# ON LIN-NI'S CONJECTURE IN GENERAL DOMAIN, N=4, 6.

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ABSTRACT. We consider the following nonlinear Neumann problem:

$$\begin{cases} \Delta u - \mu u + u^q = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth and bounded domain, $\mu > 0$ and $\nu$ denotes the outward unit normal vector of $\Omega$. Lin and Ni (1986) conjectured that when $q = \frac{n+2}{n-2}$, for $\mu$ small, all solutions are constants. We show that this conjecture is false for general domain in $n = 4, 6$.

## 1. INTRODUCTION

In this paper we consider the following nonlinear Neumann elliptic problem:

$$\begin{cases} \Delta u - \mu u + u^q = 0, \ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $1 < q < +\infty$, $\mu > 0$ and $\Omega$ is a smooth and bounded domain in $\mathbb{R}^n$, $n = 4, 6$.

Eq.(1.1) arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in mathematical biology [11], [16], or for parabolic equation in chemotaxis, e.g. Keller-Segel model [14].

Eq.(1.1) enjoys at least one solution, namely the constant solution $u \equiv \mu^{\frac{1}{q-1}}$. In a series of seminal work, Lin-Ni-Takagi [14] and Ni-Takagi [17] got interest in the potential existence of non-constant solution to Eq.(1.1). In particular, it is proved in [14],[17] that for $\mu$ large, the least energy solution concentrate at the boundary point of maximum mean curvature. In the subcritical case $1 < q < 2^* - 1$, we can use the well-known result due to Gidas-Spruck [10] to prove that for small positive $\mu$, the constant solution is the only solution. This uniqueness result incited Lin and Ni to raise the following conjecture, the extension of this result to the critical case $q = 2^* - 1$.

**Lin-Ni's Conjecture** [13]. For $\mu$ small and $q = \frac{n+2}{n-2}$, problem (1.1) admits only the constant solution.

We recall below the main results towards proving or disproving Lin-Ni's conjecture. Adimurthi-Yadava [1]-[2] and Budd-Knapp-Peletier [4] first considered the following problem

$$\begin{cases} \Delta u - \mu u + u^{\frac{n+2}{n-2}} = 0 & \text{in } B_R(0), \\ u > 0 & \text{in } B_R(0), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_R(0). \end{cases} \tag{1.2}$$

They proved the following result

**Theorem A.** ([1]-[4]) For $\mu$ sufficiently small,
(1) if n=3 or n $\geq 7$, problem (1.2) admits only the constant solution;
(2) if n=4, 5 or 6, problem (1.2) admits a nonconstant solution.

The proof of Theorem A relies on the radial symmetry of the domain. In the asymmetric case, the complete answer is not known yet, but there are a few results. In the general three-dimension domain case, Zhu [26] proved

**Theorem B.** ([25],[26]) The conjecture is true if $n = 3$ ($q = 5$) and $\Omega$ is convex.

Zhu's proof relies on a priori estimate. Later, Wei-Xu [25] gave a direct proof for Theorem B by using the integration by parts only. In comparison with the strong convexity condition assumed on the domain. Recently, under the assumption on the bound of the energy and a weaker convexity condition (mean convex domains) Druet-Robert-Wei [9] showed the following result:

**Theorem C.** ([9]) Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n = 3$ or $n \geq 7$. Assume that $H(x) > 0$ for all $x \in \partial\Omega$, where $H(x)$ is the mean curvature of $x$, $x \in \partial\Omega$. Then for all $\mu > 0$, there exists $\mu_0(\Omega, \Lambda) > 0$ such that for all $\mu \in (0, \mu_0(\Omega, \Lambda))$ and for any $u \in C^2(\overline{\Omega})$, we have that

$$\left\{\begin{array}{ll} -\Delta u + \mu u = n(n-2)u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \\ \int_\Omega u^{2^*} dx \leq \Lambda & \end{array}\right\} \Rightarrow u \equiv \left(\frac{\mu}{n(n-2)}\right)^{\frac{n-2}{4}}.$$

It should be mentioned that the assumption of the bounded energy is necessary in obtaining Theorem C. Without this technical assumption, it was proved that the solutions to (1.1) may accumulate with infinite energy when the mean curvature is negative somewhere (see Wang-Wei-Yan [21]). More precisely, Wang-Wei-Yan gave a negative answer to Lin-Ni's conjecture in all dimensions ($n \geq 3$) for non-convex domain by assuming that $\Omega$ is a smooth and bounded domain satisfying the following conditions:

$(H_1)$ $y \in \Omega$ if and only if $(y_1, y_2, y_3, \cdots, -y_i, \cdots, y_n) \in \Omega$, $\forall i = 3, \cdots, n$.
$(H_2)$ If $(r, 0, y'') \in \Omega$, then $(r\cos\theta, r\sin\theta, y'') \in \Omega$, $\forall \theta \in (0, 2\pi)$, where $y'' = (y_3, \cdots, y_n)$.
$(H_3)$ Let $T := \partial\Omega \cap \{y_3 = \cdots = y_n = 0\}$. There exists a connected component $\Gamma$ of $T$ such that $H(x) \equiv \gamma < 0$, $\forall x \in \Gamma$.

**Theorem D.** ([21]) Suppose $n \geq 3$, $q = \frac{n+2}{n-2}$ and $\Omega$ is a bounded smooth domain satisfying $(H_1)$-$(H_3)$. Let $\mu$ be any fixed positive number. Then problem (1.1) has infinitely many positive solutions, whose energy can be made arbitrarily large.

Wang-Wei-Yan [22] also gave a negative answer to Lin-Ni's conjecture in some convex domain including the balls for $n \geq 4$.

**Theorem E.** ([22]) Suppose $n \geq 4$, $q = \frac{n+2}{n-2}$ and $\Omega$ satisfies $(H_1)$-$(H_2)$. Let $\mu$ be any fixed positive number. Then problem (1.1) has infinitely many positive solutions, whose energy can be made arbitrarily large.

Theorem A-E reveal that Lin-Ni's conjecture depends very sensitively not only on the dimension, but also on the shape of the domain. A natural question is: what about the general domains? Inspired by the result of Theorem A. We expect to give a negative answer to the case $n = 4, 5, 6$. The only approach in this direction is given by Rey-Wei [19]. They disproved the conjecture in the five-dimensional case by establishing a nontrivial solution which blows up at $K$ interior points in $\Omega$ provided $\mu$ is sufficiently small.

The purpose of this paper is to establish a result similar to (2) of Theorem A in general four, and six-dimensional domains by establishing a nontrivial solution which blows up at a single point in $\Omega$ provided $\mu$ is sufficiently small. Namely, we consider the problem

$$\Delta u - \mu u + u^{\frac{n+2}{n-2}} = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1.3}$$

where $n = 4, 6$ and $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$ and $\mu > 0$ very small. Our main result is stated as follows

**Main Theorem.** For problem (1.3) in $n = 4, 6$, there exists $\mu_0 > 0$ such that for all $0 < \mu < \mu_0$, equation (1.3) possesses a nontrivial solution which blows up at an interior point of $\Omega$.

In order to make this statement more precise, we introduce the following notation. Let $G(x, Q)$ be the Green's function defined as

$$\Delta_x G(x, Q) + \delta_Q - \frac{1}{|\Omega|} = 0 \text{ in } \Omega, \ \frac{\partial G}{\partial \nu} = 0 \text{ on } \partial\Omega, \ \int_\Omega G(x, Q)dx = 0. \tag{1.4}$$

We decompose

$$G(x, Q) = K(|x - Q|) - H(x, Q),$$

where

$$K(r) = \frac{1}{c_n r^{n-2}}, \ c_n = (n-2)|S^{n-1}|, \tag{1.5}$$

is the fundamental solution of the Laplacian operator in $\mathbb{R}^n$($|S^{n-1}|$ denotes the area of the unit sphere), $n = 4, 6$.

For the reason of normalization, we consider the following equation:

$$\Delta u - \mu u + n(n-2)u^{\frac{n+2}{n-2}} = 0, \ u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.6}$$

We recall that, according to [5], the functions

$$U_{\Lambda,Q} = (\frac{\Lambda}{\Lambda^2 + |x - Q|^2})^{\frac{n-2}{2}}, \ \Lambda > 0, \ Q \in \mathbb{R}^n, \tag{1.7}$$

are the only solutions to the problem

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \ u > 0 \text{ in } \mathbb{R}^n. \tag{1.8}$$

Our main result can be stated precisely as follows:

**Theorem 1.1.** *Let $\Omega$ be any smooth bounded domain in $\mathbb{R}^n$.*

(1). *For $n = 4$, there exists $\mu_1 > 0$ such that for $0 < \mu < \mu_1$, problem (1.6) has a nontrivial solution*

$$u_\mu = U_{e^{-\frac{c_1}{\mu^2}}\Lambda, Q^\mu} + O(\mu^{-1} e^{-\frac{c_1}{\mu^2}}),$$

*where $c_1$ is some constant depending on the domain, to be determined later, $\Lambda$ will be some generic constant. The point $Q^\mu$ depends on the domain and parameter $\Lambda$, and will be given in Section 6.*

(2). *For $n = 6$, there exists $\mu_2 > 0$ such that for $0 < \mu < \mu_2$, problem (1.6) has a nontrivial solution*

$$u_\mu = U_{\mu\Lambda, Q^\mu} + O(\mu),$$

*where $\Lambda \to \Lambda_0$, and $\Lambda_0 > 0$ is some generic constant. The point $Q^\mu$ depends on the domain and parameter $\Lambda$, and will be given in Section 6.*

We set

$$\Omega_\varepsilon := \Omega/\varepsilon = \{z | \varepsilon z \in \Omega\}, \tag{1.9}$$

and

$$\mu = \begin{cases} (\frac{c_1}{-\ln \varepsilon})^{\frac{1}{2}}, & n = 4, \\ \varepsilon \ , & n = 6. \end{cases} \tag{1.10}$$

Through the transformation $u(x) \longmapsto \varepsilon^{-\frac{n-2}{2}} u(x/\varepsilon)$, (1.6) yields the rescaled problem we will work with

$$\Delta u - \mu\varepsilon^2 u + n(n-2) u^{\frac{n+2}{n-2}} = 0, \ u > 0 \text{ in } \Omega_\varepsilon, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Omega_\varepsilon. \tag{1.11}$$

We set

$$S_\varepsilon[u] := -\Delta u + \mu\varepsilon^2 u - n(n-2) u_+^{\frac{n+2}{n-2}}, \ u_+ = \max(u, 0), \tag{1.12}$$

and introduced the following functional

$$J_\varepsilon[u] := \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{1}{2} \mu\varepsilon^2 \int_{\Omega_\varepsilon} u^2 - \frac{(n-2)^2}{2} \int_{\Omega_\varepsilon} |u|^{\frac{2n}{n-2}}, \ u \in H^1(\Omega_\varepsilon). \tag{1.13}$$

The paper is organized as follows: In Section 2, we construct suitable approximated bubble solution $W$, and list their properties. In Section 3, we solve the linearized problem at $W$ in a finite-codimensional space. Then, in Section 4, we are able to solve the nonlinear problem in that space. In section 5, we study the remaining finite-dimensional problem and solve it in Section 6, finding critical points of the reduced energy functional. Some numerical results may be found in the last Section.

## 2. Approximate bubble solutions

In this section, we construct suitable approximate solution, in the neighborhood of which solutions in Theorem 1.1 will be found.

Let $\varepsilon$ be as defined in (1.10). For any $Q \in \Omega$ with $d(Q/\varepsilon, \partial\Omega_\varepsilon)$ large, $U_{\Lambda, Q/\varepsilon}$ in (1.7) provides an approximate solution of (1.11). Because of the appearance of the additional linear term $\mu\varepsilon^2 u$, we need to add an extra term to get a better approximation. To this end, for $n = 4$, we consider the following equation

$$\Delta \bar{\Psi} + U_{1,0} = 0 \text{ in } \mathbb{R}^4, \ \bar{\Psi}(0) = 1. \tag{2.1}$$

Then

$$\bar{\Psi}(|y|) = -\frac{1}{2} \ln|y| + I + O\Big(\frac{1}{|y|}\Big), \ \bar{\Psi}^{'} = -\frac{1}{2|y|}\Big(1 + O\Big(\frac{\ln(1+|y|)}{|y|^2}\Big)\Big) \text{ as } |y| \to \infty, \tag{2.2}$$

where $I$ is a constant. Let

$$\Psi_{\Lambda,Q} = \frac{\Lambda}{2}\ln\frac{1}{\Lambda\varepsilon} + \Lambda\bar{\Psi}(\frac{y-Q}{\Lambda}). \tag{2.3}$$

Then

$$\Delta\Psi_{\Lambda,Q} + U_{\Lambda,Q} = 0.$$

Note that we have

$$|\Psi_{\Lambda,Q}(y)|,\ |\partial_\Lambda\Psi_{\Lambda,Q}(y)| \le C\Big|\ln\frac{1}{\varepsilon(1+|y-Q|)}\Big|,\ |\partial_{Q_i}\Psi_{\Lambda,Q}(y)| \le \frac{C}{1+|y-Q|}. \tag{2.4}$$

For $n=6$, let $\Psi(|y|)$ be the radial solution of

$$\Delta\Psi + U_{1,0} = 0 \text{ in } \mathbb{R}^n, \quad \Psi \to 0 \text{ as } |y| \to +\infty. \tag{2.5}$$

Then, it is easy to check that

$$\Psi(y) = \frac{1}{4|y|^2}(1+O(\frac{1}{|y|^2})) \text{ as } |y| \to +\infty. \tag{2.6}$$

For $Q \in \Omega_\varepsilon$, we set

$$\Psi_{\Lambda,Q}(y) = \Psi(\frac{y-Q}{\Lambda}).$$

Then

$$\Delta\Psi_{\Lambda,Q}(y) + U_{\Lambda,Q} = 0 \text{ in } \mathbb{R}^6.$$

It is easy to check that

$$|\Psi_{\Lambda,Q}(y)|,\ |\partial_\Lambda\Psi_{\Lambda,Q}(y)| \le \frac{C}{(1+|y-Q|)^2},\ |\partial_{Q_i}\Psi_{\Lambda,Q}(y)| \le \frac{C}{(1+|y-Q|)^3}. \tag{2.7}$$

In order to obtain approximate solutions which satisfy the boundary condition, we need an extra correction term. For this purpose, we define

$$\hat{U}_{\Lambda,Q/\varepsilon}(z) = -\Psi_{\Lambda,Q/\varepsilon}(z) - c_n\mu^{-1}\varepsilon^{n-4}\Lambda^{\frac{n-2}{2}}H(\varepsilon z,Q) + R_{\varepsilon,\Lambda,Q}(z)\chi(\varepsilon z), \tag{2.8}$$

where $R_{\varepsilon,\Lambda,Q}$ is defined by $\Delta R_{\varepsilon,\Lambda,Q} - \varepsilon^2 R_{\varepsilon,\Lambda,Q} = 0$ in $\Omega_\varepsilon$ and

$$\mu\varepsilon^2\frac{\partial R_{\varepsilon,\Lambda,Q}}{\partial\nu} = -\frac{\partial}{\partial\nu}\Big[U_{\Lambda,Q/\varepsilon} - \mu\varepsilon^2\Psi_{\Lambda,Q/\varepsilon} - c_n\varepsilon^{n-2}\Lambda^{\frac{n-2}{2}}H(\varepsilon z,Q)\Big] \text{ on } \partial\Omega_\varepsilon, \tag{2.9}$$

where $\chi(x)$ is a smooth cut-off function in $\Omega$ such that $\chi(x)=1$ for $d(x,\partial\Omega) < \delta/4$ and $\chi(x)=0$ for $d(x,\partial\Omega) > \delta/2$.

We notice (2.2) and (2.6), an expansion of $U_{\Lambda,Q/\varepsilon}$ and the definition of $H$ imply that the normal derivative of $R_{\varepsilon,Q}$ is of order $\varepsilon^{n-3}$ on the boundary of $\Omega_\varepsilon$, from which we deduce [1]

$$\big|R_{\varepsilon,\Lambda,Q}\big| + \big|\varepsilon^{-1}\nabla_z R_{\varepsilon,\Lambda,Q}\big| + \big|\varepsilon^{-2}\nabla_z^2 R_{\varepsilon,\Lambda,Q}\big| \le \begin{cases} C\Lambda, & n=4,\\ C\varepsilon^2, & n=6.\end{cases} \tag{2.10}$$

A similar estimate also holds for the derivatives of $R_{\varepsilon,\Lambda,Q}$ with respect to $\Lambda,\ Q$.

Now we are able to define the approximate bubble solutions. Since it is different in analyzing between the case $n=4$ and the case $n=6$, we will tact them respectively in the following. For $n=4$, let

$$\Lambda_{4,1} \le \Lambda \le \Lambda_{4,2},\ Q \in \mathcal{M}_{\delta_4} := \{x\in\Omega|\ d(x,\partial\Omega) > \delta_4\}, \tag{2.11}$$

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[1]For $n=4$, the parameter $\Lambda$ is located in a range that depends on $\varepsilon$. Therefore, we have to take $\Lambda$ into consideration, and we note that each component on the right hand side of (2.9) exactly carry $\Lambda$ as a factor.

where $\Lambda_{4,1} = \exp(-\frac{1}{2})\varepsilon^{\beta}$, $\Lambda_{4,2} = \exp(-\frac{1}{2})\varepsilon^{-\beta}$, $\beta$ is a small constant with a generic constant $\delta_4$, to be determined later. In viewing of the rescaling, we write

$$\bar{Q} = \frac{1}{\varepsilon}Q,$$

and we define our approximate solutions as

$$W_{\varepsilon,\Lambda,Q} = U_{\Lambda,\bar{Q}} + \mu\varepsilon^2\hat{U}_{\Lambda,\bar{Q}} + \frac{c_4\Lambda}{|\Omega|}\mu^{-1}\varepsilon^2. \tag{2.12}$$

For $n = 6$, let

$$\begin{aligned} \sqrt{\frac{|\Omega|}{c_6}(\frac{1}{96} - \Lambda_6\varepsilon^{\frac{2}{3}})} \leq &\Lambda \leq \sqrt{\frac{|\Omega|}{c_6}(\frac{1}{96} + \Lambda_6\varepsilon^{\frac{2}{3}})}, \\ Q \in \mathcal{M}_{\delta_6} &:= \{x \in \Omega |\ d(x, \partial\Omega) > \delta_6\}, \\ \frac{1}{48} - \eta_6\varepsilon^{\frac{1}{3}} \leq &\eta \leq \frac{1}{48} + \eta_6\varepsilon^{\frac{1}{3}}, \end{aligned} \tag{2.13}$$

where $\Lambda_6$ and $\eta_6$ are some constants that may depend on the domain, $\delta_6$ is a small constant, which is determined later. Our approximate solution for $n = 6$ is the following

$$W_{\varepsilon,\Lambda,Q,\eta} = U_{\Lambda,\bar{Q}} + \mu\varepsilon^2\hat{U}_{\Lambda,\bar{Q}} + \eta\mu^{-1}\varepsilon^4. \tag{2.14}$$

**Remark**: We want to mention more for the setting of $\eta$ in the case $n = 6$. For the case $n = 4$, we just set $\eta$ to cancel the Laplacian of $H(x, Q)$, which makes the error of the approximate solution small enough to control. While, in the case $n = 6$, we can not proceed in the same way. Furthermore, if we simply set $\eta$ to be the solution of the quadratic equation

$$24\eta^2 - \eta + c_6\Lambda^2/(|\Omega|) = 0.$$

It could make the error small, However, we shall fail in getting the a-priori estimate because of the parameter we obtained in the reduced problem. This is the main difficulty in studying the Lin-Ni problem for $n = 6$.

For convenience, in the following, we write $W$, $U$, $\hat{U}$, $R$, and $\Psi$ instead of $W_{\varepsilon,\Lambda,Q}$, $U_{\varepsilon,\bar{Q}}$, $\hat{U}_{\Lambda,\bar{Q}}$, $R_{\varepsilon,\Lambda,Q}$ and $\Psi_{\Lambda,\bar{Q}}$ respectively in the following. By construction, the normal derivative of $W$ vanishes on the boundary of $\Omega_\varepsilon$, and $W$ satisfies

$$-\Delta W + \mu\varepsilon^2 W = \begin{cases} 8U^3 + \mu^2\varepsilon^4\hat{U} - \mu\varepsilon^2\Delta(R_{\varepsilon,\Lambda,Q}\chi), & n = 4, \\ 24U^2 + \mu^2\varepsilon^4\hat{U} - \mu\varepsilon^2\Delta(R_{\varepsilon,\Lambda,Q}\chi) + \varepsilon^6(\eta - \frac{c_6\Lambda^2}{|\Omega|}), & n = 6. \end{cases} \tag{2.15}$$

We note that $W$ depends smoothly on $\Lambda$, $\bar{Q}$. Setting, for $z \in \Omega_\varepsilon$,

$$\langle z - \bar{Q}\rangle = (1 + |z - \bar{Q}|^2)^{\frac{1}{2}}.$$

A simple computation shows

$$|W(z)| \leq \begin{cases} C(\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}} + \langle z - \bar{Q}\rangle^{-2}), & n = 4, \\ C(\varepsilon^3 + \langle z - \bar{Q}\rangle^{-4}), & n = 6, \end{cases} \tag{2.16}$$

$$|D_\Lambda W(z)| \leq \begin{cases} C(\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}} + \langle z - \bar{Q}\rangle^{-2}), & n = 4, \\ C\langle z - \bar{Q}\rangle^{-4}, & n = 6, \end{cases} \tag{2.17}$$

$$|D_{\bar{Q}}W(z)| \le \begin{cases} C(\langle z-\bar{Q}\rangle^{-3}), & n=4, \\ C(\langle z-\bar{Q}\rangle^{-5}), & n=6, \end{cases} \tag{2.18}$$

and

$$|D_\eta W(z)| = O(\varepsilon^3),\ n=6. \tag{2.19}$$

According to the choice of $W$, we have the following error and energy estimates, we leave the proof in Section 7.

**Lemma 2.1.** *For $n=4$, we have*

$$\begin{aligned}\big|S_\varepsilon[W](z)\big| &\le C\Big(\langle z-\bar{Q}\rangle^{-4}\varepsilon^2(-\ln\varepsilon)^{\frac12} + \langle z-\bar{Q}\rangle^{-2}\varepsilon^4(-\ln\varepsilon)\Big) \\ &\quad + C\Big(\big(\frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac12}} + \frac{\varepsilon^4}{(-\ln\varepsilon)}|\ln(\frac{1}{\varepsilon(1+|z-\bar{Q}|)})|\big)\Lambda\Big),\end{aligned} \tag{2.20}$$

$$\begin{aligned}\big|D_\Lambda S_\varepsilon[W](z)\big| &\le C\Big(\langle z-\bar{Q}\rangle^{-4}\varepsilon^2(-\ln\varepsilon)^{\frac12} + \langle z-\bar{Q}\rangle^{-2}\varepsilon^4(-\ln\varepsilon) \\ &\quad + \frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac12}} + \frac{\varepsilon^4}{(-\ln\varepsilon)}|\ln(\frac{1}{\varepsilon(1+|z-\bar{Q}|)})|\Big),\end{aligned} \tag{2.21}$$

$$\begin{aligned}\big|D_{\bar{Q}} S_\varepsilon[W](z)\big| &\le C\Big(\langle z-\bar{Q}\rangle^{-5}\varepsilon^2(-\ln\varepsilon)^{\frac12} + \langle z-\bar{Q}\rangle^{-3}\varepsilon^4(-\ln\varepsilon) \\ &\quad + \langle z-\bar{Q}\rangle^{-1}\frac{\varepsilon^4}{(-\ln\varepsilon)}\Big),\end{aligned} \tag{2.22}$$

*and*

$$\begin{aligned}J_\varepsilon[W] &= 2\int_{\mathbb{R}^4} U_{1,0}^4 + \frac{c_4\Lambda^2}{4}\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac12}\ln\frac{1}{\Lambda\varepsilon} - \frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac12} \\ &\quad + \frac12 c_4^2\Lambda^2\varepsilon^2 H(Q,Q) + O\big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac12}\Lambda^2\big) + O(\varepsilon^4(-\ln\varepsilon)^2).\end{aligned} \tag{2.23}$$

*For $n=6$, we have*

$$S_\varepsilon[W](z) = -\varepsilon^6\big(24\eta^2 - \eta + \frac{c_6\Lambda^2}{|\Omega|}\big) + O(1)\varepsilon^3\langle z-\bar{Q}\rangle^{-4}, \tag{2.24}$$

$$\big|D_\Lambda S_\varepsilon[W](z)\big| = O(1)\big(\langle z-\bar{Q}\rangle^{-4}\varepsilon^3 + \varepsilon^6\big), \tag{2.25}$$

$$\big|D_\eta S_\varepsilon[W](z)\big| = O(1)\big(\langle z-\bar{Q}\rangle^{-4}\varepsilon^3 + \varepsilon^{6\frac13}\big), \tag{2.26}$$

$$\big|D_{\bar{Q}} S_\varepsilon[W](z)\big| \le C\langle z-\bar{Q}\rangle^{-5}\varepsilon^3, \tag{2.27}$$

*and*

$$\begin{aligned}J_\varepsilon[W] &= 4\int_{\mathbb{R}^6} U_{1,0}^3 + \Big(\frac12\eta^2|\Omega| - c_6\eta\Lambda^2 + \frac{1}{48}c_6\Lambda^2 - 8\eta^3|\Omega|\Big)\varepsilon^3 + \frac12 c_6^2\Lambda^4\varepsilon^4 H(Q,Q) \\ &\quad + \frac12\big(\eta - \frac{c_6\Lambda^2}{|\Omega|}\big)\varepsilon^4\int_\Omega \frac{\Lambda^2}{|x-Q|^4} + O(\varepsilon^5).\end{aligned} \tag{2.28}$$

## 3. Finite-Dimensional Reduction

According to our general strategy, we first consider the linearized problem at $W$, and we solve it in a finite-codimensional space, i.e., the orthogonal space to the finite-dimensional subspace generated by the derivatives of $W$ with respect to the parameters $\Lambda$ and $\bar{Q}_i$ in the case $n=4$, and the orthogonal space to the finite-dimensional subspace generated by the derivatives of $W$ with respect to the parameters $\Lambda$, $\bar{Q}_i$ and $\eta$ in the case $n=6$. Equipping $H^1(\Omega_\varepsilon)$ with the scalar product

$$(u,v)_\varepsilon=\int_{\Omega_\varepsilon}(\nabla u\cdot\nabla v+\mu\varepsilon^2 uv). \tag{3.1}$$

For the case $n=4$. Orthogonality to the functions

$$Y_0=\frac{\partial W}{\partial\Lambda},\ Y_i=\frac{\partial W}{\partial\bar{Q}_i},1\le i\le 4, \tag{3.2}$$

in that space is equivalent to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle\cdot,\cdot\rangle$, to the functions $Z_i, 0\le i\le 4$, defined as

$$\begin{cases}Z_0=-\Delta\frac{\partial W}{\partial\Lambda}+\mu\varepsilon^2\frac{\partial W}{\partial\Lambda},\\ Z_i=-\Delta\frac{\partial W}{\partial\bar{Q}_i}+\mu\varepsilon^2\frac{\partial W}{\partial\bar{Q}_i},\ 1\le i\le 4.\end{cases} \tag{3.3}$$

Straightforward computations provide us with the estimate:

$$|Z_i(z)|\le C(\varepsilon^4+\langle z-\bar{Q}\rangle^{-6}). \tag{3.4}$$

Then, we consider the following problem: given $h$, finding a solution $\phi$ which satisfies

$$\begin{cases}-\Delta\phi+\mu\varepsilon^2\phi-24W^2\phi=h+\Sigma_{i=0}^4 c_iZ_i & \text{in } \Omega_\varepsilon,\\ \frac{\partial\phi}{\partial\nu}=0 & \text{on } \partial\Omega_\varepsilon,\\ \langle Z_i,\phi\rangle=0, & 0\le i\le 4,\end{cases} \tag{3.5}$$

for some numbers $c_i$.

While for the case $n=6$. Orthogonality to the functions

$$Y_0=\frac{\partial W}{\partial\Lambda},\ Y_i=\frac{\partial W}{\partial\bar{Q}_i},1\le i\le 6,\ Y_7=\frac{\partial W}{\partial\eta}, \tag{3.6}$$

in that space is equivalent to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle\cdot,\cdot\rangle$, to the functions $Z_i, 0\le i\le 7$, defined as

$$\begin{cases}Z_0=-\Delta\frac{\partial W}{\partial\Lambda}+\mu\varepsilon^2\frac{\partial W}{\partial\Lambda},\\ Z_i=-\Delta\frac{\partial W}{\partial\bar{Q}_i}+\mu\varepsilon^2\frac{\partial W}{\partial\bar{Q}_i},\ 1\le i\le 6,\\ Z_7=-\Delta\frac{\partial W}{\partial\eta}+\mu\varepsilon^2\frac{\partial W}{\partial\eta}.\end{cases} \tag{3.7}$$

Direct computations provide us the following estimate:

$$|Z_i(z)|\le C(\varepsilon^6+\langle z-\bar{Q}\rangle^{-8}),\ 0\le i\le 6,\ Z_7(z)=O(\varepsilon^6). \tag{3.8}$$

Then, we consider the following problem: given $h$, finding a solution $\phi$ which satisfies

$$\begin{cases}-\Delta\phi+\mu\varepsilon^2\phi-48W\phi=h+\Sigma_{i=0}^7 d_iZ_i & \text{in } \Omega_\varepsilon,\\ \frac{\partial\phi}{\partial\nu}=0 & \text{on } \partial\Omega_\varepsilon,\\ \langle Z_i,\phi\rangle=0, & 0\le i\le 7,\end{cases} \tag{3.9}$$

for some numbers $d_i$.

Existence and uniqueness of $\phi$ will follow from an inversion procedure in suitable weighted function space. To this end, we define

$$\begin{cases} \|\phi\|_* = \|\langle z-\bar{Q}\rangle\phi(z)\|_\infty, \ \|f\|_{**} = \varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}|\overline{f}| + \|\langle z-\bar{Q}\rangle^3 f(z)\|_\infty, \ n=4, \\ \|\phi\|_{***} = \|\langle z-\bar{Q}\rangle^2\phi(z)\|_\infty, \ \|f\|_{****} = \|\langle z-\bar{Q}\rangle^4 f(z)\|_\infty, \ n=6, \end{cases} \tag{3.10}$$

where $\|f\|_\infty = \max_{z\in\Omega_\varepsilon}|f(z)|$ and $\overline{f} = |\Omega_\varepsilon|^{-1}\int_{\Omega_\varepsilon} f(z)\mathrm{d}z$ denotes the average of $f$ in $\Omega_\varepsilon$.

Before stating an existence result for $\phi$ in (3.5) and (3.9), we need the following lemma:

**Lemma 3.1.** *Let $u$ and $f$ satisfy*

$$-\Delta u = f, \ \frac{\partial u}{\partial\nu} = 0, \ \bar{u} = \bar{f} = 0.$$

*Then*

$$|u(x)| \le C\int_{\Omega_\varepsilon}\frac{|f(y)|}{|x-y|^{n-2}}\mathrm{d}y. \tag{3.11}$$

*Proof.* The proof is similar to Lemma 3.1 in [19], we omit it here. □

As a consequence, we have

**Corollary 3.2.** *For $n=4$, suppose $u$ and $f$ satisfy*

$$-\Delta u + \mu\varepsilon^2 u = f \text{ in } \Omega_\varepsilon, \ \frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega_\varepsilon.$$

*Then*

$$\|u\|_* \le C\|f\|_{**}. \tag{3.12}$$

*For $n=6$, suppose $u$ and $f$ satisfy*

$$-\Delta u + c\mu\varepsilon^2 u = f \text{ in } \Omega_\varepsilon, \ \frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega_\varepsilon, \ \overline{u} = \overline{f} = 0,$$

*where $c$ is an arbitrary constant. Then*

$$\|u\|_{***} \le C\|f\|_{****}. \tag{3.13}$$

*Proof.* For $n=4$, integrating the equation yields $\bar{f} = \mu\varepsilon^2\bar{u}$. We may rewrite the original equation as

$$\Delta(u-\bar{u}) = \mu\varepsilon^2(u-\bar{u}) - (f-\bar{f}).$$

With the help of Lemma 3.1, we get

$$|u(y)-\bar{u}| \le C\mu\varepsilon^2\int_{\Omega_\varepsilon}\frac{|u(x)-\bar{u}|}{|x-y|^2}\mathrm{d}x + C\int_{\Omega_\varepsilon}\frac{|f(x)-\bar{f}|}{|x-y|^2}\mathrm{d}x.$$

Since

$$\langle y-\bar{Q}\rangle\int_{\mathbb{R}^4}\frac{1}{|x-y|^2}\langle x-\bar{Q}\rangle^{-3}\mathrm{d}x < \infty,$$

we obtain

$$\begin{aligned} \|\langle y-\bar{Q}\rangle|u-\bar{u}|\|_\infty &\le C\mu\varepsilon^2\|\langle y-\bar{Q}\rangle^3|u-\bar{u}|\|_\infty + C\|\langle y-\bar{Q}\rangle^3|f-\bar{f}|\|_\infty \\ &\le C\mu\|\langle y-\bar{Q}\rangle|u-\bar{u}|\|_\infty + C\|\langle y-\bar{Q}\rangle^3|f-\bar{f}|\|_\infty, \end{aligned}$$

which gives

$$\|\langle y-\bar{Q}\rangle|u-\bar{u}|\|_\infty \le C\|\langle y-\bar{Q}\rangle^3|f-\bar{f}|\|_\infty,$$

whence

$$\|\langle y-\bar{Q}\rangle u\|_\infty \le C\|\langle y-\bar{Q}\rangle\|_\infty|\bar{u}| + C\varepsilon^{-3}|\bar{f}| + \|\langle y-\bar{Q}\rangle^3 f\|_\infty \le C\|f\|_{**}.$$

Hence we finish the proof of the case $n=4$.

For $n=6$, by the help of Lemma 3.1,

$$|\langle y-\bar{Q}\rangle^2 u| \le C\int_{\Omega_\varepsilon}\frac{\langle y-\bar{Q}\rangle^2(|\mu\varepsilon^2 u|+|f|)}{|x-y|^4}\mathrm{d}x \le C(|\mu\ln\varepsilon|\|u\|_{***}+\|f\|_{****}),$$

where we used some similar estimates appeared in $n=4$. From the above inequality, we obtain $\|u\|_{***}\le\|f\|_{****}$. Hence we finish the proof. □

We now state the main result in this section

**Proposition 3.3.** *There exists $\varepsilon_0>0$ and a constant $C>0$, independent of $\varepsilon$, $\Lambda$, $Q$ satisfying (2.11) and independent of $\varepsilon$, $\eta$, $\Lambda$, $Q$ satisfying (2.13), such that for all $0<\varepsilon<\varepsilon_0$ and all $h\in L^\infty(\Omega_\varepsilon)$, problem (3.5) and (3.9) has a unique solution $\phi=L_\varepsilon(h)$. Furthermore, for equation (3.5) and (3.9), we have the following estimates,*

$$\begin{aligned}&\|L_\varepsilon(h)\|_* \le C\|h\|_{**},\ |c_i|\le C\|h\|_{**} \text{ for } 0\le i\le 4,\\ &\|L_\varepsilon(h)\|_{***}\le C\|h\|_{****},\ |d_i|\le C\|h\|_{****} \text{ for } 0\le i\le 6.\end{aligned} \tag{3.14}$$

*Moreover, the map $L_\varepsilon(h)$ is $C^1$ with respect to $\Lambda$, $\bar{Q}$ of the $L^\infty_*$-norm in $n=4$ and with respect to $\Lambda$, $\bar{Q}$, $\eta$ of the $L^\infty_{***}$-norm in $n=6$, i.e.,*

$$\|D_{(\Lambda,\bar{Q})}L_\varepsilon(h)\|_* \le C\|h\|_{**} \text{ in } n=4, \quad \|D_{(\eta,\Lambda,\bar{Q})}L_\varepsilon(h)\|_{***}\le C\varepsilon^{-1}\|h\|_{****} \text{ in } n=6. \tag{3.15}$$

The argument goes the same as the Proposition 3.1 in [19], for convenience of the reader, we list the proof here. First, we need the following Lemma

**Lemma 3.4.** *For $n=4$, assume that $\phi_\varepsilon$ solves (3.5) for $h=h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as $\varepsilon$ goes to zero, so does $\|\phi_\varepsilon\|_*$. While for $n=6$, assume that $\phi_\varepsilon$ solves (3.9) for $h=h_\varepsilon$. If $\|h_\varepsilon\|_{****}$ goes to zero as $\varepsilon$ goes to zero, so does $\|\phi_\varepsilon\|_{***}$.*

*Proof.* We prove this lemma by contradiction. First we consider for the case $n=4$, let $\|\phi_\varepsilon\|_*=1$. Multiplying the first equation in (3.5) by $Y_j$ and integrating in $\Omega_\varepsilon$ we find

$$\sum_i c_i\langle Z_i,Y_j\rangle = \langle -\Delta Y_j+\mu\varepsilon^2 Y_j-24W^2Y_j,\phi_\varepsilon\rangle - \langle h_\varepsilon,Y_j\rangle.$$

We can easily get the following equalities from the definition of $Z_i, Y_j$

$$\begin{aligned}&\langle Z_0,Y_0\rangle = \|Y_0\|_\varepsilon^2=\gamma_0+o(1),\\ &\langle Z_i,Y_i\rangle=\|Y_i\|_\varepsilon^2=\gamma_1+o(1),\ 1\le i\le 4,\end{aligned} \tag{3.16}$$

where $\gamma_0,\gamma_1$ are strictly positive constants, and

$$\langle Z_i,Y_j\rangle = o(1),\ i\ne j. \tag{3.17}$$

On the other hand, in view of the definition of $Y_j$ and $W$, straightforward computations yield

$$\langle -\Delta Y_j+\mu\varepsilon^2Y_j-24W^2Y_j,\phi_\varepsilon\rangle = o(\|\phi_\varepsilon\|_*)$$

and

$$\langle h_\varepsilon,Y_j\rangle = O(\|h_\varepsilon\|_{**}).$$

Consequently, inverting the quasi diagonal linear system solved by the $c_i$'s we find

$$c_i = O(\|h_\varepsilon\|_{**}) + o(\|\phi_\varepsilon\|_*). \tag{3.18}$$

In particular, $c_i = o(1)$ as $\varepsilon$ goes to zero.

Since $\|\phi_\varepsilon\|_* = 1$, elliptic theory shows that along some subsequence, the functions $\phi_{\varepsilon,0} = \phi_\varepsilon(y - \bar{Q})$ converge uniformly in any compact subset of $\mathbb{R}^4$ to a nontrivial solution of

$$-\Delta\phi_0 = 24U_{\Lambda,0}^2\phi_0.$$

A bootstrap argument (see e.g.Proposition 2.2 of [23]) implies $|\phi_0(y)| \le C(1+|y|)^{-2}$. As consequence, $\phi_0$ can be written as

$$\phi_0 = \alpha_0\frac{\partial U_{\Lambda,0}}{\partial\Lambda} + \sum_i \alpha_i\frac{\partial U_{\Lambda,0}}{\partial y_i}$$

(see [18]). On the other hand, equalities $\langle Z_i, \phi_\varepsilon\rangle = 0$ yield

$$\int_{\mathbb{R}^4} -\Delta\frac{\partial U_{\Lambda,0}}{\partial\Lambda}\phi_0 = \int_{\mathbb{R}^4} U_{\Lambda,0}^2\frac{\partial U_{\Lambda,0}}{\partial\Lambda}\phi_0 = 0,$$
$$\int_{\mathbb{R}^4} -\Delta\frac{\partial U_{\Lambda,0}}{\partial y_i}\phi_0 = \int_{\mathbb{R}^4} U_{\Lambda,0}^2\frac{\partial U_{\Lambda,0}}{\partial y_i}\phi_0 = 0,\ 1\le i\le 4.$$

As we also have

$$\int_{\mathbb{R}^4}\left|\nabla\frac{\partial U_{\Lambda,0}}{\partial\Lambda}\right|^2 = \gamma_0 > 0,\ \int_{\mathbb{R}^4}\left|\nabla\frac{\partial U_{\Lambda,0}}{\partial y_i}\right|^2 = \gamma_1 > 0,\ 1\le i\le 4,$$

and

$$\int_{\mathbb{R}^4}\nabla\frac{\partial U_{\Lambda,0}}{\partial\Lambda}\nabla\frac{\partial U_{\Lambda,0}}{\partial y_i} = \int_{\mathbb{R}^4}\nabla\frac{\partial U_{\Lambda,0}}{\partial y_i}\nabla\frac{\partial U_{\Lambda,0}}{\partial y_j} = 0,\ i\ne j,$$

the $\alpha_i's$ solve a homogeneous quasi diagonal linear system, yielding $\alpha_i = 0, 0\le i\le 4$, and $\phi_0 = 0$. So $\phi_\varepsilon(z-\bar{Q}) \to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. Next, we will show $\|\phi_\varepsilon\|_* = o(1)$ by using the equation (3.5).

Using (3.5) and Corollary 3.2, we have

$$\|\phi_\varepsilon\|_* \le C(\|W^2\phi_\varepsilon\|_{**} + \|h\|_{**} + \sum_i |c_i|\|Z_i\|_{**}). \tag{3.19}$$

Then we estimate the right hand side of (3.19) term by term. By the help of (2.16), we deduce that

$$|\langle z-\bar{Q}\rangle^3 W^2\phi_\varepsilon| \le C\varepsilon^4(-\ln\varepsilon)\langle z-\bar{Q}\rangle^2\|\phi_\varepsilon\|_* + \langle z-\bar{Q}\rangle^{-1}|\phi_\varepsilon|. \tag{3.20}$$

Since $\|\phi_\varepsilon\|_* = 1$, the first term on the right hand side of (3.20) is dominated by $\varepsilon^2(-\ln\varepsilon)$. The last term goes uniformly to zero in any ball $B_R(\bar{Q})$, and is dominated by $\langle z-\bar{Q}\rangle^{-2}\|\phi_\varepsilon\|_* = \langle z-\bar{Q}\rangle^{-2}$, which, through the choice of $R$, can be made as small as possible in $\Omega_\varepsilon\backslash B_R(\bar{Q})$. Consequently,

$$|\langle z-\bar{Q}\rangle^3 W^2\phi_\varepsilon| = o(1) \tag{3.21}$$

as $\varepsilon$ goes to zero, uniformly in $\Omega_\varepsilon$. On the other hand, we can also get

$$\begin{aligned}\varepsilon^{-3}(-\ln\varepsilon)^{\frac12}\overline{W^2\phi_\varepsilon} &\le C\varepsilon(-\ln\varepsilon)^{\frac12}\int_{\Omega_\varepsilon}(\langle z-\bar{Q}\rangle^{-4} + \varepsilon^4(-\ln\varepsilon))|\phi_\varepsilon| \\ &\le C\varepsilon(-\ln\varepsilon)^{\frac12}\int_{\Omega_\varepsilon}(\langle z-\bar{Q}\rangle^{-5} + \varepsilon^4(-\ln\varepsilon)\langle z-\bar{Q}\rangle^{-1})\|\phi_\varepsilon\|_* \\ &= o(1).\end{aligned}$$

Finally, we obtain

$$\|W^2\phi_\varepsilon\|_{**} = o(1).$$

In view of the formula (3.4), we have

$$\langle z-\bar{Q}\rangle^3|Z_i| \le C(\langle z-\bar{Q}\rangle^3\varepsilon^4(\frac{1}{-\ln\varepsilon}) + \langle z-\bar{Q}\rangle^{-3}) = O(1).$$

and

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}\overline{Z_i} \le C\varepsilon(-\ln\varepsilon)^{\frac{1}{2}}\int_{\Omega_\varepsilon}|\langle z-\bar{Q}\rangle^{-6}+\varepsilon^4|\mathrm{d}x = o(1).$$

Hence, $\|Z_i\|_{**} = O(1)$. Therefore, we have

$$\|\phi_\varepsilon\|_* \le C(\|W^2\phi_\varepsilon\|_{**} + \|h\|_{**} + \sum_i |c_i|\|Z_i\|_{**}) = o(1), \tag{3.22}$$

which contradicts our assumption that $\|\phi_\varepsilon\|_* = 1$.

For $n = 6$. We still assume that $\|\phi_\varepsilon\|_{***} = 1$. Using the similar arguments in previous case, we obtain the following

$$d_i = O(\|h\|_{****}) + o(\|\phi\|_{***}) \text{ for } 0 \le i \le 6,\ \ d_7 = O(\varepsilon^{-2}\|h\|_{****}) + O(\varepsilon^{-1}\|\phi\|_{***}). \tag{3.23}$$

and $\phi_\varepsilon(z-\bar{Q}) \to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. Next, we will show $\|\phi_\varepsilon\|_{***} = o(1)$ by using the equation (3.9). At first, we write the equation (3.9) into the following

$$-\Delta\phi_\varepsilon + \mu\varepsilon^2(1-48\eta)\phi_\varepsilon = h + \sum_i d_iZ_i + 48U\phi_\varepsilon + 48\varepsilon^3\hat{U}\phi_\varepsilon. \tag{3.24}$$

Since $\int_{\Omega_\varepsilon}\phi = 0$, as a result, we can find the integral for both sides of (3.24) in $\Omega_\varepsilon$ are 0. Using Corollary 3.2 again, we have

$$\|\phi_\varepsilon\|_{***} \le C(\|(U+\varepsilon^3\hat{U})\phi_\varepsilon\|_{****} + \|h\|_{****} + \sum_i |d_i|\|Z_i\|_{****}). \tag{3.25}$$

From the formula of $U$ and $\hat{U}$, it is not difficult to show

$$U + \varepsilon^3\hat{U} \le C\langle z-\bar{Q}\rangle^{-4}.$$

Similar to the case $n = 4$, we could show $\|\langle z-\bar{Q}\rangle^{-4}\phi_\varepsilon\|_{****} = o(1)$, $\|Z_i\|_{****} = O(1)$ for $0 \le i \le 6$ and $\|Z_7\|_{****} = O(\varepsilon^2)$. Therefore, by the above facts and (3.23), we conclude

$$\|\phi_\varepsilon\|_{***} \le o(1) + C\|h\|_{****} + o(1)\|\phi_\varepsilon\|_{***} = o(1)$$

which contradicts the previous assumption that $\|\phi_\varepsilon\|_{***} = 1$. Hence, we finish the proof. □

*Proof of Proposition 3.3.* Since the proof of the case $n = 4$ and $n = 6$ are almost the same, we only give the proof for the former one. We set

$$H = \{\phi \in H^1(\Omega_\varepsilon) \mid \langle Z_i, \phi\rangle = 0,\ 0 \le i \le 4\},$$

equipped with the scalar product $(\cdot,\cdot)_\varepsilon$. Problem (3.5) is equivalent to find $\phi \in H$ such that

$$(\phi,\theta)_\varepsilon = \langle 24W^2\phi + h, \theta\rangle, \quad \forall\theta \in H,$$

that is

$$\phi = T_\varepsilon(\phi) + \tilde{h}, \tag{3.26}$$

where $\tilde{h}$ depends on $h$ linearly, and $T_\varepsilon$ is a compact operator in $H$. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of

$Id - T_\varepsilon$ is reduced to 0. We notice that any $\phi_\varepsilon \in \mathrm{Ker}(Id - T_\varepsilon)$ solves (3.5) with $h = 0$. Thus, we deduce from Lemma 3.4 that $\|\phi_\varepsilon\|_* = o(1)$ as $\varepsilon$ goes to zero. As $\mathrm{Ker}(Id - T_\varepsilon)$ is a vector space and is $\{0\}$. The inequalities (3.14) follow from Lemma 3.4 and (3.18). This completes the proof of the first part of Proposition 3.3.

The smoothness of $L_\varepsilon$ with respect to $\Lambda$ and $\bar{Q}$ is a consequence of the smoothness of $T_\varepsilon$ and $\tilde{h}$, which occur in the implicit definition (3.26) of $\phi \equiv L_\varepsilon(h)$, with respect to these variables. Inequality (3.15) is obtained by differentiating (3.5), writing the derivatives of $\phi$ with respect $\Lambda$ and $\bar{Q}$ as linear combinations of the $Z_i$'s and an orthogonal part, and estimating each term by using the first part of the proposition, one can see [6],[12] for detailed computations. □

## 4. Finite-dimensional reduction:a nonlinear problem

In this section, we turn our attention to the nonlinear problem, which we solve in the finite-dimensional subspace orthogonal to the $Z_i$. Let $S_\varepsilon[u]$ be as defined at (1.12). Then (1.11) is equivalent to

$$S_\varepsilon[u] = 0 \text{ in } \Omega_\varepsilon, \quad u_+ \neq 0, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon. \tag{4.1}$$

Indeed, if $u$ satisfies (4.1), the Maximal Principle ensures that $u > 0$ in $\Omega_\varepsilon$ and (1.12) is satisfied. Observing that

$$S_\varepsilon[W+\phi] = -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - n(n-2)(W+\phi)^{\frac{n+2}{n-2}}$$

may be written as

$$S_\varepsilon[W+\phi] = -\Delta\phi + \mu\varepsilon^2\phi - n(n+2)W^{\frac{4}{n-2}}\phi + R^\varepsilon - n(n-2)N_\varepsilon(\phi) \tag{4.2}$$

with

$$N_\varepsilon(\phi) = (W+\phi)^{\frac{n+2}{n-2}} - W^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}W^{\frac{4}{n-2}}\phi \tag{4.3}$$

and

$$R^\varepsilon = S_\varepsilon[W] = -\Delta W + \mu\varepsilon^2 W - n(n-2)W^{\frac{n+2}{n-2}}. \tag{4.4}$$

From Lemma 2.1 we get

$$\begin{cases} \|R^\varepsilon\|_{**} \le C\varepsilon\Lambda + \varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}, \ \|D_{(\Lambda,\bar{Q})}R^\varepsilon\|_{**} \le C\varepsilon, & n=4, \\ \|R^\varepsilon\|_{****} \le C\varepsilon^{2\frac{2}{3}}, \ \|D_{(\Lambda,\bar{Q},\eta)}R^\varepsilon\|_{****} \le C\varepsilon^2, & n=6. \end{cases} \tag{4.5}$$

We now consider the following nonlinear problem: finding $\phi$ such that, for some numbers $c_i$,

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 8(W+\phi)^3 = \sum_i c_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \le i \le 4 \end{cases} \tag{4.6}$$

for $n = 4$, and finding $\phi$ such that, for some numbers $d_i$,

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 24(W+\phi)^2 = \sum_i d_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \le i \le 7 \end{cases} \tag{4.7}$$

for $n=6$. The first equation in (4.6) and (4.7) can be also written as

$$-\Delta\phi+\mu\varepsilon^2\phi-24W^2\phi=8N_\varepsilon(\phi)-R^\varepsilon+\sum_i c_iZ_i,$$
$$-\Delta\phi+\mu\varepsilon^2\phi-48W\phi=24N_\varepsilon(\phi)-R^\varepsilon+\sum_i d_iZ_i. \tag{4.8}$$

In order to employ the contraction mapping theorem to prove that (4.6) and (4.7) are uniquely solvable in the set where $\|\phi\|_*$ and $\|\phi\|_{***}$ are small respectively, we need to estimate $N_\varepsilon$ in the following lemma.

**Lemma 4.1.** *There exists $\varepsilon_1>0$, independent of $\Lambda,\bar{Q},\eta$ and $C$ independent of $\varepsilon,\Lambda,\bar{Q},\eta$ such that for $\varepsilon\le\varepsilon_1$ and*

$$\|\phi\|_*\le C\varepsilon\Lambda \text{ for } n=4,\quad \|\phi\|_{***}\le C\varepsilon^{2\frac{2}{3}} \text{ for } n=6.$$

*Then,*

$$\|N_\varepsilon(\phi)\|_{**}\le C\varepsilon\Lambda\|\phi\|_* \text{ for } n=4,\quad \|N_\varepsilon(\phi)\|_{****}\le C\varepsilon\|\phi\|_{***} \text{ for } n=6. \tag{4.9}$$

*For*

$$\|\phi_i\|_*\le C\varepsilon\Lambda \text{ for } n=4,\quad \|\phi_i\|_{***}\le C\varepsilon^{2\frac{2}{3}} \text{ for } n=6,\qquad i=1,2.$$

*Then,*

$$\|N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)\|_{**}\le C\varepsilon\Lambda\|\phi_1-\phi_2\|_* \text{ for } n=4,$$
$$\|N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)\|_{****}\le C\varepsilon\|\phi_1-\phi_2\|_{***} \text{ for } n=6. \tag{4.10}$$

*Proof.* Since the proof of these two cases are similar, we only consider $n=4$ here. From (4.3), we see

$$|N_\varepsilon(\phi)|\le C(W\phi^2+|\phi|^3). \tag{4.11}$$

Using (2.15), we gain

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac12}\overline{W\phi^2+|\phi|^3}=\varepsilon(-\ln\varepsilon)^{\frac12}\int_{\Omega_\varepsilon}(W\phi^2+|\phi|^3),$$

where the integration term on the right hand side of the above equality can be estimated as

$$\begin{aligned}\left|W\phi^2+|\phi|^3\right|&\le C\left(\left(\langle z-\bar{Q}\rangle^{-2}+\varepsilon^2(-\ln\varepsilon)^{\frac12}\right)|\phi|^2+|\phi|^3\right)\\&\le C\Big(\big(\langle z-\bar{Q}\rangle^{-4}+\varepsilon^2(-\ln\varepsilon)^{\frac12}\langle z-\bar{Q}\rangle^{-2}\big)\|\phi\|_*^2+\langle z-\bar{Q}\rangle^{-3}\|\phi\|_*^3\Big)\\&\le C\Big(\big(\varepsilon\langle z-\bar{Q}\rangle^{-4}+\varepsilon^3(-\ln\varepsilon)^{\frac12}\langle z-\bar{Q}\rangle^{-2}\big)\Lambda\Big)\|\phi\|_*.\end{aligned}$$

As a consequence,

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac12}\overline{W\phi^2+|\phi|^3}\le C\varepsilon^2(-\ln\varepsilon)^{\frac32}\Lambda\|\phi\|_*\le C\varepsilon\Lambda\|\phi\|_*.$$

On the other hand,

$$\|\langle z-\bar{Q}\rangle^3(W\phi^2+|\phi|^3)\|_\infty\le C\varepsilon\Lambda\|\phi\|_*.$$

Thus, (4.9) follows. Concerning (4.10), we write

$$N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)=\partial_\vartheta N_\varepsilon(\vartheta)(\phi_1-\phi_2)$$

for some $\vartheta=x\phi_1+(1-x)\phi_2,\ x\in[0,1]$. From

$$\partial_\vartheta N_\varepsilon(\vartheta)=3[(W+\vartheta)^2-W^2],$$

we deduce that

$$\partial_\vartheta N_\varepsilon(\vartheta) \le C(|W||\vartheta| + \vartheta^2) \tag{4.12}$$

and the proof of (4.10) is similar to the previous one. □

**Proposition 4.2.** *For the case $n = 4$, there exists $C$, independent of $\varepsilon$ and $\Lambda$, $Q$ satisfying (2.11), such that for small $\varepsilon$ problem (4.6) has a unique solution $\phi = \phi(\Lambda, \bar{Q}, \varepsilon)$ with*

$$\|\phi\|_* \le C\varepsilon\Lambda. \tag{4.13}$$

*Moreover, $(\Lambda, \bar{Q}) \to \phi(\Lambda, \bar{Q}, \varepsilon)$ is $C^1$ with respect to the $*$-norm, and*

$$\|D_{(\Lambda,\bar{Q})}\phi\|_* \le C\varepsilon. \tag{4.14}$$

*For the case $n = 6$, there exists $C$, independent of $\varepsilon$ and $\Lambda$, $\eta$, $Q$ satisfying (2.13), such that for small $\varepsilon$ problem (4.7) has a unique solution $\phi = \phi(\Lambda, \eta, \bar{Q}, \varepsilon)$ with*

$$\|\phi\|_{***} \le C\varepsilon^{\frac{8}{3}}. \tag{4.15}$$

*Moreover, $(\Lambda, \eta, \bar{Q}) \to \phi(\Lambda, \eta, \bar{Q}, \varepsilon)$ is $C^1$ with respect to the $***$-norm, and*

$$\|D_{(\Lambda,\eta,\bar{Q})}\phi\|_{***} \le C\varepsilon^{\frac{5}{3}}. \tag{4.16}$$

*Proof.* We only give the proof of $n = 4$, the other case can be argued similarly. In the same spirit of [6], we consider the map $A_\varepsilon$ from $\mathcal{F}$=$\{\phi \in H^1(\Omega_\varepsilon) | \|\phi\|_* \le C^{'}\varepsilon\Lambda\}$ to $H^1(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi) = L_\varepsilon(8N_\varepsilon(\phi) + R^\varepsilon).$$

Here $C^{'}$ is a large number, to be determined later, and $L_\varepsilon$ is given by Proposition 3.3. We note that finding a solution $\phi$ to problem (4.6) is equivalent to finding a fixed point of $A_\varepsilon$. On the one hand, we have for $\phi \in \mathcal{F}$, using (4.5), Proposition 3.3 and Lemma 4.1,

$$\begin{aligned}\|A_\varepsilon(\phi)\|_* &\le 8\|L_\varepsilon(N_\varepsilon(\phi))||_* + \|L_\varepsilon(R^\varepsilon)\|_* \le C_1(\|N_\varepsilon(\phi)\|_{**} + \varepsilon\Lambda)\\ &\le C_2C^{'}\varepsilon^2\Lambda + C_1\varepsilon\Lambda \le C^{'}\varepsilon\Lambda\end{aligned}$$

for $C^{'} = 2C_1$ and $\varepsilon$ small enough, implying that $A_\varepsilon$ sends $\mathcal{F}$ into itself. On the other hand, $A_\varepsilon$ is a contraction. Indeed, for $\phi_1$ and $\phi_2$ in $\mathcal{F}$, we write

$$\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \le C\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \le C\varepsilon\Lambda\|\phi_1 - \phi_2\|_* \le \frac{1}{2}\|\phi_1 - \phi_2\|_*$$

for $\varepsilon$ small enough. The contraction Mapping Theorem implies that $A_\varepsilon$ has a unique fixed point in $\mathcal{F}$, that is, problem (4.6) has a unique solution $\phi$ such that $\|\phi\|_* \le C^{'}\varepsilon\Lambda$.

In order to prove that $(\Lambda, \bar{Q}) \to \phi(\Lambda, \bar{Q})$ is $C^1$, we remark that if we set for $\psi \in F$,

$$B(\Lambda, \bar{Q}, \psi) \equiv \psi - L_\varepsilon(8N_\varepsilon(\psi) + R^\varepsilon),$$

then $\phi$ is defined as

$$B(\Lambda, \bar{Q}, \phi) = 0. \tag{4.17}$$

We have

$$\partial_\psi B(\Lambda, \bar{Q}, \psi)[\theta] = \theta - 8L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi)).$$

Using Proposition 3.3 and (4.12) we write

$$\begin{aligned}\|L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi))\|_* &\le C\|\theta(\partial_\psi N_\varepsilon)(\psi)\|_{**} \le \|\langle z - \bar{Q}\rangle^{-1}(\partial_\psi N_\varepsilon)(\psi)\|_{**}\|\theta\|_*\\ &\le C\|\langle z - \bar{Q}\rangle^{-1}(W_+|\psi| + |\psi|^2)\|_{**}\|\theta\|_*.\end{aligned}$$

Using (2.16), (3.10) and $\psi \in \mathcal{F}$, we obtain

$$\|L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi))\|_* \le C\varepsilon\|\theta\|_*.$$

Consequently, $\partial_\psi B(\Lambda, \bar{Q}, \phi)$ is invertible with uniformly bounded inverse. Then the fact that $(\Lambda, \bar{Q}) \mapsto \phi(\Lambda, \bar{Q})$ is $C^1$ follows from the fact that $(\Lambda, \bar{Q}, \psi) \mapsto L_\varepsilon(N_\varepsilon(\psi))$ is $C^1$ and the implicit function theorem.

Finally, let us consider (4.14). Differentiating (4.17) with respect to $\Lambda$, we find

$$\partial_\Lambda \phi = (\partial_\psi B(\Lambda, \xi, \phi))^{-1}\Big((\partial_\Lambda L_\varepsilon)(N_\varepsilon(\phi)) + L_\varepsilon((\partial_\Lambda N_\varepsilon)(\phi)) + L_\varepsilon(\partial_\Lambda R^\varepsilon)\Big).$$

Then by Proposition 3.3,

$$\|\partial_\Lambda \phi\|_* \le C(\|N_\varepsilon(\phi)\|_{**} + \|(\partial_\Lambda N_\varepsilon)(\phi)\|_{**} + \|\partial_\Lambda R^\varepsilon\|_{**}).$$

From Lemma 4.1 and (4.13), we know that $\|N_\varepsilon(\phi)\|_{**} \le C\varepsilon^2$. Concerning the next term, we notice that according to the definition of $N_\varepsilon$,

$$|\partial_\Lambda N_\varepsilon(\phi)| = 3\phi^2|\partial_\Lambda W|.$$

Note that

$$|\partial_\Lambda W(z)| \le C(\langle z - \bar{Q}\rangle^{-2} + \varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}),$$

we have

$$\|\partial_\Lambda N_\varepsilon(\phi)\|_{**} \le C\varepsilon.$$

Finally, using (4.5), we obtain

$$\|\partial_\Lambda \phi\|_* \le C\varepsilon.$$

The derivative of $\phi$ with respect to $\bar{Q}$ may be estimated in the same way. This concludes the proof. □

## 5. Finite-dimensional reduction: reduced energy

Let us define a reduced energy functional as

$$I_\varepsilon(\Lambda, Q) \equiv J_\varepsilon[W_{\Lambda,\bar{Q}} + \phi_{\varepsilon,\Lambda,\bar{Q}}] \tag{5.1}$$

for $n = 4$ and

$$I_\varepsilon(\Lambda, \eta, Q) \equiv J_\varepsilon[W_{\Lambda,\eta,\bar{Q}} + \phi_{\varepsilon,\Lambda,\eta,\bar{Q}}] \tag{5.2}$$

for $n = 6$. We have

**Proposition 5.1.** *The function $u = W_{\Lambda,\bar{Q}} + \phi_{\varepsilon,\Lambda,\bar{Q}}$ is a solution to problem (1.11) for $n = 4$ if and only if $(\Lambda, \bar{Q})$ is a critical point of $I_\varepsilon$. The function $u = W_{\Lambda,\eta,\bar{Q}} + \phi_{\varepsilon,\Lambda,\eta,\bar{Q}}$ is a solution to problem (1.11) for $n = 6$ if and only if $(\Lambda, \eta, \bar{Q})$ is a critical point of $I_\varepsilon$.*

*Proof.* Here we only give the proof for the case $n = 6$, the other case can be proved in the same way. We notice that $u = W + \phi$ being a solution of (1.11) is equivalent to being a critical point of $J_\varepsilon$, which is also equivalent to the vanish of the $d_i$'s in (4.7) or, in view of

$$\begin{aligned}
&\langle Z_0, Y_0\rangle = \|Y_0\|_\varepsilon^2 = \gamma_0 + o(1),\\
&\langle Z_i, Y_i\rangle = \|Y_i\|_\varepsilon^2 = \gamma_1 + o(1),\ 1 \le i \le 6,\\
&\langle Z_7, Y_7\rangle = \|Y_7\|_\varepsilon^2 = \gamma_2\varepsilon^3,
\end{aligned} \tag{5.3}$$

where $\gamma_0, \gamma_1, \gamma_2$ are strictly positive constants, and

$$\langle Z_i, Y_j\rangle = o(1), i \ne j, 0 \le i, j \le 6, \quad \langle Z_i, Y_j\rangle = O(\varepsilon^3), i \ne j, i = 7 \text{ or } j = 7. \tag{5.4}$$

We have

$$J'_\varepsilon[W + \phi][Y_i] = 0, \quad 0 \le i \le 7. \tag{5.5}$$

On the other hand, we deduce from (5.2) that $I'_\varepsilon(\Lambda, \eta, Q) = 0$ is equivalent to the cancellation of $J'_\varepsilon(W + \phi)$ applied to the derivative of $W + \phi$ with respect to $\Lambda$, $\eta$ and $\bar{Q}$. By the definition of $Y_i$'s and Proposition 4.2, we have

$$\frac{\partial(W + \phi)}{\partial\Lambda} = Y_0 + y_0, \quad \frac{\partial(W + \phi)}{\partial\bar{Q}_i} = Y_i + y_i, \quad 1 \le i \le 6, \quad \frac{\partial(W + \phi)}{\partial\eta} = Y_7 + y_7$$

with $\|y_i\|_{***} = O(\varepsilon^2),\ 0 \le i \le 7$. We write

$$-\Delta(W + \phi) + \mu\varepsilon^2(W + \phi) - 24(W + \phi)^2 = \sum_{j=0}^{7} \alpha_j Z_j$$

and denote $a_{ij} = \langle y_i, Z_j\rangle$. Since $J'_\varepsilon[W + \phi][\theta] = 0$ for $\langle\theta, Z_i\rangle = (\theta, Y_i)_\varepsilon = 0,\ 0 \le i \le 7$, it turns out that $I'_\varepsilon(\Lambda, \eta, \bar{Q}) = 0$ is equivalent to

$$([b_{ij}] + [a_{ij}])[\alpha_j] = 0,$$

where $b_{ij} = \langle Y_i, Z_j\rangle$. Using the estimate $\|y_i\|_{***} = O(\varepsilon^2)$ and the expression of $Z_i, Y_i, 0 \le i \le 7$, we directly obtain

$$\begin{aligned}
&b_{00} = \gamma_0 + o(1), \quad b_{ii} = \gamma_1 + o(1) \text{ for } 1 \le i \le 6, \quad b_{77} = \gamma_2\varepsilon^3,\\
&b_{ij} = o(1) \text{ for } 0 \le i \ne j \le 6, \quad b_{ij} = O(\varepsilon^3) \text{ for } i = 7 \text{ or } j = 7, i \ne j,\\
&a_{ij} = O(\varepsilon^2) \text{ for } 0 \le i \le 7, 0 \le j \le 6, \quad a_{i7} = O(\varepsilon^4) \text{ for } 0 \le i \le 7.
\end{aligned}$$

Then it is easy to see the matrix $[b_{ij} + a_{ij}]$ is invertible by the above estimates of each components, hence $\alpha_i = 0$. We see that $I'_\varepsilon(\Lambda, \eta, \bar{Q}) = 0$ means exactly that (5.5) is satisfied. □

With Proposition 5.1, it remains to find critical points of $I_\varepsilon$. First, we establish an expansion of $I_\varepsilon$.

**Proposition 5.2.** *In the case $n = 4$, for $\varepsilon$ sufficiently small, we have*

$$I_\varepsilon(\Lambda, \eta, Q) = J_\varepsilon[W] + \varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}\sigma_{\varepsilon,4}(\Lambda, Q) \tag{5.6}$$

*where $\sigma_{\varepsilon,4} = O(\Lambda^2) + o(1)$ and $D_\Lambda(\sigma_{\varepsilon,4}) = O(\Lambda) + o(1)$ as $\varepsilon$ goes to $0$, uniformly with respect to $\Lambda$, $Q$ satisfying (2.11).*

*In the case $n = 6$, for $\varepsilon$ sufficiently small, we have*

$$I_\varepsilon(\Lambda, \eta, Q) = J_\varepsilon[W] + \varepsilon^4\sigma_{\varepsilon,6}(\Lambda, \eta, Q) \tag{5.7}$$

*where $\sigma_{\varepsilon,6} = o(1)$ and $D_{\Lambda,\eta}(\sigma_{\varepsilon,6}) = o(1)$ as $\varepsilon$ goes to $0$, uniformly with respect to $\Lambda$, $\eta$, $Q$ satisfying (2.13).*

*Proof.* We only consider the case $n=4$ here, the case $n=6$ can be argued similarly with minor changes. In view of (5.1), a Taylor expansion and the fact that $J_\varepsilon'[W+\phi][\phi]=0$ yield

$$\begin{aligned} I_\varepsilon(\Lambda,Q)-J_\varepsilon[W]=&J_\varepsilon[W+\phi]-J_\varepsilon[W]=-\int_0^1 J_\varepsilon''(W+t\phi)[\phi,\phi](t)\mathrm{d}t\\ &=-\int_0^1(\int_{\Omega_\varepsilon}(|\nabla\phi|^2+\mu\varepsilon^2\phi^2-24(W+t\phi)^2\phi^2))t\mathrm{d}t, \end{aligned}$$

whence

$$\begin{aligned} &I_\varepsilon(\Lambda,Q)-J_\varepsilon[W]\\ &=-\int_0^1\Big(8\int_{\Omega_\varepsilon}(N_\varepsilon(\phi)\phi+3[W^2-(W+t\phi)^2]\phi^2)\Big)t\mathrm{d}t-\int_{\Omega_\varepsilon}R^\varepsilon\phi. \end{aligned}\tag{5.8}$$

The first term on the right hand side of (5.8) can be estimated as

$$\Big|\int_{\Omega_\varepsilon}N_\varepsilon(\phi)\phi\Big|\le C\int_{\Omega_\varepsilon}|\phi|^4+|W\phi^3|=O(\varepsilon^4\ln\varepsilon).$$

Similarly, for the second term on the right hand side of (5.8), we obtain

$$\Big|\int_{\Omega_\varepsilon}[W^2-(W+t\phi)^2]\phi^2\Big|\le C\int_{\Omega_\varepsilon}|\phi|^4+|W\phi^3|=O(\varepsilon^4\ln\varepsilon).$$

Concerning the last one, recalling

$$\begin{aligned} |R^\varepsilon|_*=|S_\varepsilon[W]|=&O\Big(\varepsilon^4(-\ln\varepsilon)\langle z-\bar Q\rangle^{-2}+\varepsilon^2(-\ln\varepsilon)^{\frac12}\langle z-\bar Q\rangle^{-4}\Big)\\ &+O(\Lambda)\Big(\frac{\varepsilon^4}{(-\ln\varepsilon)}\big|\ln\frac{1}{\varepsilon(1+|z-\bar Q|)}\big|+\frac{\varepsilon^4}{(-\ln\varepsilon)}\Big) \end{aligned}$$

uniformly in $\Omega_\varepsilon$. A simple computation shows that

$$\Big|\int_{\Omega_\varepsilon}R^\varepsilon\phi\Big|=O\Big(\varepsilon^2(-\ln\varepsilon)^{\frac12}\Lambda^2+\varepsilon^3(-\ln\varepsilon)^{\frac12}\Big),$$

where we used $\|\phi\|_*=O(\varepsilon\Lambda)$. This concludes the proof of the first part of Proposition (5.6).

An estimate for the derivatives with respect to $\Lambda$ is established exactly in the same way, differentiating the right side in (5.8) and estimating each term separately, using (4.3), (4.5) and Lemma 2.1. □

## 6. Proof of Theorem 1.1

In this section, we prove the existence of a critical point of $I_\varepsilon(\Lambda,Q)$ and $I_\varepsilon(\Lambda,\eta,Q)$, and then prove Theorem 1.1 by Proposition 5.1. According to Proposition 5.2 and Lemma 2.1. Setting

$$K_\varepsilon(\Lambda,Q)=\frac{I_\varepsilon(\Lambda,Q)-2\int_{\mathbb{R}^n}U^4}{(-\frac{\ln\varepsilon}{c_1})^{\frac12}\varepsilon^2}\tag{6.1}$$

and

$$K_\varepsilon(\Lambda,\eta,Q)=\frac{I_\varepsilon(\Lambda,\eta,Q)-4\int_{\mathbb{R}^n}U^3}{\varepsilon^3} \tag{6.2}$$

Then, we have when $n=4$,

$$\begin{aligned}K_\varepsilon(\Lambda,Q)=&\frac{1}{4}c_4\Lambda^2\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})-\frac{c_4^2\Lambda^2}{2|\Omega|}+\frac{1}{2}c_4^2\Lambda^2H(Q,Q)(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\\&+O\big(\frac{\Lambda^2}{-\ln\varepsilon}+\varepsilon\big),\end{aligned} \tag{6.3}$$

and when $n=6$,

$$\begin{aligned}K_\varepsilon(\Lambda,\eta,Q)=&\Big(\frac{1}{2}\eta^2|\Omega|-c_6\Lambda^2\eta+\frac{1}{48}c_6\Lambda^2-8\eta^3|\Omega|\Big)+\frac{1}{2}c_6^2\Lambda^4H(Q,Q)\varepsilon\\&+\frac{1}{2}\big(\eta-\frac{c_6\Lambda^2}{|\Omega|}\big)\varepsilon\int_\Omega\frac{\Lambda^2}{|x-Q|^4}+o(\varepsilon).\end{aligned} \tag{6.4}$$

Then we begin to consider $K_\varepsilon(\Lambda,Q)$, finding its critical points with respect to $\Lambda,Q$, and $K_\varepsilon(\Lambda,\eta,Q)$ with its critical points with respect to the parameters $\Lambda,\eta,Q$.

First, we consider $K_\varepsilon(\Lambda,Q)$ for $n=4$. For the setting of the parameters $\Lambda,Q$, we see that $\Lambda,Q$ are located on a compact set, we can obtain a maximal value of $K_\varepsilon(\Lambda,Q)$. We claim that:

**Claim:** The maximal point of $K_\varepsilon(\Lambda,Q)$ with respect to $\Lambda,Q$ can not happen on the boundary of the parameters.

If we can prove this claim, then we could obtain an interior critical point of $K_\varepsilon(\Lambda,Q)$. Before proving the claim, we first consider

$$F_\varepsilon(\Lambda)=\frac{1}{4}c_4\Lambda^2\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})-\frac{c_4^2\Lambda^2}{2|\Omega|}.$$

Note that

$$\frac{\partial}{\partial\Lambda}[F_\varepsilon(\Lambda)]=\frac{1}{2}c_4\Lambda\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})-\frac{1}{4}c_4\Lambda(\frac{c_1}{-\ln\varepsilon})-\frac{c_4^2\Lambda}{|\Omega|},$$

Choosing $c_1=\frac{2c_4}{|\Omega|}$, we could obtain that there exists

$$\Lambda^*=\exp(-\frac{1}{2})\in(\exp(-\frac{1}{2})\varepsilon^\beta,\exp(-\frac{1}{2})\varepsilon^{-\beta})$$

with some proper fixed constant $\beta\in(0,\frac{1}{3})$, such that

$$\frac{\partial}{\partial\Lambda}F_\varepsilon\mid_{\Lambda=\Lambda^*}=0.$$

It can be also found that such $\Lambda^*$ provides the maximal value of $F_\varepsilon(\Lambda)$ in $[\Lambda_{4,1},\Lambda_{4,2}]$, where $\Lambda_{4,1}=\exp(-\frac{1}{2})\varepsilon^\beta,\Lambda_{4,2}=\exp(-\frac{1}{2})\varepsilon^{-\beta}$. In order to prove the claim, we need to take $\Lambda$ into consideration for the expansion of the energy, going through the first part of the Appendix, we have

$$\begin{aligned}K_\varepsilon(\Lambda,Q)=&\frac{1}{4}c_4\Lambda^2\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})-\frac{c_4^2\Lambda^2}{2|\Omega|}+\frac{1}{2}c_4^2\Lambda^2H(Q,Q)(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\\&+O(\frac{\Lambda^2}{-\ln\varepsilon}+\varepsilon).\end{aligned}$$

Now, we come back to prove the claim, choosing $\Lambda = \Lambda^*$ and $Q = p$. (Here $p$ refers to the point where $H(Q,Q)$ obtain its maximal value, it is possible to find such a point. Indeed, we notice a fact $H(Q,Q) \to -\infty$ as $d(Q,\partial\Omega) \to 0$ see [19] and references therein for a proof of this fact. Therefore we could find such $p$.)

First, we prove that the maximal value can not happen on $\partial\mathcal{M}_{\delta_4}$. We choose $\delta_4$ such that $\omega_1 < \max_{\partial\mathcal{M}_{\delta_4}} H < \omega_2$ for some proper constant $\omega_1, \omega_2$ sufficiently negative, then we fixed $\mathcal{M}_{\delta_4}$. It is easy to see that $K_\varepsilon(\Lambda, Q) < K_\varepsilon(\Lambda, p)$, where $Q$ lies on the boundary of $\mathcal{M}_{\delta_4}$ and $\Lambda \in (\Lambda_{4,1}, \Lambda_{4,2})$. For $\Lambda = \Lambda_{4,1}$ or $\Lambda_{4,2}$, we go to the arguments below. Therefore, we prove that the maximal point can not lie on the boundary of $\mathcal{M}_{\delta_4} \times [\Lambda_{4,1}, \Lambda_{4,2}]$.

Next, we show $K_\varepsilon(\Lambda^*, p) > K_\varepsilon(\Lambda_{4,2}, Q)$. It is easy to see that

$$F_\varepsilon[\Lambda_{4,2}] \le c\varepsilon^{-2\beta},$$

where $c < 0$. Then we can find $c_1 < 0$ such that $K_\varepsilon(\Lambda_{4,2}, Q) \le c_1\varepsilon^{-2\beta}$ for any $Q \in \mathcal{M}_{\delta_4}$, since the other terms compared to $\varepsilon^{-2\beta}$ are higher order term. On the other hand, for the choice of $\Lambda^*, p$, we see that $K_\varepsilon(\Lambda^*, p) = O(1)$. Therefore, we prove that $K_\varepsilon(\Lambda^*, p) > K_\varepsilon(\Lambda_{4,2}, Q)$ for any $Q \in \mathcal{M}_{\delta_4}$.

It remains to prove that the maximal value can not happen at $\Lambda = \Lambda_{4,1}$. We choose $\Lambda = \varepsilon^{\beta/2}, Q = p$, direct computation yields.

$$K_\varepsilon(\varepsilon^{\beta/2}, p) = \frac{\beta c_4^2 \varepsilon^\beta}{4|\Omega|}(1 + o(1)), \quad K_\varepsilon(\Lambda_{4,1}, Q) = \frac{\beta c_4^2 \varepsilon^{2\beta}}{2|\Omega|}(1 + o(1)).$$

It is to see $K_\varepsilon(\varepsilon^{\beta/2}, p) > K_\varepsilon(\Lambda_{4,1}, Q)$ for any $Q \in \mathcal{M}_{\delta_4}$ when $\varepsilon$ is sufficiently small. Hence, we finish the proof of the claim. In other words, we could obtain an interior maximal point in $[\Lambda_{4,1}, \Lambda_{4,2}] \times \mathcal{M}_{\delta_4}$. Therefore, we show the existence of the critical points of $K_\varepsilon(\Lambda, Q)$ with respect to $\Lambda, Q$.

For $n = 6$. We set $\eta = \frac{1}{48} + a\varepsilon^{\frac{1}{3}}$, $\frac{c_6\Lambda^2}{|\Omega|} = \frac{1}{96} + b\varepsilon^{\frac{2}{3}}$, then

$$K_\varepsilon(a, b, Q) := K_\varepsilon(\Lambda, \eta, Q) = \frac{1}{6912}|\Omega| + \big[F(Q) - (8a^3 + ab)|\Omega|\big]\varepsilon + o(\varepsilon), \tag{6.5}$$

where

$$F(x) = \frac{|\Omega|}{18432}\Big(|\Omega|H(x,x) + \frac{1}{c_6}\int_\Omega \frac{1}{|x-y|^4}\mathrm{d}y\Big),$$

$-\eta_6 \le a \le \eta_6$ and $-\Lambda_6 \le b \le \Lambda_6$.

We set $C_0 = F(p_0)$, $p_0$ refers to the point where $F(x)$ obtains its maximal value. Indeed, we have $H(Q,Q) \to -\infty$ as $d(Q,\partial\Omega) \to 0$ and $I(x) = \int_\Omega \frac{1}{|x-y|^4}\mathrm{d}y$ is uniformly bounded in $\Omega$. Hence, we can always find such point $p_0$. Let us introduce another five constants $C_i, i = 1,2,3,4,5$, with $C_2 < C_1 < C_0$, $0 < C_3 < C_4 < \eta_6$ and $0 < C_3 < C_5 < \Lambda_6$, the value of these five constants will be determined later.

We set

$$\Sigma_0 = \Big\{ -C_4 \le a \le C_4,\ -C_5 \le b \le C_5,\ Q \in \mathcal{N}_{C_2}\Big\}, \tag{6.6}$$

where $\mathcal{N}_{C_i} = \{q : F(q) > C_i\}, i = 1,2$ and $\delta_6$ is chosen such that $\mathcal{N}_{C_2} \subset \mathcal{M}_{\delta_6}$.

We also define

$$B = \{(a,b,Q) \mid (a,b) \in B_{C_3}(0),\ Q \in \overline{\mathcal{N}_{C_1}}\},\ B_0 = \{(a,b) \mid (a,b) \in B_{C_3}(0)\} \times \partial\mathcal{N}_{C_1}, \tag{6.7}$$

where $B_r(0) := \{0 \le a^2 + b^2 \le r\}$.

It is trivial to see that $B_0 \subset B \subset \Sigma_0$, $B$ is compact. Let $\Gamma$ be the class of continuous functions $\varphi : B \to \Sigma_0$ with the property that $\varphi(y) = y, y = (a, b, Q)$ for all $y \in B_0$. Define the min-max value $c$ as

$$c = \min_{\varphi\in\Gamma} \max_{y\in B} K_\varepsilon(\varphi(y)).$$

We now show that $c$ defines a critical value. To this end, we just have to verify the following conditions

(T1) $\max_{y\in B_0} K_\varepsilon(\varphi(y)) < c,\ \forall\varphi \in \Gamma$,

(T2) For all $y \in \partial\Sigma_0$ such that $K_\varepsilon(y) = c$, there exists a vector $\tau_y$ tangent to $\partial\Sigma_0$ at $y$ such that

$$\partial_{\tau_y} K_\varepsilon(y) \neq 0.$$

Suppose (T1) and (T2) hold. Then standard deformation argument ensures that the min-max value $c$ is a (topologically nontrivial) critical value for $K_\varepsilon(\Lambda, \eta, Q)$ in $\Sigma_0$. (Similar notion has been introduced in [7] for degenerate critical points of mean curvature.)

To check (T1) and (T2), we define $\varphi(y) = \varphi(a, b, Q) = (\varphi_a, \varphi_b, \varphi_Q)$ where $(\varphi_a, \varphi_b) \in [-C_4, C_4] \times [-C_5, C_5]$ and $\varphi_Q \in \mathcal{N}_{C_2}$.

For any $\varphi \in \Gamma$ and $Q \in \mathcal{N}_{C_2}$, the map $Q \to \varphi_Q(a, b, Q)$ is a continuous function from $\mathcal{N}_{C_1}$ to $\mathcal{N}_{C_2}$ such that $\varphi_Q(a, b, Q) = Q$ for $Q \in \partial\mathcal{N}_{C_1}$. Let $\mathcal{D}$ be the smallest ball which contain $\mathcal{N}_{C_1}$, we extend $\varphi_Q$ to a continuous function $\tilde\varphi_Q$ from $\mathcal{D}$ to $\mathcal{D}$ where $\tilde\varphi(Q)$ is defined as follows:

$$\tilde\varphi_Q(x) = \varphi(x),\ x \in \mathcal{N}_{C_1}, \quad \tilde\varphi_Q(x) = Id,\ x \in \mathcal{D} \setminus \mathcal{N}_{C_1}.$$

Then we claim there exists $Q' \in \mathcal{D}$ such that $\tilde\varphi_Q(Q') = p_0$. Otherwise $\frac{\tilde\varphi_Q - p_0}{|\tilde\varphi_Q - p_0|}$ provides a continuous map from $\mathcal{D}$ to $\mathbb{S}^5$, which is impossible in algebraic topology. Hence, there exists $Q' \in \mathcal{D}$ such that $\tilde\varphi_Q(Q') = p_0$. By the definition of $\tilde\varphi$, we can further conclude $Q' \in \mathcal{N}_{C_1}$. Whence

$$\begin{aligned} \max_{y\in B} K_\varepsilon(\varphi(y)) &\geq K_\varepsilon(\varphi_a(a, b, Q'), \varphi_b(a, b, Q'), p_0) \\ &\geq \frac{1}{6912}|\Omega| + (C_0 - C_6|\Omega|)\varepsilon + o(\varepsilon), \end{aligned} \tag{6.8}$$

where $C_6 = 8C_4^3 + C_4C_5$ which stands for the maximal value of $8a^3 + ab$ in $[-C_4, C_4] \times [-C_5, C_5]$. As a consequence

$$c \geq \frac{1}{6912}|\Omega| + (C_0 - C_6|\Omega|)\varepsilon + o(\varepsilon). \tag{6.9}$$

For $(a, b, Q) \in B_0$, we have $F(\varphi_Q(a, b, Q)) = C_1$. So we have

$$K_\varepsilon(a, b, Q) \leq \frac{1}{6912}|\Omega| + (C_1 + C_7|\Omega|)\varepsilon + o(\varepsilon), \tag{6.10}$$

where $C_7 = \max_{(a,b)\in B_{C_3}(0)} 8a^3 + ab < 8C_3^3 + C_3^2$.

If we choose $C_0 - C_1 > 8C_4^3 + C_4C_5 + 8C_3^3 + C_3^2 > C_6 + C_7$. Then $\max_{y\in B_0} K_\varepsilon(\varphi(y)) < c$ holds. So (T1) is verified.

To verify (T2), we observe that

$$\partial\Sigma_0 =: \{a, b, Q \mid a = -C_4 \text{ or } a = C_4 \text{ or } b = -C_5 \text{ or } b = C_5 \text{ or } Q \in \partial\mathcal{N}_{C_2}\}.$$

Since $C_4, C_5$ are arbitrary, we choose $0 < 24C_4^2 < C_5$. Then on $a = -C_4$ or $a = C_4$, we choose $\tau_y = \frac{\partial}{\partial b}$, on $b = -C_5$ or $b = C_5$, we choose $\tau_y = \frac{\partial}{\partial a}$. By our setting on

$C_4, C_5$, we could show $\partial_{\tau_y} K_\varepsilon(y) \neq 0$. It only remains to consider the case $Q \in \partial\mathcal{N}_{C_2}$. If $Q \in \partial\mathcal{N}_{C_2}$, then

$$K_\varepsilon(a,b,Q) \leq \frac{1}{6912}|\Omega| + (C_2 + C_7|\Omega|)\varepsilon + o(\varepsilon), \tag{6.11}$$

which is obviously less than $c$ for $C_2 < C_1$. So (T2) is also verified.

In conclusion, we proved that for $\varepsilon$ sufficiently small, $c$ is a critical value, i.e., a critical point $(a,b,Q) \in \Sigma_0$ of $K_\varepsilon$ exists. Which means $K_\varepsilon$ indeed has critical points respect to $\Lambda, \eta, Q$ in (2.13).

*Proof of Theorem 1.1.* For $n = 4$, we proved that for $\varepsilon$ small enough, $I_\varepsilon$ has a critical point $(\Lambda^\varepsilon, Q^\varepsilon)$. Let $u_\varepsilon = W_{\Lambda^\varepsilon, \bar{Q}^\varepsilon, \varepsilon}$. Then $u_\varepsilon$ is a nontrivial solution to problem (1.12) for $n = 4$. The strong maximal principle shows $u_\varepsilon > 0$ in $\Omega_\varepsilon$. Let $u_\mu = \varepsilon^{-1} u_\varepsilon(x/\varepsilon)$. By our construction, $u_\mu$ has all the properties stated in Theorem 1.1.

For $n = 6$, we proved that for $\varepsilon$ small enough, $I_\varepsilon$ has a critical point $(\Lambda^\varepsilon, \eta^\varepsilon, Q^\varepsilon)$. Let $u_\varepsilon = W_{\Lambda^\varepsilon, \eta^\varepsilon, \bar{Q}^\varepsilon, \varepsilon}$. Then $u_\varepsilon$ is a nontrivial solution to problem (1.12) for $n = 6$. The strong maximal principle shows $u_\varepsilon > 0$ in $\Omega_\varepsilon$. Let $u_\mu = \varepsilon^{-2} u_\varepsilon(x/\varepsilon)$. By our construction, $u_\mu$ has all the properties stated in Theorem 1.1. □

## 7. Appendix A: Proof of Lemma 2.1

We divide the proof into two parts. First, we study the case $n = 4$. From the definition of $W$, (2.10) and (2.15), we know that

$$\begin{aligned} S_\varepsilon[W] =& -\Delta W + \mu\varepsilon^2 W - 8W^3 \\ =& \, 8U^3 + \varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)\hat{U} - \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Delta(R_{\varepsilon,\Lambda,Q}\chi) - 8W^3 \\ =& \, O\Big(\varepsilon^4(-\ln\varepsilon)\langle z - \bar{Q}\rangle^{-2} + \varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}\langle z - \bar{Q}\rangle^{-4}\Big) \\ &+ O(\Lambda)\Big(\frac{\varepsilon^4}{(-\ln\varepsilon)}\Big|\ln\frac{1}{\varepsilon(1+|z-\bar{Q}|)}\Big| + \frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac{1}{2}}}\Big). \end{aligned}$$

The estimates for $D_\Lambda S_\varepsilon[W]$ and $D_{\bar{Q}} S_\varepsilon[W]$ can be computed in the same way.

We now turn to the proof of the energy estimate (2.23). From (2.15) and (2.16) we deduce that

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla W|^2 + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon} W^2 =& 8\int_{\Omega_\varepsilon} U^3 W + \varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)\int_{\Omega_\varepsilon}\hat{U}W \\ &- \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\Delta(R\chi)W. \end{aligned} \tag{7.1}$$

Concerning the first term on the right hand side of (7.1), we have

$$\int_{\Omega_\varepsilon} U^3 W = \int_{\Omega_\varepsilon} U^4 + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}U^3 + \frac{c_4\Lambda}{|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\int_{\Omega_\varepsilon} U^3. \tag{7.2}$$

We note that

$$\int_{\Omega_\varepsilon} U^4 = \int_{\mathbb{R}^4} U_{1,0}^4 + O(\varepsilon^4), \quad \int_{\Omega_\varepsilon} U^3 = \frac{c_4\Lambda}{8} + O(\varepsilon^2).$$

Then, we get

$$\int_{\Omega_\varepsilon} U^3 W = \int_{\mathbb{R}^4} U_{1,0}^4 + \frac{c_4^2\Lambda^2}{8|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}} + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}U^3 + O\big(\varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\big),$$

for the third term on the right hand side of the above equality, we have

$$\begin{aligned}\int_{\Omega_\varepsilon}\hat{U}U^3 =& -\int_{\Omega_\varepsilon}\Psi U^3 - c_4\Lambda\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\int_{\Omega_\varepsilon} H(x,Q)U^3 + \int_{\Omega_\varepsilon}(R\chi)U^3\\ =& -\frac{c_4\Lambda^2}{16}\ln\frac{1}{\Lambda\varepsilon} - \frac{c_4^2\Lambda^2}{8}\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}H(Q,Q) + O(\Lambda^2).\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} U^3 W =& \int_{\mathbb{R}^4} U_{1,0}^4 + \frac{c_4^2\Lambda^2}{8|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}} - \frac{c_4\Lambda^2}{16}\ln\frac{1}{\Lambda\varepsilon}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\\ &- \frac{c_4^2\Lambda^2}{8}\varepsilon^2 H(Q,Q) + O\big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Lambda^2 + \varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\big). \qquad (7.3)\end{aligned}$$

For the second term on the right hand side of (7.1)

$$\int_{\Omega_\varepsilon}\hat{U}W = \int_{\Omega_\varepsilon}\hat{U}U + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}^2 + \frac{c_4\Lambda}{|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}.$$

By noting that

$$\begin{aligned}&\int_{\Omega_\varepsilon}\hat{U}U = O\big(\varepsilon^{-2}\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\Lambda^2\big),\ \int_{\Omega_\varepsilon}\hat{U}^2 = O\big(\varepsilon^{-4}(-\ln\varepsilon)\Lambda^2\big),\\ &\int_{\Omega_\varepsilon}\hat{U} = \varepsilon^{-4}\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\int_\Omega\frac{\Lambda}{|x-Q|^2} + O(\varepsilon^{-4}\Lambda),\end{aligned}$$

where we used $\int_\Omega G(x,Q) = 0$. Then, we obtain

$$\varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)\int_{\Omega_\varepsilon}\hat{U}W = \frac{c_4\Lambda^2}{|\Omega|}\varepsilon^2\int_\Omega\frac{1}{|x-Q|^2} + O\big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Lambda^2\big). \qquad (7.4)$$

For the last term on the right hand side of (7.1),

$$\begin{aligned}\int_{\Omega_\varepsilon}\Delta(R\chi)W =& \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\frac{c_4\Lambda}{|\Omega|}\int_{\Omega_\varepsilon}\Delta(R\chi) + O(\Lambda^2)\\ =& \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\frac{c_4\Lambda}{|\Omega|}\int_{\partial\Omega_\varepsilon}\frac{\partial(R\chi)}{\partial\nu} + O(\Lambda^2)\\ =& \big(\frac{c_1}{-\ln\varepsilon}\big)^{-1}\frac{c_4\Lambda}{|\Omega|}\int_{\partial\Omega_\varepsilon}\frac{\partial\big(U - \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Psi - c_4\Lambda\varepsilon^2 H\big)}{\partial\nu} + O(\Lambda^2)\\ =& \big(\frac{c_1}{-\ln\varepsilon}\big)^{-1}\frac{c_4\Lambda}{|\Omega|}\int_{\Omega_\varepsilon}\Delta\big(U - \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Psi - c_4\Lambda\varepsilon^2 H\big) + O(\Lambda^2)\\ =& \big(\frac{c_1}{-\ln\varepsilon}\big)^{-1}\frac{c_4\Lambda}{|\Omega|}\int_{\Omega_\varepsilon}\big(-8U^3 + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}U + c_4\Lambda\varepsilon^4\frac{1}{|\Omega|}\big) + O(\Lambda^2)\\ =& \big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\frac{c_4\Lambda^2}{|\Omega|}\int_\Omega\frac{1}{(\varepsilon^2\Lambda^2 + |x-Q|^2)} + O(\Lambda^2 + \varepsilon^2(-\ln\varepsilon)). \qquad (7.5)\end{aligned}$$

(7.3)-(7.5) implies

$$\frac{1}{2}\int_{\Omega_\varepsilon}\Big(|\nabla W|^2+\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac12}W^2\Big)=4\int_{\mathbb{R}^4}U_{1,0}^4+\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac12}\frac{c_4^2\Lambda^2}{2|\Omega|}-\frac{c_4^2\Lambda^2}{2}H(Q,Q)\varepsilon^2 -\frac{c_4\Lambda^2}{4}\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac12}\ln\frac{1}{\Lambda\varepsilon}+O(\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac12}\Lambda^2) +O\big(\varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac12}\big). \tag{7.6}$$

At last, we compute the term $\int_{\Omega_\varepsilon}W^4$.

$$\begin{aligned}\int_{\Omega_\varepsilon}W^4=&\int_{\Omega_\varepsilon}U^4+4\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac12}\int_{\Omega_\varepsilon}U^3\hat U+4\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac12}\frac{c_4\Lambda}{|\Omega|}\int_{\Omega_\varepsilon}U^3\\&+O\big(\varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)^{-2}\big)\\=&\int_{\mathbb{R}^4}U_{1,0}^4-\frac{c_4\Lambda^2}{4}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac12}\ln\frac{1}{\Lambda\varepsilon}-\frac{c_4^2\Lambda^2}{2}\varepsilon^2H(Q,Q)\\&+\frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac12}+O\big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac12}\Lambda^2\big)\\&+O\big(\varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)^{-2}\big).\end{aligned}\tag{7.7}$$

Combining (7.6) and (7.7), we obtain

$$\begin{aligned}J_\varepsilon[W]=&\frac12\int_{\Omega_\varepsilon}|\nabla W|^2+\frac{\mu\varepsilon^2}{2}\int_{\Omega_\varepsilon}W^2-2\int_{\Omega_\varepsilon}W^4\\=&2\int_{\mathbb{R}^4}U_{1,0}^4+\frac{c_4\Lambda^2}{4}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac12}\ln\frac{1}{\Lambda\varepsilon}-\frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac12}\\&+\frac12 c_4^2\Lambda^2\varepsilon^2H(Q,Q)+O\big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac12}\Lambda^2\big)\\&+O(\varepsilon^4(-\ln\varepsilon)^2).\end{aligned}\tag{7.8}$$

In the end of this section, we prove (2.24)-(2.28). From the definition of $W$, (2.10) and (2.15), we know that

$$\begin{aligned}S_\varepsilon[W]=&-\Delta W+\varepsilon^3W-24W^2\\=&\ 24U^2+\varepsilon^6\hat U-\varepsilon^3\Delta(R\chi)+\varepsilon^6\big(\eta-\frac{c_6\Lambda^2}{|\Omega|}\big)-24U^2-24\eta^2\varepsilon^6+O\big(\varepsilon^3\langle z-\bar Q\rangle^{-4}\big)\\=&\ -\varepsilon^6\big(24\eta^2-\eta+\frac{c_6\Lambda^2}{|\Omega|}\big)+O\big(\varepsilon^3\langle z-\bar Q\rangle^{-4}\big)\\=&\ O\big(\langle z-\bar Q\rangle^{-3\frac23}\varepsilon^3\big).\end{aligned}$$

The estimates for $D_\Lambda S_\varepsilon[W]$, $D_{\bar Q}S_\varepsilon[W]$ and $D_\eta S_\varepsilon[W]$ can be derived in the same way. Now we are in the position to compute the energy. From (2.15) and (2.16), we deduce that

$$\int_{\Omega_\varepsilon}|\nabla W|^2+\varepsilon^3\int_{\Omega_\varepsilon}W^2=\int_{\Omega_\varepsilon}(-\Delta W+\varepsilon^3 W)W$$
$$=\int_{\Omega_\varepsilon}\Big(24U^2+\varepsilon^6\hat{U}-\varepsilon^3\Delta(R\chi)+\varepsilon^6\big(\eta-\frac{c_6\Lambda^2}{|\Omega|}\big)\Big)W. \quad (7.9)$$

Concerning the first term on the right hand side of (7.9), we have

$$\int_{\Omega_\varepsilon}U^2W=\int_{\Omega_\varepsilon}U^3+\varepsilon^3\int_{\Omega_\varepsilon}\hat{U}U^2+\eta\varepsilon^3\int_{\Omega_\varepsilon}U^2$$
$$=\int_{\mathbb{R}^6}U_{1,0}^3+\frac{1}{24}c_6\eta\Lambda^2\varepsilon^3-\varepsilon^3\int_{\Omega_\varepsilon}U^2\Psi-c_6\Lambda^2\varepsilon^4\int_{\Omega_\varepsilon}U^2H+O(\varepsilon^5)$$
$$=\int_{\mathbb{R}^6}U_{1,0}^3+\frac{1}{24}c_6\eta\Lambda^2\varepsilon^3-\frac{1}{24}c_6^2\Lambda^4\varepsilon^4H(Q,Q)-\frac{1}{576}c_6\Lambda^2\varepsilon^3+O(\varepsilon^5). \quad (7.10)$$

For the second, third and fourth term on the right hand side of (7.9), following the similar steps as we did in case $n=4$.

$$\varepsilon^6\int_{\Omega_\varepsilon}\hat{U}W=\varepsilon^6\int_{\Omega_\varepsilon}\hat{U}(U+\varepsilon^3\hat{U}+\eta\varepsilon^3)=-\eta\Lambda^2\varepsilon^4\int_\Omega\frac{1}{|x-Q|^4}+O(\varepsilon^5), \quad (7.11)$$

$$-\varepsilon^3\int_{\Omega_\varepsilon}\Delta(R\chi)W=\varepsilon^3\eta\int_{\Omega_\varepsilon}\Delta(U-\varepsilon^3\Psi-c_6\varepsilon^4\Lambda^2H)+O(\varepsilon^5)=\varepsilon^6\eta\int_{\Omega_\varepsilon}U+O(\varepsilon^5)$$
$$=\eta\Lambda^2\varepsilon^4\int_\Omega\frac{1}{|x-Q|^4}+O(\varepsilon^5), \quad (7.12)$$

and

$$\varepsilon^6\big(\eta-\frac{c_6\Lambda^2}{|\Omega|}\big)\int_{\Omega_\varepsilon}W=\big(\eta^2|\Omega|-c_6\eta\Lambda^2\big)\varepsilon^3+\big(\eta-\frac{c_6\Lambda^2}{|\Omega|}\big)\varepsilon^4\int_\Omega\frac{\Lambda^2}{|x-Q|^4}+O(\varepsilon^5). \quad (7.13)$$

(7.10)-(7.13) implies

$$\frac{1}{2}\int_{\Omega_\varepsilon}|\nabla W|^2+\frac{\varepsilon^3}{2}\int_{\Omega_\varepsilon}W^2=12\int_{\mathbb{R}^6}U_{1,0}^3+\big(\frac{1}{2}\eta^2|\Omega|-\frac{1}{48}c_6\Lambda^2\big)\varepsilon^3-\frac{c_6^2\Lambda^4}{2}H(Q,Q)\varepsilon^4$$
$$+\frac{1}{2}\big(\eta-\frac{c_6\Lambda^2}{|\Omega|}\big)\varepsilon^4\int_\Omega\frac{\Lambda^2}{|x-Q|^4}+O(\varepsilon^5). \quad (7.14)$$

Then,

$$\int_{\Omega_\varepsilon}W^3=\int_{\mathbb{R}^6}U_{1,0}^3+3\varepsilon^3\int_{\Omega_\varepsilon}U^2\hat{U}+3\varepsilon^3\int_{\Omega_\varepsilon}U^2\eta+3\varepsilon^6\int_{\Omega_\varepsilon}U\eta^2+3\varepsilon^9\int_{\Omega_\varepsilon}\hat{U}\eta^2$$
$$+\varepsilon^9\int_{\Omega_\varepsilon}\eta^3+O(\varepsilon^5)$$
$$=\int_{\mathbb{R}^6}U_{1,0}^3+\frac{1}{8}c_6\eta\Lambda^2\varepsilon^3-\frac{1}{192}c_6\Lambda^2\varepsilon^3+\eta^3|\Omega|\varepsilon^3-\frac{1}{8}c_6^2\Lambda^4H(Q,Q)\varepsilon^4$$
$$+O(\varepsilon^5). \quad (7.15)$$

Combining (7.14)-(7.15), we gain the energy

$$J_\varepsilon[W] = 4\int_{\mathbb{R}^6} U_{1,0}^3 + \Big(\frac{1}{2}\eta^2|\Omega| - c_6\eta\Lambda^2 + \frac{1}{48}c_6\Lambda^2 - 8\eta^3|\Omega|\Big)\varepsilon^3 + \frac{1}{2}c_6^2\Lambda^4 H(Q,Q)\varepsilon^4 + \frac{1}{2}\big(\eta - \frac{c_6\Lambda^2}{|\Omega|}\big)\varepsilon^4\int_\Omega \frac{\Lambda^2}{|x-Q|^4} + O(\varepsilon^5). \tag{7.16}$$

Hence, we finish the whole proof of Lemma 2.1. □

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