives*

$$\lim_{i\to+\infty} \int_{B_1^+} y^a \frac{|\nabla w_1^{R_i}|^2}{|z|^a} = C(s) b^2,$$

where $C(s)$ is a constant depending only on s.

Substituting this into (6.5) *we get*

$$C(s)^2 b^4 = \lim_{R\to+\infty} J(R).$$

Thus b does not depend on the choice of subsequence R_i.

After a scaling $(u(z), v(z)) \mapsto (\lambda^s u(\lambda z), \lambda^s v(\lambda z)$ with a suitable λ, which leaves the equation (1.12) *invariant, we can assume $b = 1$. That is, as $R \to +\infty$,*

$$R^{-s}u(Rz) \to r^s\left(\cos\frac{\theta}{2}\right)^{2s}, \quad R^{-s}v(Rz) \to r^s\left(\sin\frac{\theta}{2}\right)^{2s}.$$

By (6.1) and Proposition 6.1, there exist two constants $c_1, c_2 > 0$, such that

$$\{u > c_1 r^s\} \cap \partial^+ B_r^+ \cap \{y \geq c_2|x|\} \neq \emptyset.$$

Since u is positive L_a-harmonic in $\mathbb{R}^2_+$, by applying the Harnack inequality to a chain of balls (with the number of balls depending only on ε), for any $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that

$$u(z) \geq c(\varepsilon)|z|^s, \quad v(z) \geq c(\varepsilon)|z|^s \quad \text{in } \{y \geq \varepsilon|x|\}. \tag{6.6}$$

Lemma 6.7. *For any $\varepsilon, \delta > 0$, there exists a constant $R(\varepsilon, \delta)$ such that*

$$v(z) \leq \delta|z|^s, \quad \text{in } \{(x, y) : x \geq R(\varepsilon, \delta), 0 \leq y \leq \varepsilon(x - R(\varepsilon, \delta))\}. \tag{6.7}$$

Proof. By Proposition 6.1 and Proposition 6.3, in the definition of blow down sequence we can take

$$u_R(z) = R^{-s}u(Rz), \quad v_R(z) = R^{-s}v(Rz).$$

As $R \to +\infty$, (u_R, v_R) converges to $(r^s(\cos\frac{\theta}{2})^{2s}, r^s(\sin\frac{\theta}{2})^{2s})$ uniformly in B_1^+. Thus we can choose an ε depending only on δ so that for all R large, $v_R \leq \delta$ in $B_1^+ \cap \{0 \leq y \leq \varepsilon x^+\}$. $\square$

Lemma 6.8. *For any $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that*

$$u(z) \geq c(\varepsilon)|z|^s, \quad \text{in } \{y \geq \varepsilon x_-\}. \tag{6.8}$$

Proof. In view of (6.6) we only need to give a lower bound in the domain $\mathcal{C} := \{(x, y) : x \geq R_0, 0 \leq y \leq \varepsilon(x - R_0)\}$, where R_0 is large but fixed.

$u - v$ is L_a-harmonic in $\mathcal{C}$, satisfying the following boundary conditions (thanks to Lemma 6.4)

$$\begin{cases} u - v \geq c(\varepsilon)|z|^s, \text{ on } \{y = \varepsilon(x - R_0)\} \cap \mathcal{C}, \\ \partial_y^a(u - v) = uv^2 - vu^2 \leq 0, \text{ on } \{y = 0\} \cap \partial\mathcal{C}. \end{cases}$$

We claim that $u - v \geq c(\varepsilon, R_0)r^s$ in $\mathcal{C}$.

First, let $\psi(\theta)$ be the solution of

$$\begin{cases} -L_\theta^a \psi = d(d + a)\psi, \text{ in } \{-\varepsilon < \theta < \varepsilon\}, \\ \psi > 0, \text{ in } \{-\varepsilon < \theta < \varepsilon\}, \\ \psi(-\varepsilon) = \psi(\varepsilon) = 0. \end{cases}$$

Here d is determined by

$$d(d + a) = \min \frac{\int_{-\varepsilon}^{\varepsilon} \psi'(\theta)^2 \left|\sin\theta\right|^a d\theta}{\int_{-\varepsilon}^{\varepsilon} \psi(\theta)^2 \left|\sin\theta\right|^a d\theta},$$

in the class of functions satisfying $\psi(-\varepsilon) = \psi(\varepsilon) = 0$.

This minima can be bounded from below by

$$c \min_{\eta \in C_0^\infty((-\varepsilon,\varepsilon))} \frac{\int_{-\varepsilon}^{\varepsilon} |x|^a \eta'(x)^2 dx}{\int_{-\varepsilon}^{\varepsilon} |x|^a \eta(x)^2 dx} \geq \frac{c}{\varepsilon^2} \min_{\eta \in C_0^\infty((-1,1))} \frac{\int_{-1}^{1} |x|^a \eta'(x)^2 dx}{\int_{-1}^{1} |x|^a \eta(x)^2 dx} \geq \frac{c}{\varepsilon^2}.$$

In particular, if ε is small enough, $d > s$. Note that $\phi := r^d\psi(\theta)$ is a positive L_a-harmonic function in the cone $\{|\theta| < 2\varepsilon\}$. Moreover, since ψ is even in θ (by the uniqueness of the first eigenfunction), ϕ is even in y.

For ε sufficiently small, we have got a positive L_a-harmonic function ϕ in the cone $\{|y| \leq 2\varepsilon x\}$, satisfying $\phi \geq |z|^{2s}$ in $\mathcal{C}$. Apparently, $\partial_y^a |z|^s = \partial_y^a \phi = 0$ on $\{y = 0\}$. Then we can apply the maximum principle to

$$\frac{u - v - c(\varepsilon)|z|^s}{\phi},$$

to deduce that it is nonnegative in $\mathcal{C}$. $\square$

Proposition 6.9 (Decay estimate). *For all $x > 0$, $v(x, 0) \leq C(1 + x)^{-3s}$. For all $x < 0$, $u(x, 0) \leq C(1 + |x|)^{-3s}$.*

Proof. For any $\lambda > 0$ large, let

$$u^\lambda(x, y) := \lambda^{-s} u(\lambda x, \lambda y), \quad v^\lambda(x, y) := \lambda^{-s} v(\lambda x, \lambda y).$$

By the previous lemma and Proposition 6.1,

$$u^\lambda \geq c, \quad v^\lambda \leq C \quad \text{in } B^+_{1/2}(1, 0).$$

The equation for v^λ is

$$\begin{cases} L_a v^\lambda = 0, \text{ in } B^+_{1/2}(1, 0), \\ \partial_y^a v^\lambda = \lambda^{4s} \left(u^\lambda\right)^2 v^\lambda \geq c\lambda^{4s} v^\lambda, \text{ on } \partial B^+_{1/2}(1, 0). \end{cases}$$

By Lemma A.3,

$$v^\lambda(1, 0) \leq C\lambda^{-4s}.$$

This then gives the estimate for $v(\lambda, 0)$. $\square$

Before proving a similar decay estimate for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$, we first give an upper bound for the gradient of u and v.

Proposition 6.10. *There exists a constant C such that,*

$$\left|\frac{\partial u}{\partial x}(x, y)\right| + \left|\frac{\partial v}{\partial x}(x, y)\right| \leq C(1 + |x| + |y|)^{s-1}.$$

$$\left|y^a \frac{\partial u}{\partial y}(x, y)\right| + \left|y^a \frac{\partial v}{\partial y}(x, y)\right| \leq C(1 + |x| + |y|)^{-s}.$$

Proof. For all λ large, consider (u^λ, v^λ) introduced in the proof of the previous proposition. It satisfies

$$\begin{cases} L_a u^\lambda = L_a v^\lambda = 0, \text{ in } B_1^+, \\ \partial_y^a u^\lambda = \lambda^{4s} u^\lambda \left(v^\lambda\right)^2, \quad \partial_y^a v^\lambda = \lambda^{4s} v^\lambda \left(u^\lambda\right)^2, \quad \text{on } \partial^0 B_1^+. \end{cases} \tag{6.9}$$

By Proposition 6.1, u^λ and v^λ are uniformly bounded in $\overline{B_1^+}$. Then by the gradient estimate Theorem 2.1,

$$\sup_{\{y \geq |x|/2\} \cap (B_1^+ \setminus B_{1/2}^+)} |\nabla u^\lambda| + |\nabla v^\lambda| \leq C.$$

Rescaling back this gives the claimed estimates in the part $\{y \geq |x|/2\}$. (Note that here y^a is comparable to $(|x| + y)^a$.)

Next we consider the part $D := \{0 \le y < x\} \cap (B_1^+ \setminus B_{1/2}^+)$. Here by differentiating (6.9) we obtain

$$\begin{cases} L_a \dfrac{\partial u^\lambda}{\partial x} = L_a \dfrac{\partial v^\lambda}{\partial x} = 0, \text{ in } D, \\ \partial_y^a \dfrac{\partial u^\lambda}{\partial x} = \lambda^{4s} \left(v^\lambda\right)^2 \dfrac{\partial u^\lambda}{\partial x} + 2\lambda^{4s} u^\lambda v^\lambda \dfrac{\partial v^\lambda}{\partial x}, \quad \text{on } \partial^0 D, \\ \partial_y^a \dfrac{\partial v^\lambda}{\partial x} = \lambda^{4s} \left(u^\lambda\right)^2 \dfrac{\partial v^\lambda}{\partial x} + 2\lambda^{4s} u^\lambda v^\lambda \dfrac{\partial u^\lambda}{\partial x}, \quad \text{on } \partial^0 D. \end{cases}$$

By Corollary 6.2,

$$\int_D y^a \left(\left| \frac{\partial u^\lambda}{\partial x} \right|^2 + \left| \frac{\partial v^\lambda}{\partial x} \right|^2 \right) \le C,$$

for a constant C independent of λ.

By Proposition 6.9, $v^\lambda \le C\lambda^{-4s}$ on $\partial^0 D$. Thus the coefficient $2\lambda^{4s} u^\lambda v^\lambda$ is uniformly bounded on $\partial^0 D$. Although $\lambda^{4s}\left(u^\lambda\right)^2$ is not uniformly bounded, it has a favorable sign. Then standard Moser iteration (see for example [20, Theorem 1.2]) gives

$$\sup_{\{0 \le y < x/2\} \cap (B_{3/4}^+ \setminus B_{2/3}^+)} \left| \frac{\partial u^\lambda}{\partial x} \right| + \left| \frac{\partial v^\lambda}{\partial x} \right| \le C,$$

for a constant C independent of λ.

Finally, similar to the proof of Lemma 4.3, we have

$$\begin{cases} L_{-a} \left(y^a \dfrac{\partial u^\lambda}{\partial y} \right) = L_{-a} \left(y^a \dfrac{\partial v^\lambda}{\partial y} \right) = 0, \text{ in } D, \\ y^a \dfrac{\partial u^\lambda}{\partial y} = \lambda^{4s} u^\lambda \left(v^\lambda\right)^2 \in (0, C), \quad \text{on } \partial^0 D, \\ y^a \dfrac{\partial v^\lambda}{\partial y} = \lambda^{4s} v^\lambda \left(u^\lambda\right)^2 \in (0, C), \quad \text{on } \partial^0 D. \end{cases}$$

Moreover, by Corollary 6.2,

$$\int_D y^{-a} \left(\left| y^a \frac{\partial u^\lambda}{\partial y} \right|^2 + \left| y^a \frac{\partial v^\lambda}{\partial y} \right|^2 \right) \le C,$$

for a constant C independent of λ.

Then by applying the Moser iteration to $(y^a \frac{\partial u^\lambda}{\partial y} - C)_+$ and $(y^a \frac{\partial u^\lambda}{\partial y} + C)_-$, we see

$$\sup_{\{0 \le y < x/2\} \cap (B_{3/4}^+ \setminus B_{2/3}^+)} \left| y^a \frac{\partial u^\lambda}{\partial y} \right| + \left| y^a \frac{\partial v^\lambda}{\partial y} \right| \le C,$$

for a constant C independent of λ. □

Written in polar coordinates, this reads as

Corollary 6.11. *There exists a constant C such that,*

$$\left| \frac{\partial u}{\partial r}(r, \theta) \right| + \left| \frac{\partial v}{\partial r}(r, \theta) \right| \le C(1 + r)^{s-1}.$$

$$\left|\frac{(\sin\theta)^a}{r}\frac{\partial u}{\partial\theta}(r,\theta)\right|+\left|\frac{(\sin\theta)^a}{r}\frac{\partial v}{\partial\theta}(r,\theta)\right|\leq C(1+r)^{s-1}.$$

Proof. We have

$$\frac{\partial u}{\partial r}=\cos\theta\frac{\partial u}{\partial x}+(\sin\theta)^{1-a}(\sin\theta)^a\frac{\partial u}{\partial y},$$

$$\frac{(\sin\theta)^a}{r}\frac{\partial u}{\partial\theta}(r,\theta)=-(\sin\theta)^{1+a}\frac{\partial u}{\partial x}+\cos\theta\,(\sin\theta)^a\frac{\partial u}{\partial y}.$$

Since $1+a>0$ and $1-a>0$, $(\sin\theta)^{1-a}$ and $(\sin\theta)^{1+a}$ are bounded. Then this corollary follows from the previous proposition. $\square$

Finally we give a further decay estimate for $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$.

Proposition 6.12. *For all $x>0$, $\left|\frac{\partial v}{\partial x}(x,0)\right|\leq C(1+x)^{-3s-1}$. For all $x<0$, $\left|\frac{\partial u}{\partial x}(x,0)\right|\leq C(1+|x|)^{-3s-1}$.*

Proof. We use notations introduced in the proof of Proposition 6.9.
By differentiating the equation for v^λ, we obtain

$$\begin{cases} L_a\left(\frac{\partial v^\lambda}{\partial x}\right)_+\geq 0, \text{ in } B^+_{1/2}(1,0),\\ \partial^a_y\left(\frac{\partial v^\lambda}{\partial x}\right)_+\geq\lambda^{4s}\left(u^\lambda\right)^2\left(\frac{\partial v^\lambda}{\partial x}\right)_+-2\lambda^{4s}u^\lambda v^\lambda\left|\frac{\partial u^\lambda}{\partial x}\right|\left(\frac{\partial v^\lambda}{\partial x}\right)_+, \text{ on } \partial^0B^+_{1/2}(1,0).\end{cases}$$

By Proposition 6.9, $v^\lambda\leq C\lambda^{-4s}$ on $\partial^0B^+_{1/2}(1,0)$. By the previous proposition $\left|\frac{\partial u^\lambda}{\partial x}\right|\leq C$ in $\overline{B^+_{1/2}(1,0)}$. Lemma 6.8 also implies that $u^\lambda\geq c$ in $\overline{B^+_{1/2}(1,0)}$. Hence on $\partial^0B^+_{1/2}(1,0)$,

$$\partial^a_y\left(\frac{\partial v^\lambda}{\partial x}\right)_+\geq\left(c\lambda^{4s}-C\right)\left(\frac{\partial v^\lambda}{\partial x}\right)_+.$$

Applying Lemma A.3, we get

$$\frac{\partial v^\lambda}{\partial x}(1,0)\leq C\lambda^{-4s}.$$

The same estimate holds for the negative part. This then implies the bound for $\left|\frac{\partial v}{\partial x}(\lambda,0)\right|$. $\square$

7. Refined asymptotics at infinity

In this section we prove a refined asymptotic expansion of the solution (u,v). See Proposition 7.4 below. Here we need $s>\frac{1}{4}$. The refined asymptotic is needed for the method of moving planes in the next section.

7.1. Let
$$x = e^t \cos\theta, \quad y = e^t \sin\theta, \quad t \in \mathbb{R},\ \theta \in [0, \pi]$$
and
$$\bar{u}(t,\theta) = e^{-st} u(e^t\cos\theta, e^t\sin\theta), \quad \bar{v}(t,\theta) = e^{-st} v(e^t\cos\theta, e^t\sin\theta).$$
The equation (1.12) can be transformed to the one for $(\bar{u}, \bar{v})$,
$$\begin{cases} \bar{u}_{tt} + \bar{u}_t + s(1-s)\bar{u} + L^a_\theta \bar{u} = 0, \text{ in } (-\infty,+\infty)\times(0,\pi), \\ \bar{v}_{tt} + \bar{v}_t + s(1-s)\bar{v} + L^a_\theta \bar{v} = 0, \text{ in } (-\infty,+\infty)\times(0,\pi), \\ \lim\limits_{\theta\to 0 \ or \ \pi} \partial^a_\theta \bar{u} = \pm e^{4st}\bar{u}\bar{v}^2, \text{ on } (-\infty,+\infty)\times\{0,\pi\}, \\ \lim\limits_{\theta\to 0 \ or \ \pi} \partial^a_\theta \bar{v} = \pm e^{4st}\bar{v}\bar{u}^2, \text{ on } (-\infty,+\infty)\times\{0,\pi\}, \end{cases} \tag{7.1}$$
where we take the positive sign $+$ at $\{0\}$ and the negative one $-$ at $\{\pi\}$.

By Proposition 6.1,
$$0 \le \bar{u},\ \bar{v} \le C, \quad \text{in } [1,+\infty)\times[0,\pi]. \tag{7.2}$$
By Proposition 6.9,
$$\begin{cases} \bar{u} \le Ce^{-4st}, \text{ on } [1,+\infty)\times\{\pi\}, \\ \bar{v} \le Ce^{-4st}, \text{ on } [1,+\infty)\times\{0\}. \end{cases} \tag{7.3}$$
Combining Proposition 6.1 and Proposition 6.9, we also have
$$\begin{cases} 0 \le \partial^a_\theta \bar{u} \le Ce^{-4st}, \text{ on } [1,+\infty)\times\{0\} \\ 0 \ge \partial^a_\theta \bar{v} \ge -Ce^{-4st}, \text{ on } [1,+\infty)\times\{\pi\}. \end{cases} \tag{7.4}$$

What we have shown in Remark 6.6 is equivalent to the following statement.

Lemma 7.1. *As $t \to +\infty$, $\bar{u}(t,\theta) \to (\cos\frac{\theta}{2})^{2s}$ and $\bar{v}(t,\theta) \to (\sin\frac{\theta}{2})^{2s}$ uniformly in $[0,\pi]$.*

The next task is to get an exact convergence rate.

Proposition 7.2. *There exists a constant C so that*
$$\left|u(r,\theta) - r^s(\cos\frac{\theta}{2})^{2s}\right| + \left|v(r,\theta) - r^s(\sin\frac{\theta}{2})^{2s}\right| \le C(1+r)^{s-\min\{1,4s\}}.$$

In the following we denote
$$\sigma := \min\{1, 4s\}.$$

Proof. Let $\phi(t,\theta) := \bar{u}(t,\theta) - (\cos\frac{\theta}{2})^{2s}$. There exists a constant M such that
$$\begin{cases} \phi_{tt} + \phi_t + s(1-s)\phi + L^a_\theta \phi = 0, \text{ in } [1,+\infty)\times(0,\pi), \\ 0 \le \phi(t,\pi) \le Me^{-4st}, \\ 0 \le \partial^a_\theta \phi(t,0) \le Me^{-4st}. \end{cases} \tag{7.5}$$
Moreover, $|\phi| \le M$ in $[1,+\infty)\times[0,\pi]$ and it converges to 0 uniformly as $t \to +\infty$.

Let $\psi(t,\theta) := (\phi(t,\theta) - Me^{-4st})_+$. It satisfies

$$\begin{cases} \psi_{tt} + \psi_t + s(1-s)\psi + L^a_\theta \psi \geq -Ce^{-4st}, \text{ in } [1,+\infty) \times (0,\pi), \\ \psi(t,\pi) = 0, \\ \partial^a_\theta \psi(t,0) \geq 0. \end{cases} \tag{7.6}$$

We still have $0 \leq \psi \leq M$ in $[1,+\infty) \times [0,\pi]$, and ψ converges to 0 uniformly as $t \to +\infty$.

Define

$$f(t) := \int_0^\pi \psi(t,\theta) \left(\cos\frac{\theta}{2}\right)^{2s} (\sin\theta)^a \, d\theta \geq 0.$$

Multiplying (7.6) by $\left(\cos\frac{\theta}{2}\right)^{2s}$ and integrating on $(0,\pi)$ with respect to the measure $(\sin\theta)^a \, d\theta$, we obtain

$$f''(t) + f'(t) \geq f''(t) + f'(t) - \partial^a_\theta \psi(t,0) \geq -Ce^{-4st}. \tag{7.7}$$

This implies that

$$\left(e^t f'(t) + \frac{C}{1-4s} e^{(1-4s)t}\right)' \geq 0.$$

Consequently,

$$e^t f'(t) + \frac{C}{1-4s} e^{(1-4s)t} \geq e f'(1) + \frac{C}{1-4s} e^{1-4s} \geq -C, \quad \forall\, t \geq 1.$$

In other words,

$$f'(t) \geq -Ce^{-t} - \frac{C}{1-4s} e^{-4st} \geq -Ce^{-\sigma t} \quad \text{on } [1,+\infty).$$

This then implies that

$$f(t) - \frac{C}{\sigma} e^{-\sigma t}$$

is nondecreasing in t. Because

$$\lim_{t \to +\infty} \left[f(t) - \frac{C}{\sigma} e^{-\sigma t} \right] = 0,$$

we obtain

$$f(t) \leq Ce^{-\sigma t}, \quad \forall\, t \in [1,+\infty).$$

A similar estimate holds for ϕ_-.

Now we have got, for all $t \geq 1$,

$$\int_0^\pi \left| \bar{u}(t,\theta) - \left(\cos\frac{\theta}{2}\right)^{2s} \right| \left(\cos\frac{\theta}{2}\right)^{2s} (\sin\theta)^a \, d\theta \leq Ce^{-\sigma t}.$$

Then by standard estimates we get, for any $h > 0$,

$$\sup_{\theta \in (0,\pi-h)} \left| \bar{u}(t,\theta) - \left(\cos\frac{\theta}{2}\right)^{2s} \right| \leq \frac{C}{h} e^{-\sigma t}.$$

Next we extend this bound to $(\pi - h, \pi)$. Let

$$\varphi := \left(\phi - \max\{\frac{C}{h}e^{-\sigma t}, Me^{-4st}\}\right)_+, \quad \text{on } [1, +\infty) \times [\pi - h, \pi].$$

It satisfies

$$\begin{cases} \varphi_{tt} + \varphi_t + s(1-s)\varphi + L^a_\theta \varphi \geq -Ce^{-\sigma t}, \text{ in } (1, +\infty) \times (\pi - h, \pi), \\ \varphi(t, \pi) = \varphi(t, \pi - h) = 0. \end{cases} \tag{7.8}$$

Let

$$g(t) := \int_{\pi-h}^{\pi} \varphi(t, \theta)^2 (\sin\theta)^a \, d\theta.$$

We claim that the following Poincare inequality holds.

Claim. There exists a constant c, which is independent of h, so that

$$\frac{\int_{\pi-h}^{\pi} \left(\frac{\partial \varphi}{\partial \theta}(t,\theta)\right)^2 (\sin\theta)^a \, d\theta}{\int_{\pi-h}^{\pi} \varphi(t,\theta)^2 (\sin\theta)^a \, d\theta} \geq \frac{c}{h^2}.$$

This is because the left hand side can be bounded from below by

$$c \min_{\eta \in C_0^\infty((0,h))} \frac{\int_0^h x^a \eta'(x)^2 dx}{\int_0^h x^a \eta(x)^2 dx} \geq \frac{c}{h^2} \min_{\eta \in C_0^\infty((0,1))} \frac{\int_0^1 x^a \eta'(x)^2 dx}{\int_0^1 x^a \eta(x)^2 dx}.$$

Multiplying (7.8) by $\varphi (\sin\theta)^a$ and integrating on $(\pi - h, \pi)$ leads to

$$g''(t) + g'(t) - 2\left[\frac{c}{h^2} - s(1-s)\right] g(t) \geq -Ce^{-\sigma t} g(t)^{\frac{1}{2}} \geq -Ce^{-2\sigma t} - s(1-s)g(t).$$

Thus

$$g''(t) + g'(t) - \left[\frac{2c}{h^2} - 3s(1-s)\right] g(t) \geq -Ce^{-2\sigma t}.$$

Now we fix an h small so that

$$\frac{2c}{h^2} - 3s(1-s) > 4\sigma^2.$$

Because $g(t) \to 0$ as $t \to +\infty$, by the comparison principle,

$$g(t) \leq C\left(e^{-2\sigma t} + e^{-\frac{1+\sqrt{1+4\left(\frac{c}{h^2}-2s(1-s)\right)}}{2}t}\right) \leq Ce^{-2\sigma t}.$$

Then standard elliptic estimates imply that

$$\begin{aligned} \sup_{[\pi-h,\pi]} \left|\bar{u}(t,\theta) - \left(\cos\frac{\theta}{2}\right)^{2s}\right| &\leq \max\{\frac{C}{h}e^{-\sigma t}, Me^{-4st}\} + Ce^{-\sigma t} \\ &\leq Ce^{-\sigma t}. \end{aligned}$$

Coming back to u this gives the claimed estimate. $\square$

7.2. Now consider $\bar{u}_t$. By differentiation in t, $\bar{u}_t$ still satisfies the first equation in (7.1). Moreover, we have the following boundary conditions. At $\theta = \pi$, by Proposition 6.9 and Proposition 6.12,

$$\bar{u}_t(t,\pi) = -se^{-st}u(e^t,\pi) + e^{(1-s)t}u_r(e^t,\pi) = O(e^{-4st}).$$

Note that $\bar{u}_t(t,0)$ is bounded in $t \in [1,+\infty)$, which can be deduced from Proposition 6.9 and Proposition 6.10. At $\theta = 0$, because $|\bar{v}(t,0)| + |\bar{v}_t(t,0)| \le Ce^{-4st}$,

$$\begin{aligned} \partial_\theta^a \bar{u}_t &= 4se^{4st}\bar{u}\bar{v}^2 + e^{4st}\bar{v}^2\bar{u}_t + 2e^{4st}\bar{u}\bar{v}\bar{v}_t \\ &= O(e^{-4st}). \end{aligned}$$

Then as in the proof of the previous subsection, we have

Lemma 7.3. *As $t \to +\infty$,*

$$\sup_{\theta\in[0,\pi]} \left|\bar{u}_t(t,\theta)\right| + \left|\bar{v}_t(t,\theta)\right| \le Ce^{-\min\{1,4s\}t}. \tag{7.9}$$

Proof. For any $h > 0$, in $\{h < \theta < \pi - h\}$, by Proposition 7.2, (7.9) follows by applying the interior gradient estimates to $\bar{u}(t,\theta) - (\cos\theta/2)^{2s}$ and $\bar{v}(t,\theta) - (\sin\theta/2)^{2s}$. In the part $\{0 < \theta < h\}$ or $\{\pi - h < \theta < \pi\}$, if we have chosen h sufficiently small, the proof is exactly the same as in the last part of the proof of Proposition 7.2. $\square$

7.3. Now we assume $s > 1/4$. This implies $\sigma = 1$. Here we improve Proposition 7.2 to

Proposition 7.4. *There exist two constants α and β so that we have the expansion*

$$u(r,\theta) = r^s\left(\cos\frac{\theta}{2}\right)^{2s} + \alpha r^{s-1}\left(\cos\frac{\theta}{2}\right)^{2s} + o(r^{s-1}),$$

$$v(r,\theta) = r^s\left(\sin\frac{\theta}{2}\right)^{2s} + \beta r^{s-1}\left(\sin\frac{\theta}{2}\right)^{2s} + o(r^{s-1}).$$

Proof. Let

$$\tilde{u}(t,\theta) := e^t\left[\bar{u}(t,\theta) - \left(\cos\frac{\theta}{2}\right)^{2s}\right],$$

and $\tilde{v}$ be defined similarly.

By Proposition 7.2, $\tilde{u}$ is bounded on $[1,+\infty)\times[0,\pi]$. Moreover,

$$\tilde{u}_t(t,\theta) = e^t\left[\bar{u}(t,\theta) - \left(\cos\frac{\theta}{2}\right)^{2s}\right] + e^t\bar{u}_t(t,\theta),$$

is also uniformly bounded, thanks to the estimate in Lemma 7.3.

$\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_t + s(1-s)\tilde{u} + L_\theta^a\tilde{u} = 0, \text{ in } [1,+\infty)\times(0,\pi), \\ \left|\partial_\theta^a\tilde{u}(t,0)\right| \le Me^{-(4s-1)t}, \\ \left|\tilde{u}(t,\pi)\right| \le Me^{-(4s-1)t}. \end{cases} \tag{7.10}$$

Thus for any $t_i \to +\infty$, we can assume that $\tilde{u}(t_i + t, \theta)$ converges to a limit function $\tilde{u}^\infty$, weakly in $L^2_{loc}(\mathbb{R} \times [0, \pi])$. Here $\tilde{u}^\infty$ satisfies

$$\begin{cases} \tilde{u}^\infty_{tt} - \tilde{u}^\infty_t + s(1-s)\tilde{u}^\infty + L^a_\theta \tilde{u}^\infty = 0, \text{ in } \mathbb{R} \times (0, \pi), \\ \left|\tilde{u}^\infty(t, \theta)\right| \leq C, \\ \tilde{u}^\infty(t, \pi) = 0, \\ \partial^a_\theta \tilde{u}^\infty(t, 0) = 0. \end{cases} \tag{7.11}$$

Consider the eigenvalue problem

$$\begin{cases} -L^a_\theta \psi_j = \lambda_j \psi_j, \text{ in } (0, \pi), \\ \psi_j(\pi) = 0, \\ \partial^a_\theta \psi_j(0) = 0. \end{cases}$$

This problem has a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to +\infty$, and the corresponding eigenfunctions are denoted by ψ_j, which is normalized in $L^2((0, \pi), (\sin \theta)^a d\theta)$. Here the first eigenvalue $\lambda_1 = s(1-s)$ and the corresponding eigenfunction $\psi_1(\theta) = (\cos \frac{\theta}{2})^{2s}$ (modulo a constant) is positive in $(0, \pi)$.

Consider the decomposition

$$\tilde{u}^\infty(t, \theta) = \sum_{j=1}^{\infty} c_j(t) \psi_j(\theta).$$

Then $c_j(t)$ satisfies

$$c_j'' - c_j' + \left[s\left(1-s\right) - \lambda_j\right] c_j = 0.$$

Note that $|c_j(t)| \leq C$ for all t. Combined with the above equation, we see $c_j \equiv 0$ for all $j \geq 2$, and $c_1(t)$ is a constant.

Now we show that this constant does not depend on the sequence $t_i \to +\infty$. Let

$$f(t) := \int_0^\pi \tilde{u}(t, \theta) \left(\cos \frac{\theta}{2}\right)^{2s} (\sin \theta)^a \, d\theta.$$

By the bound on $\tilde{u}$ and $\tilde{u}_t$, $f(t)$ and

$$f'(t) = \int_0^\pi \tilde{u}_t(t, \theta) \left(\cos \frac{\theta}{2}\right)^{2s} (\sin \theta)^a \, d\theta$$

are bounded on $[1, +\infty)$. Multiplying the equation in (7.10) by $\left(\cos \frac{\theta}{2}\right)^{2s} (\sin \theta)^a$ and integrating by parts leads to

$$f''(t) - f'(t) = -\partial^a_\theta \tilde{u}(t, 0) - 2^a s \tilde{u}(t, \pi) = O(e^{-(4s-1)t}).$$

In particular, $f''(t)$ is also bounded on $[1, +\infty)$

For any $t_i \to +\infty$, we can assume that $f(t_i + t)$ converges to a limit $f_\infty(t)$ in $C^1_{loc}(\mathbb{R})$, which satisfies

$$f''_\infty(t) - f'_\infty(t) = 0.$$

Because f_∞ is bounded on $\mathbb{R}$, it must be a constant. Thus $f'_\infty \equiv 0$. This implies that $f'(t) \to 0$ as $t \to +\infty$.

Now we also have

$$\left(e^{-t}f'(t)\right)' = O(e^{-4st})$$

Integrating this on $[t, +\infty)$, we obtain

$$\left|f'(t)\right| = O(e^{-(4s-1)t}).$$

Hence there exists a constant α such that

$$\left|f(t) - \alpha\right| = O(e^{-(4s-1)t}).$$

Together with the previous analysis, we see for any $t \to +\infty$,

$$\tilde{u}(t,\theta) \to \alpha \left(\cos\frac{\theta}{2}\right)^{2s},$$

weakly in $L^2([0,\pi])$. To improve this to a uniform convergence, we use the method in the proof of Proposition 7.2 (or standard De Giorgi-Moser iteration). This gives the expansion of u. □

Remark 7.5. *This expansion is in consistence with the $s = 1$ case,*

$$u'' = uv^2, \quad v'' = vu^2, \quad \textit{on } \mathbb{R}.$$

For this problem, we have the expansion (after a translation and a rescaling)

$$u(x) = x_+ + O(1), \quad v(x) = x_- + O(1).$$

Remark 7.6. *We can also estimate the convergence rate of $\tilde{u}$, which is of order $O(e^{-\delta t})$. Hence in the expansion of u, $o(r^{s-1})$ can be replaced by $O(r^{s-1-\delta})$.*

8. Symmetry between u and v

In this section, we prove the following theorem and use it to prove the symmetry between u and v, as claimed in Theorem 1.1.

Theorem 8.1. *Let (u_i, v_i), $i = 1, 2$ be two solutions of* (1.12)*. Suppose that they satisfy*

$$u_i(r,\theta) = r^s\left(\cos\frac{\theta}{2}\right)^{2s} + \alpha_i r^{s-1}\left(\cos\frac{\theta}{2}\right)^{2s} + o(r^{s-1}), \quad i = 1, 2, \tag{8.1}$$

$$v_i(r,\theta) = r^s\left(\sin\frac{\theta}{2}\right)^{2s} + \beta_i r^{s-1}\left(\sin\frac{\theta}{2}\right)^{2s} + o(r^{s-1}), \quad i = 1, 2, \tag{8.2}$$

for four constants $\alpha_i, \beta_i, i = 1, 2$. If $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$, then

$$u_1(x + t_0, y) \equiv u_2(x,y), \qquad v_1(x + t_0, y) \equiv v_2(x,y),$$

where $t_0 = \frac{1}{s}(\alpha_2 - \alpha_1) = \frac{1}{s}(\beta_1 - \beta_2)$.

Note that (8.1) and (8.2) imply that

$$\left|u_i(r,\theta) - r^s\left(\cos\frac{\theta}{2}\right)^{2s}\right| + \left|v_i(r,\theta) - r^s\left(\sin\frac{\theta}{2}\right)^{2s}\right| \le M\left(1 + r\right)^{s-1}, \quad i = 1, 2, \tag{8.3}$$

for some constant $M > 0$.

For any $t \in \mathbb{R}$, let

$$u^t(x,y) := u_1(x+t,y), \qquad v^t(x,y) := v_1(x+t,y),$$

which is still a solution of (1.12).

In the following, it will be helpful to keep the following fact in mind. Because

$$\begin{aligned}\left|u^t - u_2\right| &\leq M\left(1+|x|+|y|\right)^{s-1} + M\left(1+|x+t|+|y|\right)^{s-1}\\ &+\left|\left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^s - \left(\frac{\sqrt{(x+t)^2+y^2}+x+t}{2}\right)^s\right|,\end{aligned}$$

$\left|u^t - u_2\right| \to 0$ as $|z| \to \infty$. Thus any positive maximum (or negative minima) of $u^t - u_2$ is attained at some point.

The first step is to show that we can start the moving plane from the infinity.

Lemma 8.2. *If t is large enough,*

$$u^t(x,y) \geq u_2(x,y), \qquad v^t(x,y) \leq v_2(x,y), \quad on \ \overline{\mathbb{R}^2_+}. \tag{8.4}$$

Proof. If t is sufficiently large, for $x \geq 0$,

$$\begin{aligned}u^t(x,0) &\geq (x+t)^s - M(x+t)^{s-1}\\ &\geq x^s + Mx^{s-1}\\ &\geq u_2(x,0).\end{aligned}$$

Similarly,

$$v^t(x,0) \leq v_2(x,0), \quad \text{on } (-\infty, -C(M)],$$

where $C(M)$ is a constant depending only on M.

It can be checked directly that for t large,

$$u^t(x,0) \geq u_2(x,0), \qquad \text{on } [-C(M), 0].$$

In fact, for $x \in [-C(M), 0]$, $\lim_{t\to+\infty} u^t(x,0) = +\infty$ uniformly (see Lemma 6.5), while $u_2(x,0)$ has an upper bound here.

Then by noting that

$$\begin{cases} L_a\left(u^t - u_2\right) = 0, \ \text{in } \mathbb{R}^2_+,\\ \left|u^t(z) - u_2(z)\right| \to 0, \ \text{as } |z| \in \mathbb{R}^2_+, \ z \to \infty,\\ u^t(x,0) - u_2(x,0) \geq 0, \ \text{on } [-C(M), +\infty)\\ \partial^a_y\left(u^t(x,0) - u_2(x,0)\right) \leq v_2(x,0)^2\left(u^t(x,0) - u_2(x,0)\right) \ \text{on } (-\infty, -C(M)),\end{cases}$$

we can apply the maximum principle to deduce that

$$u^t \geq u_2, \ \text{in } \overline{\mathbb{R}^2_+}.$$

In fact, if $\inf(u^t - u_2) < 0$, this minima is attained at some point. Because $u^t - u_2$ is L_a-harmonic, the strong maximum principle implies that this point is on the boundary, say $(x_0, 0)$. Clearly $x_0 \leq -C(M)$. Then

$$0 \leq \partial^a_y\left(u^t - u_2\right)(x_0,0) \leq v_2(x,0)^2\left(u^t(x,0) - u_2(x,0)\right) < 0.$$

This is a contradiction.

The same reasoning using

$$\begin{cases} L_a\left(v^t - v_2\right) = 0, \text{ in } \mathbb{R}^2_+, \\ \left|v^t(z) - v_2(z)\right| \to 0, \text{ as } z \in \mathbb{R}^2_+, \ z \to \infty, \\ \partial^a_y\left(v^t(x,0) - v_2(x,0)\right) \geq u^t(x,0)^2\left(v^t(x,0) - v_2(x,0)\right) \text{ on } \partial\mathbb{R}^2_+, \end{cases}$$

gives

$$v^t \leq v_2, \text{ on } \overline{\mathbb{R}^2_+}. \qquad \square$$

Now we can define t_0 to be

$$\min\{t: \ \forall s > t, u^s(x,y) \geq u_2(x,y), \quad v^s(x,y) \leq v_2(x,y), \quad \text{on } \overline{\mathbb{R}^2_+}\}. \tag{8.5}$$

By continuity, $u^{t_0} \geq u_2$, $v^{t_0} \leq v_2$.

We want to prove that $t_0 = \frac{1}{s}(\alpha_2 - \alpha_1)$. Indeed, if this is true, we have $u^{t_0} \geq u_2$ and $v^{t_0} \leq v_2$. Then we can slide from the left, by the same reasoning this procedure must stop at t_0. Thus we also have $u^{t_0} \leq u_2$ and $v^{t_0} \geq v_2$. Consequently $u^{t_0} \equiv u_2$ and $v^{t_0} \equiv v_2$.

Now assume $t_0 > \frac{1}{s}(\alpha_2 - \alpha_1)$. We will get a contradiction from this assumption. Let $\delta_0 = st_0 - (\alpha_2 - \alpha_1) > 0$. By (8.1) and (8.2),

$$u^{t_0}(x,0) = x^s + (a_1 + st_0)\, x^{s-1} + o(x^{s-1}), \quad \text{as } x \to +\infty.$$

$$v^{t_0}(x,0) = |x|^s + (b_1 - st_0)\, |x|^{s-1} + o(|x|^{s-1}), \quad \text{as } x \to -\infty.$$

Comparing with u_2 and v_2 respectively, we get a constant T_0 such that

$$u^{t_0}(x,0) \geq u_2(x,0) + \frac{\delta_0}{2}x^{s-1}, \quad \text{if } x \geq T_0, \tag{8.6}$$

and

$$v^{t_0}(x,0) \leq v_2(x,0) - \frac{\delta_0}{2}|x|^{s-1}, \quad \text{if } x \leq -T_0. \tag{8.7}$$

By (8.1), perhaps after choosing a larger T_0, for all t satisfying $|t| \leq 2|t_0|$ we have

$$\left|u^t(x,0) - x^s - (a_1 + st)\, x^{s-1}\right| \leq \frac{\delta_0}{8}x^{s-1}, \quad \text{if } x \geq T_0.$$

Thus there exists an $\varepsilon_1 > 0$ such that, for all $t \in [t_0 - \varepsilon_1, t_0]$,

$$u^t(x,0) \geq u^{t_0}(x,0) - \frac{\delta_0}{4}x^{s-1}, \quad \text{if } x \geq T_0.$$

Combining this with (8.6), we see for these t,

$$u^t(x,0) \geq u_2(x,0), \quad \text{if } x \geq T_0, \tag{8.8}$$

and similarly

$$v^t(x,0) \leq v_2(x,0), \quad \text{if } x \leq -T_0, \tag{8.9}$$

By the strong maximum principle, $u^{t_0} > u_2$ and $v^{t_0} < v_2$ strictly. In fact, if there exists a point $z_0 \in \mathbb{R}^2_+$ such that $u^{t_0}(z_0) = u_2(z_0)$, then the strong maximum principle implies that $u^{t_0} \equiv u_2$, which contradicts (8.6).

Next, by continuity we can find an $\varepsilon_2 > 0$ so that for all $t \in [t_0 - \varepsilon_2, t_0]$,

$$u^t(x,0) \geq u_2(x,0), \quad v^t(x,0) \leq v_2(x,0), \quad \text{for } x \in [-T_0, T_0].$$

Combined with (8.8) and (8.9), by choosing $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, we see for all $t \in [t_0 - \varepsilon, t_0]$,

$$u^t(x,0) - u_2(x,0) \geq 0, \quad \text{in } [-T_0, +\infty),$$
$$v^t(x,0) - v_2(x,0) \leq 0, \quad \text{in } (-\infty, T_0].$$

Then arguing as in the proof of Lemma 8.2, we know for all $t \in [t_0 - \varepsilon, t_0]$,

$$u^t \geq u_2, \quad v^t \leq v_2, \quad \text{in } \overline{\mathbb{R}^2_+}.$$

However, this contradicts the definition of t_0. Thus the assumption $t_0 > \frac{1}{s}(a_2 - a_1)$ cannot be true.

Proof of Theorem 1.1: symmetry between u and v. We first prove the symmetry between u and v. Given a solution (u, v) of (1.12), let $(u_1(x,y), v_1(x,y)) = (u(x,y), v(x,y))$ and $(u_2(x,y), v_2(x,y)) = (v(-x,y), u(-x,y))$.

By Proposition 7.4, after a scaling, we have the expansion

$$\begin{cases} u_1(r,\theta) = u(r,\theta) = r^s(\cos\frac{\theta}{2})^{2s} + \alpha r^{s-1}(\cos\frac{\theta}{2})^{2s} + o(r^{s-1}), \\ v_1(r,\theta) = v(r,\theta) = r^s(\sin\frac{\theta}{2})^{2s} + \beta r^{s-1}(\sin\frac{\theta}{2})^{2s} + o(r^{s-1}). \end{cases} \tag{8.10}$$

Hence

$$\begin{cases} u_2(r,\theta) = r^s(\cos\frac{\theta}{2})^{2s} + \beta r^{s-1}(\cos\frac{\theta}{2})^{2s} + o(r^{s-1}), \\ v_2(r,\theta) = r^s(\sin\frac{\theta}{2})^{2s} + \alpha r^{s-1}(\sin\frac{\theta}{2})^{2s} + o(r^{s-1}). \end{cases}$$

Thus we can apply Theorem 8.1 to get a constant T such that

$$u(x+2T, y) = v(-x, y), \quad v(x+2T, y) = u(-x, y).$$

That is, u and v are symmetric with respect to the line $\{x = T\}$. $\square$

Corollary 8.3. *For any solution (u, v) of* (1.12), $\frac{\partial u}{\partial x} > 0$ *and* $\frac{\partial v}{\partial x} < 0$ *on* $\overline{\mathbb{R}^2_+}$.

Proof. Let

$$u^t(x,y) := u(x+t), \qquad v^t(x,y) := v(x+t, y).$$

As in the above argument, we know for any $t \geq 0$,

$$u^t \geq u, \qquad v^t \leq v, \quad \text{on } \overline{\mathbb{R}^2_+}.$$

Thus $\frac{\partial u}{\partial x} \geq 0$ and $\frac{\partial v}{\partial x} \leq 0$.

Next, by noting that

$$\begin{cases} L_a \dfrac{\partial u}{\partial x} = L_a \dfrac{\partial v}{\partial x} = 0, \quad \text{in } \mathbb{R}^2_+, \\ \partial^a_y \dfrac{\partial u}{\partial x} = v^2 \dfrac{\partial u}{\partial x} + 2uv\dfrac{\partial v}{\partial x}, \quad \text{on } \partial\mathbb{R}^2_+, \\ \partial^a_y \dfrac{\partial v}{\partial x} = u^2 \dfrac{\partial v}{\partial x} + 2uv\dfrac{\partial u}{\partial x}, \quad \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

we can use the strong maximum principle to conclude the proof. $\square$

9. Uniqueness

In this section, we prove the uniqueness of solutions to (1.12), thus complete the proof of Theorem 1.1. We follow the main ideas of [3].

Before going into the proof, we present a technical lemma on a barrier function.

Lemma 9.1. *There exists a function $g(x) \geq 1$ on $\mathbb{R}$, satisfying*

$$\frac{(-\Delta)^s g(x)}{g(x)} \geq c|x|^{-2s}, \quad \text{as } x \to -\infty,$$

$$\frac{(-\Delta)^s g(x)}{g(x)} \geq -C, \quad \text{as } x \to +\infty.$$

Proof. Let f be the solution to the Allen-Cahn equation

$$\begin{cases} L_a f = 0, & \text{in } \mathbb{R}^2_+, \\ \partial^a_y f = \left(f^2 - 1\right) f, & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{9.1}$$

By the main result in [7], we can take f to satisfy

$$\lim_{x \to \pm\infty} f(x,0) = \mp 1, \quad \frac{\partial f}{\partial x} < 0. \tag{9.2}$$

Moreover, by [7, Theorem 2.7], we have the decay estimate

$$c|x|^{-2s} \leq 1 - f(x,0) \leq C|x|^{-2s}, \quad \text{as } x \to -\infty.$$

Then $g = f + 2$ satisfies all of the required properties. □

Let (u_i, v_i), $i = 1,2$, be two solutions of (1.12). By what we have proved in the previous section, the following expansion holds:

$$\begin{cases} u_i(r,\theta) = r^s(\cos\frac{\theta}{2})^{2s} + \alpha_i r^{s-1}(\cos\frac{\theta}{2})^{2s} + o(r^{s-1}), \\ v_i(r,\theta) = r^s(\sin\frac{\theta}{2})^{2s} + \alpha_i r^{s-1}(\sin\frac{\theta}{2})^{2s} + o(r^{s-1}). \end{cases} \tag{9.3}$$

Here α_i, $i = 1,2$ are two constants.

If $\alpha_1 = \alpha_2$, Lemma 8.1 implies that $u_1 \equiv u_2$, $v_1 \equiv v_2$ and we are done. Hence we assume, without loss of generality, that $\alpha_1 > \alpha_2$. Denote

$$t_0 := \frac{1}{s}\left(\alpha_1 - \alpha_2\right) > 0.$$

Define (u^t, v^t) as in the previous section. As before, we can show that for all $t \geq t_0$,

$$u^t > u_2, \quad v^t < v_2, \quad \text{on } \overline{\mathbb{R}^2_+}.$$

Since (u^t, v^t) has the expansion

$$\begin{cases} u^t(x,0) = x^s + (\alpha_1 + st)\, x^{s-1} + o(x^{s-1}), & \text{as } x \to +\infty, \\ v^t(x,0) = |x|^s + (\alpha_1 - st)\, |x|^{s-1} + o(|x|^{s-1}), & \text{as } x \to -\infty, \end{cases} \tag{9.4}$$

for any $t < t_0$, if $-x$ is large enough, $v^t(x,0) > v_2(x,0)$.

Lemma 9.2. *For any $t < t_0$, $\inf_{\overline{\mathbb{R}^2_+}}(u^t - u_2) < 0$.*

Proof. Assume by the contrary, for some $t < t_0$, $u^t \geq u$ on $\overline{\mathbb{R}^2_+}$. Then as in the proof of Lemma 8.2 we can apply the maximum principle to deduce that $v^t \leq v_2$. This is a contradiction. $\square$

For $t < t_0$, let

$$\tilde{w}_1^t := u^t - u_2, \quad \tilde{w}_2^t := v_2 - v^t.$$

They satisfy

$$\begin{cases} L_a \tilde{w}_1^t = L_a \tilde{w}_2^t = 0, & \text{in } \mathbb{R}^2_+, \\ \partial_y^a \tilde{w}_1^t = \left(v^t\right)^2 \tilde{w}_1^t - u_2 \left(v_2 + v^t\right) \tilde{w}_2^t, & \text{on } \partial\mathbb{R}^2_+, \\ \partial_y^a \tilde{w}_2^t = u_2^2 \tilde{w}_2^t - v^t \left(u_2 + u^t\right) \tilde{w}_1^t, & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

Then define

$$w_1^t := \frac{\tilde{w}_1^t}{g}, \quad w_2^t := \frac{\tilde{w}_2^t}{g}.$$

They satisfy

$$\begin{cases} L_a w_1^t + 2y^a \dfrac{\nabla g}{g} \nabla w_1^t = L_a w_2^t + 2y^a \dfrac{\nabla g}{g} \nabla w_2^t = 0, & \text{in } \mathbb{R}^2_+, \\ \partial_y^a w_1^t = \left[-\dfrac{\partial_y^a g}{g} + \left(v^t\right)^2 \right] w_1^t - u_2 \left(v_2 + v^t\right) w_2^t, & \text{on } \partial\mathbb{R}^2_+, \\ \partial_y^a w_2^t = \left[-\dfrac{\partial_y^a g}{g} + u_2^2 \right] w_2^t - v^t \left(u_2 + u^t\right) w_1^t, & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{9.5}$$

Previous discussion has shown that, for any $t < t_0$,

$$\inf_{\mathbb{R}^2_+} w_1^t < 0 \text{ and } \inf_{\mathbb{R}^2_+} w_2^t < 0.$$

By the strong maximum principle using the first equation in (9.5), these two infimum are attained at two points of the form $(x_{i,t}, 0)$, $i = 1, 2$.

Because $u^{t_0} > u_2$ and $v^{t_0} < v_2$ on $\partial\mathbb{R}^2_+$, we must have

$$\lim_{t \to t_0} |x_{i,t}| = \infty, \quad i = 1, 2. \tag{9.6}$$

Lemma 9.3. *As $t \to t_0$, $x_{1,t} \to -\infty$.*

Proof. Since $\alpha_1 + st_0 > \alpha_2$, by the expansion (9.4), there exist two constants $R_0 > 0$ large and ε_* small, so that for any $t \in [t_0 - \varepsilon_*, t_0]$,

$$u^t(x, 0) \geq u_2(x, 0), \quad \text{for } x \in [R_0, +\infty).$$

Next by continuity, for any $R > 0$, there exists an $\varepsilon(R) > 0$ so that, if $t \in [t_0 - \varepsilon(R), t_0]$, then

$$u^t(x, 0) \geq u_2(x, 0), \quad \text{for } x \in [-R, R_0].$$

Thus if $t \geq t_0 - \min\{\varepsilon_*, \varepsilon(R)\}$, $x_{1,t} < -R$. $\square$

Because $\partial_y^a w_1^t(x_{1,t}) \geq 0$, we have

$$w_1^t(x_{1,t},0) \geq \frac{u_2(x_{1,t},0)\left(v_2(x_{1,t},0)+v^t(x_{1,t},0)\right)}{-\frac{\partial_y^a g}{g}(x_{1,t},0)+v^t(x_{1,t},0)^2} w_2^t(x_{1,t},0). \tag{9.7}$$

Similarly,

$$w_2^t(x_{2,t},0) \geq \frac{v^t(x_{2,t},0)\left(u_2(x_{2,t},0)+u^t(x_{2,t},0)\right)}{-\frac{\partial_y^a g}{g}(x_{2,t},0)+u_2(x_{2,t},0)^2} w_1^t(x_{1,t},0). \tag{9.8}$$

By definition, $w_1^t(x_{2,t},0) \geq w_1^t(x_{1,t},0)$ and $w_2^t(x_{1,t},0) \geq w_2^t(x_{2,t},0)$. Combining this fact with (9.7) and (9.8), and noting that $w_1^t(x_{1,t},0) < 0$, we must have

$$\frac{u_2(x_{1,t},0)\left(v_2(x_{1,t},0)+v^t(x_{1,t},0)\right)}{-\frac{\partial_y^a g}{g}(x_{1,t},0)+v^t(x_{1,t},0)^2} \times \frac{v^t(x_{2,t},0)\left(u_2(x_{2,t},0)+u^t(x_{2,t},0)\right)}{-\frac{\partial_y^a g}{g}(x_{2,t},0)+u_2(x_{2,t},0)^2} > 1. \tag{9.9}$$

By Proposition 6.1 and Proposition 6.9,

$$\begin{cases} u_2(x_{1,t},0)\big(v_2(x_{1,t},0)+v^t(x_{1,t},0)\big) \leq C|x_{1,t}|^{-2s}, \\ v^t(x_{2,t},0)\big(u_2(x_{2,t},0)+u^t(x_{2,t},0)\big) \leq C|x_{1,t}|^{-2s}. \end{cases} \tag{9.10}$$

Since $x_{1,t} \to -\infty$, by Lemma 6.8 and Lemma 9.1,

$$-\frac{\partial_y^a g}{g}(x_{1,t},0)+v^t(x_{1,t},0)^2 \geq v^t(x_{1,t},0)^2 \geq c|x_{1,t}|^{2s} \tag{9.11}$$

Hence,

$$\frac{u_2(x_{1,t},0)\left(v_2(x_{1,t},0)+v^t(x_{1,t},0)\right)}{-\frac{\partial_y^a g}{g}(x_{1,t},0)+v^t(x_{1,t},0)^2} \to 0, \quad \text{as } t \to t_0. \tag{9.12}$$

Next, if $x_{2,t} \to -\infty$, by Lemma 9.1,

$$-\frac{\partial_y^a g}{g}(x_{2,t},0)+u_2(x_{2,t},0)^2 \geq -\frac{\partial_y^a g}{g}(x_{2,t},0) \geq c|x_{2,t}|^{-2s}.$$

Together with (9.10), this implies the existence of a constant C such that, as $t \to t_0$,

$$\frac{v^t(x_{2,t},0)\left(u_2(x_{2,t},0)+u^t(x_{2,t},0)\right)}{-\frac{\partial_y^a g}{g}(x_{2,t},0)+u_2(x_{2,t},0)^2} \leq C. \tag{9.13}$$

If $x_{2,t} \to +\infty$, by Lemma 6.8 and Lemma 9.1,

$$-\frac{\partial_y^a g}{g}(x_{2,t},0)+u_2(x_{2,t},0)^2 \geq -C+u_2(x_{2,t},0)^2 \geq c|x_{2,t}|^{2s} \tag{9.14}$$

Hence,

$$\frac{v^t(x_{2,t},0)\left(u_2(x_{2,t},0)+u^t(x_{2,t},0)\right)}{-\frac{\partial_y^a g}{g}(x_{2,t},0)+u_2(x_{2,t},0)^2} \to 0, \quad \text{as } t \to t_0. \tag{9.15}$$

Combining (9.12) and (9.13) (or (9.15)), we get a contradiction with (9.9).

In conclusion, the assumption $\alpha_1 \neq \alpha_2$ cannot be true. By using Theorem 8.1, we complete the proof of Theorem 1.1.

Appendix A. Basic Facts about L_a-subharmonic functions

In this appendix we present several basic facts about L_a-subharmonic functions, which are used in this paper.

The first is a mean value inequality for L_a-subharmonic function.

Lemma A.1. *Let u be a L_a-subharmonic function in $B_r \subset \mathbb{R}^{n+1}$ (centered at the origin), then*

$$u(0) \le C(n,a) r^{-n-1-a} \int_{B_r} y^a u.$$

Here $C(n,a)$ is a constant depending only on n and a.

Proof. Direct calculation gives

$$\begin{aligned} \frac{d}{dr}\left(r^{-n-a} \int_{\partial B_r} y^a u \right) &= r^{-n-a} \int_{\partial B_r} y^a \frac{\partial u}{\partial r} \\ &= r^{-n-a} \int_{B_r} \mathrm{div}\,(y^a \nabla u) \\ &\ge 0. \end{aligned}$$

Thus $r^{-n-a} \int_{\partial B_r} y^a u$ is non-decreasing in r. Integrating this in r shows that $r^{-n-1-a} \int_{B_r} y^a u$ is also non-decreasing in r. □

By standard Moser's iteration we also have the following super bound

Lemma A.2. *Let u be a L_a-subharmonic function in $B_r \subset \mathbb{R}^{n+1}$ (centered at the origin), then*

$$\sup_{B_{r/2}} u \le C(n,a) \left(r^{-n-1-a} \int_{B_r} y^a u^2 \right)^{\frac{1}{2}}.$$

Here $C(n,a)$ is a constant depending only on n and a.

Lemma A.3. *Let $M > 0$ be fixed. Any $v \in H^1(B_1^+) \cap C(\overline{B_1^+})$ nonnegative solution to*

$$\begin{cases} L_a v \ge 0, \ \ \text{in } B_1^+, \\ \partial_y^a v \ge M v \quad \text{on } \partial^0 B_1^+, \end{cases}$$

satisfies

$$\sup_{\partial^0 B_{1/2}^+} v \le \frac{C(n)}{M} \int_{B_1^+} y^a v.$$

Proof. This is essentially [23, Lemma 3.5]. We only need to note that, since

$$\partial_y^a v \ge 0 \ \ \text{on } \partial^0 B_1^+,$$

the even extension of v to B_1 is L_a-subharmonic (cf. [7, Lemma 4.1]). Then by Lemma A.2,

$$\sup_{B_{2/3}^+} v \le C(n) \int_{B_1^+} y^a v.$$ □

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