Toda system and cluster phase transition layers in an inhomogeneous phase transition model

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Abstract

We consider the following singularly perturbed elliptic problem

$$\varepsilon^2\triangle\tilde{u} + \big(\tilde{u} - a(\tilde{y})\big)(1-\tilde{u}^2) = 0 \quad \text{in } \Omega, \quad \frac{\partial \tilde{u}}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $-1 < a(\tilde{y}) < 1$, ε is a small parameter, ν denotes the outward normal of $\partial\Omega$. Assume that $\Gamma = \{\,\tilde{y} \in \Omega : a(\tilde{y}) = 0\,\}$ is a simple closed and smooth curve contained in Ω in such a way that Γ separates Ω into two disjoint components $\Omega_+ = \{\,\tilde{y} \in \Omega : a(\tilde{y}) < 0\,\}$ and $\Omega_- = \{\,\tilde{y} \in \Omega : a(\tilde{y}) > 0\,\}$ and $\frac{\partial a}{\partial \nu_0} > 0$ on Γ, where ν_0 is the outer normal of Ω_+, pointing to the interior of Ω_-. For any fixed integer $N = 2m+1 \geq 3$, we will show the existence of a clustered solution u_ε with N-transition layers near Γ with mutual distance $O(\varepsilon|\log\varepsilon|)$, provided that ε stays away from a discrete set of values at which resonance occurs. Moreover, u_ε approaches 1 in Ω_- and -1 in Ω_+. Central to our analysis is the solvability of a Toda system.

1 Introduction

Let Ω be a bounded and smooth domain in $\mathbb{R}^2$. In gradient theory of phase transition it is common to seek for critical points in $H^1(\Omega)$ of an energy of the form

$$J_\varepsilon(\tilde{u}) = \frac{\varepsilon}{2}\int_\Omega |\nabla \tilde{u}|^2 + \frac{1}{\varepsilon}\int_\Omega W(\tilde{y}, \tilde{u}) \tag{1.1}$$

where $W(\tilde{y},\cdot)$ is a double-well potential with exactly two strict local minimizers at $\tilde{u}=-1$ and $\tilde{u}=+1$, which as well correspond to trivial local minimizers of J_ε in $H^1(\Omega)$. For simplicity of exposition we shall restrict ourselves to a potential of the form

$$W(\tilde{y},u)=\int_{-1}^{u}(s^2-1)\big(s-a(\tilde{y})\big)\,\mathrm{d}s, \tag{1.2}$$

for a smooth function $a(\tilde{y})$ with

$$-1<a(\tilde{y})<1 \text{ for all } \tilde{y}\in\bar{\Omega}.$$

Critical points of J_ε correspond to solutions of the problem

$$\varepsilon^2\triangle\tilde{u}\,+\,\big(\tilde{u}-a(\tilde{y})\big)(1-\tilde{u}^2)\,=\,0\quad\text{in }\Omega,\quad\frac{\partial\tilde{u}}{\partial\nu}=0\quad\text{on }\partial\Omega, \tag{1.3}$$

where ε is a small parameter, ν denotes the outward normal of $\partial\Omega$. Function $\tilde{u}$ represents a continuous realization of the phase present in a material confined to the region Ω at the point x which, except for a narrow region, is expected to take values close to $+1$ or -1. Of interest are of course non-trivial steady state configurations in which the antiphases coexist.

The case $a\equiv 0$ corresponds to the standard Allen-Cahn equation [6]

$$\varepsilon^2\triangle\tilde{u}\,+\,\tilde{u}(1-\tilde{u}^2)=0\quad\text{in }\Omega,\quad\frac{\partial\tilde{u}}{\partial\nu}\,=\,0\quad\text{on }\partial\Omega, \tag{1.4}$$

for which extensive literature on transition layer solution is available, see for instance [4, 9, 23, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 40, 43], and the references therein for these and related issues.

In this paper, we consider the inhomogeneous Allen-Cahn equation, i.e, problem (1.3). Let us assume that $\Gamma=\{\,\tilde{y}\in\Omega: a(\tilde{y})=0\,\}$ is a simple, closed and smooth curve in Ω which separates the domain into two disjoint components

$$\Omega=\Omega_-\cup\Gamma\cup\Omega_+ \tag{1.5}$$

such that

$$a(\tilde{y})<0\text{ in }\Omega_+,\quad a(\tilde{y})>0\text{ in }\Omega_-,\quad\frac{\partial a}{\partial\nu_0}>0\text{ on }\Gamma, \tag{1.6}$$

where ν_0 is the outer normal of Ω_+, pointing to the interior of Ω_-. Observe in particular that for the potential (1.2), we have

$$W(\tilde{y},-1)<W(\tilde{y},1)\text{ in }\Omega_-,\quad W(\tilde{y},+1)<W(\tilde{y},-1)\text{ in }\Omega_+.$$

Thus, if one consider a global minimizer u_ε for J_ε, which exists by standard arguments, it should be such that its value want to minimize $W(\tilde{y},u)$, namely, u_ε should intuitively achieve as $\varepsilon\to 0$,

$$u_\varepsilon\to -1\text{ in }\Omega_-\quad u_\varepsilon\to +1\text{ in }\Omega_+. \tag{1.7}$$

A solution u_ε to problem (1.3) with these properties was constructed, and precisely described, by Fife and Greenlee[22] via matching asymptotic and bifurcation arguments. Super-subsolutions were later used by Angenent, Mallet-Paret and Peletier in the one dimensional case (see [7]) for construction and classification of stable solutions. Radial solutions were found variationally by Alikakos and Simpson in [5]. These results were extended by del Pino in [12] for general (even non smooth) interfaces in any dimension, and further constructions have been done recently by Dancer and Yan [11] and Do Nascimento [18]. In particular, it was proved in [11] that solutions with the asymptotic behavior like (1.7) are typically minimizer of J_ε. Related results can be found in [1, 2].

On the other hand, a solution exhibiting a transition layer in the *opposite direction*, namely

$$u_\varepsilon \to +1 \text{ in } \Omega_- \quad \text{and} \quad u_\varepsilon \to -1 \text{ in } \Omega_+ \quad \text{as } \varepsilon \to 0, \tag{1.8}$$

has been believed to exist for many years. Hale and Sakamoto [25] established the existence of this type of solution in the one-dimensional case, while this was done for the radial case in [13], see also [10]. The layer with the asymptotics in (1.8) in this scalar problem is meaningful in describing pattern-formation for reaction-diffusion systems such as Gierer-Meinhardt with saturation, see [13, 21, 36, 41] and the references therein.

Recently this problem has been completely solved by del Pino-Kowalczyk-Wei [15] (in the two dimensional domain case) and Mahmoudi-Malchiodi-Wei [30] (in the higher dimensional case). More precisely, defining

$$\lambda_* = \frac{1}{3\pi^2 \int_{\mathbb{R}} H_x^2 \, \mathrm{d}x} \left[\int_\Gamma \sqrt{\frac{\partial a}{\partial \nu_0}} \right]^2, \tag{1.9}$$

where $H(x)$ is the unique heteroclinic solution of

$$H'' + H - H^3 = 0 \quad \text{in } \mathbb{R},\ H(0) = 0,\ H(\pm\infty) = \pm 1, \tag{1.10}$$

in [15], M. del Pino, M. Kowalczyk and J. Wei proved the existence of a transition layer solution H_ε in the opposite direction, namely,

Theorem 1.1. *Given $c > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition*

$$\left| j^2\varepsilon - \lambda_* \right| \geq c\sqrt{\varepsilon} \text{ for all } j \in \mathbb{N}, \tag{1.11}$$

problem (1.3) has a solution H_ε satisfying

$$H_\varepsilon \to +1 \text{ in } \Omega_-,\ H_\varepsilon \to -1 \text{ in } \Omega_+ \quad \text{as } \varepsilon \to 0. \tag{1.12}$$

Moreover, the layer will approach the curve Γ as $\varepsilon \to 0$. □

We will extend Theorem 1.1 and prove the following main result for the existence of clustering transition layers in this paper. Let $\ell = |\Gamma|$ denote the total length of Γ. We consider natural parameterization $\gamma(\theta)$ of Γ with positive orientation, where θ denotes arclength parameter measured from a fixed point of Γ. Points $\tilde{y}$, which are δ_0-close Γ for sufficiently small δ_0, can be represented in the form

$$\tilde{y} = \gamma(\theta) + t\,\nu_0(\theta), \quad |t| < \delta_0,\ \theta \in [0, \ell), \tag{1.13}$$

where the map $\tilde{y} \mapsto (t, \theta)$ is a local diffeomorphism. By slight abuse of notation we denote $a(t, \theta)$ to actually mean $a(\tilde{y})$ for $\tilde{y}$ in (1.13). The main theorem reads

Theorem 1.2. *For any fixed integer $N = 2m + 1 \geq 3$, there exists a sequence $(\varepsilon_l)_l$ approaching 0^+ such that problem (1.3) has a clustered solution u_{ε_l} with N-phase transition layers with mutual distance $O(\varepsilon_l |\log \varepsilon_l|)$. Moreover, u_{ε_l} approaches 1 in Ω_- and -1 in Ω_+. Near Γ, u_{ε_l} has the form*

$$u_{\varepsilon_l} \sim \sum_{n=1}^{N} H\left(\frac{t - \varepsilon_l e_n(\theta)}{\varepsilon_l}\right),$$

where t is the sign distance to the curve Γ along the normal direction ν_0 and θ denotes arclength parameter measured from a fixed point of Γ. The functions e_n's satisfy

$$\|e_n\|_\infty \ \leq\ C|\log \varepsilon_l|^2, \qquad \min_{1\leq n\leq N-1}(e_{n+1} - e_n)\ >\ \frac{\sqrt{2}}{2}\,|\log \varepsilon_l|.$$

More precisely,

$$e_n\ =\ (-1)^{n+1} f\ +\ f_n \quad \text{with } f(\theta)\ =\ \frac{k(\theta)}{\sqrt{2}\,a_t(0,\theta)},$$

where $k(\theta)$ denotes the curvature of Γ. All functions $f_n, n = 1, \ldots, N$ solve the Toda system, for $n = 1, \ldots, N$,

$$\varepsilon_l^2 \gamma_0 f_n'' + \varepsilon_l\,(-1)^{n+1}\,\alpha_1(\theta)\,f_n - \gamma_1\left[e^{-\left(f_n - f_{n-1} + \beta_n\right)} - e^{-\left(f_{n+1} - f_n + \beta_{n+1}\right)}\right] = 0 \ \text{in } (0, \ell), \tag{1.14}$$

$$f_n'(0)\ =\ f_n'(\ell), \qquad f_n(0)\ =\ f_n(\ell), \tag{1.15}$$

for two universal constant $\gamma_0, \gamma_1\ >\ 0$ and $\alpha_1(\theta)\ =\ \frac{4}{3}a_t(0,\theta)\ >\ 0$, with the conventions $\beta_n\ =\ 2(-1)^n f,\ f_0 = -\infty,\ f_{N+1} = \infty.$ □

Remark 1: *In [11], under the condition that $a = a(r), a(r_0) = 0, a'(r_0) \neq 0$, Dancer and Yan showed the existence of solutions with $N = 2m + 1$ interfaces, provided that ε is small. Therefore they proved Theorem 1.2 in the radial case. Due to resonance phenomenon, for the non-radial case here we only show the existence of clustered solutions for a sequence of ε, rather than a whole interval $(0, \varepsilon_0)$ with ε_0 small. Even worse than that, due to the role of the nonhomogeneous term $a(\tilde{y})$ played in the interaction between mutual neighboring interfaces, we can not give an explicit gap condition for the parameter ε like (1.11) as in Theorem 1.1.*

Remark 2: *As the method for Allen-Chan model in [16] and [17], we use a kind of Toda system to describe the interaction of neighboring interfaces. The reader can also refer [44] for existence of solutions exhibiting interaction of single transition layer and a downward spike.*

Remark 3: *Theorem 1.1 may be extended to higher dimensional case, as in Mahmoudi-Malchiodi-Wei [30]. But the computations and the proofs will be quite involved. The main technical part is to study the anologue Jacobi-Toda system (1.14) on manifolds.*

Now, we detail the resonance phenomenon and the interaction between mutual neighboring interfaces, which are characterized by system (1.14)-(1.15). Take an approximate solution $\hat{\mathbf{f}} = (\hat{f}_1, \cdots, \hat{f}_N)$ to the system (1.14)-(1.15) by

$$\hat{f}_n = \big(n-(m+1)\big)\rho_\varepsilon + \bar{f}_n, \quad n=1,\cdots,N, \qquad \text{with } e^{-\sqrt{2}\,\rho_\varepsilon} = \varepsilon\alpha_1(\theta)\rho_\varepsilon,$$

and $\bar{\mathbf{f}} = (\bar{f}_1, \cdots, \bar{f}_N)$ solves the following nonlinear algebraic system

$$\delta^2(-1)^{n+1}\alpha_1(\theta)\bar{f}_n - \gamma_1 e^{-\sqrt{2}\left(\bar{f}_n-\bar{f}_{n-1}+\beta_n\right)} + \gamma_1 e^{-\sqrt{2}\left(\bar{f}_{n+1}-\bar{f}_n+\beta_{n+1}\right)} = (-1)^n\big(n-(m+1)\big),$$

with n running from 1 to N, where $\bar{f}_0 = -\infty$, $\bar{f}_{N+1} = \infty$. After some simple algebraic computations, we then linearize the nonlinear system at $\hat{\mathbf{f}} = (\hat{f}_1, \cdots, \hat{f}_N)$. Finally, we need to deal with the solvability theory for the following operators (c.f. (7.38) and (7.39))

$$\begin{aligned} L_\varepsilon^i &= \kappa\Big(\varepsilon\,\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \alpha_1\Big) + \lambda_i^\varepsilon\Big(\alpha_1+\sigma^\varepsilon\Big),\ i=1,2,\cdots,N-1, \\ L_\varepsilon^N &= \kappa\Big(\varepsilon\,\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \alpha_1\Big) + \lambda_N^\varepsilon\Big(\alpha_1+\sigma^\varepsilon\Big), \end{aligned}$$

with suitable boundary conditions, where we have denoted

$$\begin{aligned} \lambda_i^\varepsilon > c > 0,\ i=1,2,\cdots,N-1, \qquad & \lambda_N^\varepsilon = -\frac{(N-1)\kappa}{N(\alpha+\sigma^\varepsilon)}, \\ \kappa = \frac{1}{\frac{\sqrt{2}}{2}\left(\log\frac{1}{\varepsilon} - \log\log\frac{1}{\varepsilon}\right)}, \qquad & \sigma^\varepsilon = O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big). \end{aligned}$$

In fact as for the solvability theory of the operator L_ε^N, we consider the following *eigenvalue problem*

$$\varepsilon\,\gamma_0\,\Theta^{''} + \frac{1}{N}\alpha_1(\theta)\Theta = -\Lambda\,\alpha_1(\theta)\,\Theta \quad \text{in } (0,\ell), \quad \Theta(0) = \Theta(\ell), \quad \Theta'(0) = \Theta'(\ell), \tag{1.16}$$

where have denoted

$$\alpha_1(\theta) = \frac{4}{3}a_t(0,\theta) > 0, \quad \gamma_0 = \int_{\mathbb{R}} H_x^2\,\mathrm{d}x = \frac{2\sqrt{2}}{3} > 0. \tag{1.17}$$

It's well known that (1.16) has a sequence of different eigenvalues $\Lambda_1^\varepsilon < \Lambda_2^\varepsilon < \cdots$ with corresponding eigenfunctions Θ_1^ε, $\Theta_2^\varepsilon, \cdots$. It is obvious that $\Lambda_1^\varepsilon < 0$ because α_1 and γ_0 are positive. If ε is small then we have, as $j\to\infty$,

$$\Lambda_j^\varepsilon = \frac{4\pi^2}{\ell_0^2}\,(j^2\varepsilon - \frac{1}{N}\lambda_*) + O(\frac{\varepsilon}{j^2}), \tag{1.18}$$

where we use the definition of λ_* in (1.9). As a direct consequence, for all small ε and large j we have the following spectral gap estimates

$$|\Lambda^{\varepsilon}_{j+1} - \Lambda^{\varepsilon}_{j}| > C\sqrt{\varepsilon}, \tag{1.19}$$

where C is a positive constant independent of ε. Under a similar gap condition like (1.11), we also have the following spectral gap estimate for critical eigenvalues(close to zero)

$$|\Lambda^{\varepsilon}_{j}| > c\sqrt{\varepsilon},\ \ j = 1, 2, \cdots . \tag{1.20}$$

The validity of the estimate in (1.18) and (1.19) can be proved as the arguments in Lemma 2.1. The resonance character of this type also appeared in [15] for the case $N = 1$. However, it is more complicated to get an explicit gap condition for uniform solvability theory of the operators L^{i}_{ε}, $i = 1, 2, \cdots , N-1$. For more details, the reader can refer to Lemma 7.3.

The remaining part of this paper is devoted to the complete proof of Theorem 1.2. The organization is as follows: In Section 2, after setting up the problem in stretched variables (x, z), we introduce a local approximate solution by

$$\sum_{j=1}^{N} (-1)^{j+1}\, H\big(x - e_j(\varepsilon z)\big) + O(\varepsilon),$$

in which the parameters e_j's are used to characterize the locations of the phase transition layers. In Section 3, a gluing procedure reduces the nonlinear problem to a projected problem on the infinite strip $\mathfrak{S}$, (c.f.(2.6)), while in Section 4 and Section 5, we show that the projected problem has a unique solution for given parameters $e_1, \cdots , e_N$ in a chosen region. The final step is to adjust the parameters $e_1, \cdots , e_N$ which is equivalent to solving a nonlocal, nonlinear coupled second order system of differential equations for the functions $e_1, \cdots , e_N$ with suitable boundary conditions, which is equivalent to the Toda system (1.14)-(1.15). This is done in Section 6 and Section 7.

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2 Local formulation of the problem

Addition to some preliminaries, the main object is to formulate the problem in local coordinates and construct local approximate solution.

2.1 Preliminaries

As a preliminary, we consider the following *eigenvalue problem*

$$\varepsilon\,\gamma_0\,\Theta^{''} + \alpha_1(\theta)\Theta \;=\; -\Lambda\,\alpha_1(\theta)\,\Theta \quad \text{in } (0,\ell), \quad \Theta(0) \;=\; \Theta(\ell), \quad \Theta'(0) \;=\; \Theta'(\ell). \tag{2.1}$$

All critical eigenvalues (the eigenvalues whose absolute value is close to zero) of the geometric eigenvalue problem (2.1) have good estimates as follows.

Lemma 2.1. *For all small ε satisfies the gap condition (1.11), we have the following spectral gap estimates of the geometric problem (2.1): there exist two positive constants C (independent of ε) and $N^\varepsilon \in \mathbb{N}$ with $N^\varepsilon \to \infty$ as $\varepsilon \to 0$, such that*

$$\begin{aligned}\Lambda_i^\varepsilon &\leq -C\sqrt{\varepsilon}, \quad \textit{for all } i = 1, 2, \cdots, N^\varepsilon,\\ \Lambda_i^\varepsilon &\geq C\sqrt{\varepsilon}, \quad \textit{for all } i = N^\varepsilon + 1, N^\varepsilon + 2, \cdots.\end{aligned}$$

Proof. By the following Liouville transformation, for details see Chapter X, Theorem 6 in [8]

$$\ell_0 \;=\; \int_0^\ell \sqrt{\alpha_1(\theta)/(\gamma_0)}\,\mathrm{d}\,\theta, \qquad t \;=\; \frac{\pi}{\ell_0}\int_0^\theta \sqrt{\alpha_1(\theta)/(\gamma_0)}\,\mathrm{d}\,\theta,\; t \in [0,\pi),$$
$$e(t) \;=\; \Theta(\theta)\Big(\gamma_0\,\alpha_0\Big)^{1/4}, \qquad q(t) \;=\; \frac{\gamma_0\ell_0^2\alpha_1^{''}}{4\pi^2\alpha_1^2} - \frac{5\gamma_0\ell_0^2(\alpha_1^{'})^2}{16\pi^2\alpha_1^3},$$

the eigenvalue Λ satisfies the following eigenvalue problem

$$-e^{''} - q(t)\,e = \frac{\ell_0^2}{\pi^2\,\varepsilon}\,(1+\Lambda)\,e \quad \text{in } (0,\pi),\; e(0) \;=\; e(\pi),\; e^{'}(0) = e^{'}(\pi).$$

Now consider the following auxiliary eigenvalue problem

$$-y^{''} - q(t)y = \xi\,y,\; 0 < t < \pi,\; y(0) \;=\; y(\pi),\; y^{'}(0) = y^{'}(\pi).$$

The result in [29] shows that, as $n \to \infty$

$$\sqrt{\xi_n} \;=\; 2n \,+\, O(\frac{1}{n^3}).$$

Hence, if ε is small then we have, as $n \to \infty$,

$$\Lambda_n^\varepsilon = \frac{4\pi^2}{\ell_0^2}\,(n^2\varepsilon - \lambda_*) \,+\, O(\frac{\varepsilon}{n^2}), \tag{2.2}$$

where we use the definition of λ_* in (1.9). The last formula together with the gap condition (1.11) implies the estimates in the lemma. □

We finally recall the following theorem due to T. Kato([26]), which will be fundamental for us to obtain invertibility of the linearized problem in the last section.

Theorem 2.2. *Let $T(\varepsilon)$ be a differentiable family of operators from a Hilbert space X into itself, where ε belongs to an interval containing 0. Let $T(0)$ be a self-adjoint operator of the form Identity− Compact and let $\sigma(0)=\sigma_0\neq 1$ be an eigenvalue of $T(0)$. Then the eigenvalue $\sigma(\varepsilon)$ is differentiable at 0 with respect to ε. The derivative of σ is given by*

$$\frac{\partial\sigma}{\partial\varepsilon}=\Big\{\text{ eigenvalues of } P_{\sigma_0}\circ\frac{\partial T}{\partial\varepsilon}(0)\circ P_{\sigma_0}\Big\},$$

where $P_{\sigma_0}:X\to X_{\sigma_0}$ denotes the projection onto the σ_0-eigenspace X_{σ_0} of $T(0)$.

2.2 Local coordinate and first approximate solution

Let $\ell=|\Gamma|$ denote the total length of Γ. We consider natural parameterization $\gamma(\theta)$ of Γ with positive orientation, where θ denotes arclength parameter measured from a fixed point of Γ. Let $\nu_0(\theta)$ denote the outer unit normal to Γ, pointing to the interior of Ω_-. Points $\tilde y$, which are δ_0-close Γ for sufficiently small δ_0, can be represented in the form

$$\tilde y=\gamma(\theta)+t\,\nu_0(\theta),\quad |t|<\delta_0,\ \theta\in[0,\ell),\tag{2.3}$$

where the map $\tilde y\mapsto(t,\theta)$ is a local diffeomorphism. By slight abuse of notation we denote $a(t,\theta)$ to actually mean $a(\tilde y)$ for $\tilde y$ in (2.3). In this coordinate, near Γ the metric can be parameterized as

$$g_{t,\theta}=\mathrm{d}t^2\,+\,(1+kt)^2\mathrm{d}\theta^2,$$

and the Laplacian operator is

$$\triangle_{t,\theta}=\frac{\partial^2}{\partial t^2}+\frac{1}{(1+kt)^2}\frac{\partial^2}{\partial\theta^2}+\frac{k}{1+kt}\frac{\partial}{\partial t}-\frac{k^{'}t}{(1+kt)^3}\frac{\partial}{\partial\theta},$$

where $k(\theta)$ is the curvature of Γ.

We begin with the formulation of the problem in some suitable coordinates. Stretching variable $\tilde y=\varepsilon y$ and setting $\Omega_\varepsilon=\Omega/\varepsilon$, let us consider the scaling $u(y)=\tilde u(\varepsilon y)$, then problem (1.3) is thus equivalent to

$$\Delta_y u\,+\,\big(u-a(\varepsilon y)\big)(1-u^2)=0\quad\text{ in }\Omega_\varepsilon,\quad\frac{\partial u}{\partial\nu_y}=0\quad\text{ on }\partial\Omega_\varepsilon.\tag{2.4}$$

We also set up new coordinate $(x,\,z)$ near $\Gamma_\varepsilon=\Gamma/\varepsilon$ in Ω_ε,

$$\varepsilon\,y\;=\;\gamma(\varepsilon z)\,+\,\varepsilon\,x\,\nu_0(\varepsilon z),\quad |x|<\delta_0/(20\varepsilon),\ z\in[0,\ell/\varepsilon),\tag{2.5}$$

and then the metric and Laplacian operator can be written as

$$\begin{aligned}g_{x,z}&=\mathrm{d}x^2\,+\,\big(1+\varepsilon kx\big)^2\mathrm{d}z^2,\\ \triangle_{x,z}&=\frac{\partial^2}{\partial x^2}+\frac{1}{[1+\varepsilon kx]^2}\frac{\partial^2}{\partial z^2}+\frac{\varepsilon\,k}{1+\varepsilon kx}\frac{\partial}{\partial x}-\frac{\varepsilon^2\,k^{'}x}{[1+\varepsilon kx]^3}\frac{\partial}{\partial z}.\end{aligned}$$

Now, let $\mathfrak{S}$ represent the strip as, formulated in (x,z) coordinate in $\mathbb{R}^2$,

$$\mathfrak{S} \,=\, \Big\{\ (x,z) : x\in\mathbb{R}, 0\le z\le \frac{\ell}{\varepsilon}\ \Big\}. \tag{2.6}$$

Near Γ_ε, the first equation in (2.4) can be written as a problem on the whole strip $\mathfrak{S}$ by the form

$$\Upsilon(u) \,\equiv\, u_{xx} \,+\, u_{zz} \,+\, B_3(u) \,+\, B_4(u) \,+\, B_5(u) \,+\, F(u) \,=\, 0, \tag{2.7}$$

where

$$B_3(u) \;=\; \varepsilon k(\varepsilon z)\, u_x \,-\, \varepsilon^2 k^2(\varepsilon z)\, x\, u_x, \tag{2.8}$$

$$B_4(u) \;=\; -\Big[\,\varepsilon a_t(0,\varepsilon z)\, x + \frac{1}{2}\,\varepsilon^2\, a_{tt}(0,\varepsilon z)\, x^2\,\Big](1-u^2), \tag{2.9}$$

$$B_5(u) \;=\; B_0(u) \,-\, \varepsilon^3\, a_4(0,\varepsilon z)\, x^3(1-u^2), \tag{2.10}$$

with $B_0(u)$ represents all terms of order ε^3 in the term $\triangle_{x,z}u$ and a_4 is bounded smooth function. Here and in what follows we also denote

$$F(u) \,\equiv\, u \,-\, u^3.$$

To define the approximate solution we recall some basic properties of the heteroclinic solution $H(x)=\tanh\Big(\frac{x}{\sqrt{2}}\Big)$ to (1.10) in the following.

$$H(x) \,=\, 1 \,-\, A_0\, e^{-\sqrt{2}\,|x|} \,+\, O\big(e^{-2\sqrt{2}\,|x|}\big), \quad x>1, \tag{2.11}$$

$$H(x) \,=\, -1 \,+\, A_0\, e^{-\sqrt{2}\,|x|} \,+\, O\big(e^{-2\sqrt{2}\,|x|}\big), \quad x<-1, \tag{2.12}$$

$$H^{'}(x) \,=\, \sqrt{2}\, A_0\, e^{-\sqrt{2}\,|x|} \,+\, O\big(e^{-2\sqrt{2}\,|x|}\big), \quad |x|>1, \tag{2.13}$$

where A_0 is a universal constant. It is trivial to derive that

$$1-H^2(x) \,=\, \sqrt{2}\, H_x(x), \qquad \int_{\mathbb{R}}(1-H^2)H_x\, \mathrm{d}x \,=\, \sqrt{2}\int_{\mathbb{R}} H_x^2\, \mathrm{d}x \,=\, \frac{4}{3}. \tag{2.14}$$

For a fixed odd integer $N=2m+1\ge 3$, we assume that the location of the N phase transition layers are characterized by periodic functions $x=e_j(\varepsilon z)$, $1\le j\le N$ in the coordinate (x,z). These functions can be defined as the following

$$e_j\,:\,(0,\ell)\to\mathbb{R}, \tag{2.15}$$

$$||e_j||_{H^2(0,\ell)} \,<\, C\,|\log\varepsilon|^2, \tag{2.16}$$

$$e_{j+1}(\zeta)-e_j(\zeta) \,>\, \frac{\sqrt{2}}{2}\,|\log\varepsilon| \,-\, \frac{\sqrt{2}}{2}\,\log|\log\varepsilon|. \tag{2.17}$$

For convenience of the notation we will also set

$$e_0(\zeta)=-\delta_0/\varepsilon-e_1(\zeta) \ \text{ and } \ e_{N+1}(\zeta)=\delta_0/\varepsilon-e_N(\zeta).$$

Set

$$H_j(x,z) \equiv (-1)^{j+1} H\big(x - e_j(\varepsilon z)\big),$$

and define **the first approximate solution** to (2.7) by

$$u_0(x,z) \equiv \sum_{j=1}^{N} H_j(x,z).$$

Our first goal is to compute the error of approximation in a δ_0/ε neighborhood of Γ_ε, namely the quantities

$$\begin{aligned} E_0 &\equiv \Upsilon(u_0) \\ &= u_{0,xx} + u_{0,zz} + B_3(u_0) + B_4(u_0) + B_5(u_0) + F(u_0), \end{aligned} \tag{2.18}$$

where

$$u_{0,xx} + u_{0,zz} = \sum_{j=1}^{N} H_{j,xx} - \varepsilon^2 \sum_{j=1}^{N} e_j^{''} H_{j,x} + \varepsilon^2 \sum_{j=1}^{N} (e_j^{'})^2 H_{j,xx},$$

$$B_3(u_0) = \varepsilon k \sum_{j=1}^{N} H_{j,x} - \varepsilon^2 k^2 \sum_{j=1}^{N} (x - e_j)\, H_{j,x} - \varepsilon^2 k^2 \sum_{j=1}^{N} e_j\, H_{j,x}.$$

We now turn to computing other nonlinear terms in E_0. For every fixed n, $1 \le n \le N$, we consider the following set

$$A_n = \left\{ (x,z) \in \big(-\frac{\delta_0}{\varepsilon}, \frac{\delta_0}{\varepsilon}\big) \times \big(0, \frac{\ell}{\varepsilon}\big) \,\middle|\, \frac{e_{n-1}(\varepsilon z) + e_n(\varepsilon z)}{2} \le x \le \frac{e_n(\varepsilon z) + e_{n+1}(\varepsilon z)}{2} \right\}.$$

For $(x,z) \in A_n$, we write

$$\begin{aligned} F(u_0) &= F(H_n) + F^{'}(H_n)(u_0 - H_n) + \frac{1}{2} F^{''}(H_n)(u_0 - H_n)^2 + \max_{j\neq n} O(e^{-3\sqrt{2}\,|e_j - x|}) \\ &= \sum_{j=1}^{N} F(H_j) + F^{'}(H_n)(u_0 - H_n) - \sum_{j\neq n} F(H_j) + \frac{1}{2} F^{''}(H_n)(u_0 - H_n)^2 \\ &\qquad + \max_{j\neq n} O(e^{-3\sqrt{2}\,|e_j - x|}). \end{aligned}$$

As the computations in [16], we obtain, for $(x,z) \in A_n$, $n = 1, \ldots, N$

$$\begin{aligned} F(u_0) &= \sum_{j=1}^{N} F(H_j) + \frac{1}{2} F^{''}(H_n)(u_0 - H_n)^2 + 3\Big[1 - H_n^2\Big](u_0 - H_n) \\ &\qquad - \frac{1}{2} \sum_{j\neq n} F^{''}(\sigma_{nj})(\sigma_{nj} - H_j)^2 + \max_{j\neq n} O(e^{-3\sqrt{2}\,|e_j - x|}). \end{aligned}$$

In the above formula, we define σ_{nj} as follows. If n is even, $\sigma_{nj} = (-1)^j$ for $j < n$ and $\sigma_{nj} = (-1)^{j+1}$ for $j > n$. If n is odd, $\sigma_{nj} = (-1)^{j+1}$ for $j < n$ and $\sigma_{nj} = (-1)^j$ for $j > n$.

Also, it is derived that, for $(x,z)\in A_n$, $n=1,\dots,N$

$$
\begin{aligned}
B_4(u_0) \;=\;& -\varepsilon a_t\, x \sum_{j=1}^N \Big[1 - \big(H_j\big)^2 \Big] + \varepsilon a_t\, x \sum_{j\neq n} \Big(1 - \big(H_j\big)^2 \Big)\\
& + \varepsilon a_t\, x \Big[\big(u_0\big)^2 - H_n^2 \Big] - \frac{1}{2}\varepsilon^2 a_{tt} \sum_{j=1}^N (x-e_j)^2 (1-H_j^2)\\
& - \frac{1}{2}\varepsilon^2 a_{tt} \sum_{j=1}^N e_j^2 (1-H_j^2) - \varepsilon^2 a_{tt} \sum_{j=1}^N (x-e_j)\, e_j (1-H_j^2)\\
& + \frac{1}{2}\varepsilon^2 a_{tt}\, x^2 \sum_{j\neq n} (1-H_j^2) + \frac{1}{2}\varepsilon^2 a_{tt}\, x^2 \Big[\big(u_0\big)^2 - H_n^2 \Big].
\end{aligned}
$$

Substitute (2.14) to above formula, it follows then for $(x,z)\in A_n$, $n=1,\dots,N$:

$$
\begin{aligned}
E_0 \;=\;& \varepsilon k \sum_{j=1}^N H_{j,x} - \sqrt{2}\,\varepsilon a_t \sum_{j=1}^N e_j\, (-1)^{j+1} H_{j,x} - \sqrt{2}\,\varepsilon a_t \sum_{j=1}^N (x-e_j)\, (-1)^{j+1} H_{j,x}\\
& - \frac{\sqrt{2}}{2}\varepsilon^2 a_{tt} \sum_{j=1}^N (x-e_j)^2\, (-1)^{j+1} H_{j,x} - \frac{\sqrt{2}}{2}\varepsilon^2 a_{tt} \sum_{j=1}^N e_j^2\, (-1)^{j+1} H_{j,x} - \varepsilon^2 k^2 \sum_{j=1}^N e_j\, H_{j,x}\\
& + \sqrt{2}\,\varepsilon a_t\, x \sum_{j\neq n} (-1)^{j+1} H_{j,x} + \varepsilon a_t\, x \Big[\big(u_0\big)^2 - H_n^2 \Big] + \frac{1}{2}\varepsilon^2 a_{tt}\, x^2 \Big[\big(u_0\big)^2 - H_n^2 \Big]\\
& - \sqrt{2}\,\varepsilon^2 a_{tt} \sum_{j=1}^N (x-e_j)\, e_j\, (-1)^{j+1} H_{j,x} + \frac{\sqrt{2}}{2}\varepsilon^2 a_{tt}\, x^2 \sum_{j\neq n} (-1)^{j+1} H_{j,x}\\
& - \varepsilon^2 k^2 \sum_{j=1}^N (x-e_j)\, H_{j,x} - \varepsilon^2 \sum_{j=1}^N e_j^{''} H_{j,x} + \varepsilon^2 \sum_{j=1}^N (e_j^{'})^2 H_{j,xx} + 3\Big[1 - H_n^2 \Big](u_0 - H_n)\\
& + \frac{1}{2} F^{''}(H_n)(u_0-H_n)^2 - \frac{1}{2} F^{''}(\sigma_{nj})(\sigma_{nj}-H_j)^2 + \max_{j\neq n} O(e^{-3\sqrt{2}\,|e_j - x|}) + O(\varepsilon^3)\\
\equiv\;& E_{01} + E_{02},
\end{aligned}
$$

where

$$
\begin{aligned}
E_{01}(x,z) \;=\;& \varepsilon k \sum_{j=1}^N H_{j,x} - \sqrt{2}\,\varepsilon a_t \sum_{j=1}^N e_j\, (-1)^{j+1} H_{j,x} - \sqrt{2}\,\varepsilon a_t \sum_{j=1}^N (x-e_j)\, (-1)^{j+1} H_{j,x}, \qquad (2.19)\\
E_{02} \;=\;& E_0 - E_{01}. \qquad (2.20)
\end{aligned}
$$

From the above expression for E_0 we see that, given the sizes for e_n's in (2.15)-(2.17) and the properties of the function H in (2.11), denoting by $\chi_{A_n}(x,z)$ the characteristic function of the set A_n, we have

$$
\begin{aligned}
E_0(x,z) \;=\;& E_{01} + 3 \sum_{n=1}^N \chi_{A_n} \Big[1 - H_n^2 \Big](u_0 - H_n)\\
& + \sum_{n=1}^N \chi_{A_n} \Big[O(\varepsilon^2 |\log\varepsilon|^2) e^{-\sqrt{2}\,|e_n - x|} + O(1) \max_{j\neq n} e^{-2\sqrt{2}\,|e_j - x|} \Big] + O(\varepsilon^3).
\end{aligned}
$$

For further application, the evaluation of the first approximation is of importance. Using the condition (2.15)-(2.17), a tedious computation implies,

$$\Big\| E_{01} \Big\|_{L^2(S_{\delta_0/\varepsilon})} = O(\varepsilon^{\frac{1}{2}} |\log\varepsilon|^q), \qquad \Big\| E_{02} \Big\|_{L^2(S_{\delta_0/\varepsilon})} = O(\varepsilon^{\frac{3}{2}} |\log\varepsilon|^q),$$

where

$$S_{\delta_0/\varepsilon} = \{-\delta_0/\varepsilon < x < \delta_0/\varepsilon,\ 0 < z < 1/\varepsilon\}.$$

The discrepancy between the order of the components E_{01} and E_{02} in the error E_1 makes it necessary to improve the original approximation u_0 and eliminate the $O(\varepsilon)$-part E_{01} of the error. In the language of formal asymptotic expansions one can say that we need to find a correction layer expansion of our solution, which will be done in the next subsection.

2.3 Improvement of approximate solution

We now want to construct a further approximation to a solution which eliminates the terms of order ε in the error E_0. Firstly, we take e_j as the form

$$e_j = (-1)^{j+1} f + f_j, \quad f(\theta) = \frac{k(\theta)}{\sqrt{2}\, a_t(0,\theta)}. \tag{2.21}$$

From the properties of e_j's in (2.15)-(2.17), the unknown functions f_j's should be defined as the following

$$f_j : (0,\ell) \to \mathbb{R}, \tag{2.22}$$

$$||f_j||_{H^2(0,\ell)} < C\,|\log\varepsilon|^2, \tag{2.23}$$

$$f_{j+1}(\zeta) - f_j(\zeta) > \frac{\sqrt{2}}{2}|\log\varepsilon| - \frac{\sqrt{2}}{2}\log|\log\varepsilon|. \tag{2.24}$$

For convenience of the notations we will also set

$$f_0(\zeta) = -\delta_0/\varepsilon - f_1(\zeta) \ \text{ and } \ f_{N+1}(\zeta) = \delta_0/\varepsilon - f_N(\zeta),$$

$$\mathbf{f} \equiv (f_1, \cdots, f_N) \text{ with } f_1, \cdots, f_N \text{ satisfying } (2.22) - (2.24).$$

Secondly, since

$$\int_{\mathbb{R}} x H_x^2 \,\mathrm{d}x = 0, \tag{2.25}$$

it is well known that there exists a unique solution(odd) to the following problem

$$\Psi_{xx} + F^{'}(H)\Psi = x\,H_x \quad \text{in } \mathbb{R}, \quad \Psi(\pm\infty) = 0, \quad \int_{\mathbb{R}} \Psi H_x \,\mathrm{d}x = 0. \tag{2.26}$$

Therefore, we can define our correction layer by

$$\phi^*(x,z) \equiv \varepsilon \sum_{j=1}^{N} \phi_j^*(x,z), \quad \phi_j^*(x,z) = (-1)^{j+1}\sqrt{2}\, a_t(0,\varepsilon z)\Psi(x-e_j),$$

and take the **basic approximation** to a solution to the problem near the curve Γ_ε by

$$u_1 = u_0 + \phi^*.$$

The new error can be computed as the following

$$\begin{aligned} E_1 &= E_0 + \phi^*_{xx} + \phi^*_{zz} + F^{'}(u_0)\phi^* + B_3(\phi^*) \\ &\quad + B_4(u_0+\phi^*) - B_4(u_0) + B_5(u_0+\phi^*) - B_5(u_0) + N(\phi^*), \end{aligned}$$

where

$$\begin{aligned} B_3(\phi^*) &= \sqrt{2}\,\varepsilon^2\, k\, a_t \sum_{j=1}^{N} (-1)^{j+1}\, \Psi_x + O(\varepsilon^3), \\ N(\phi^*) &= -3\, u_0 \left(\phi^*\right)^2 - \left(\phi^*\right)^3. \end{aligned}$$

We compute two main components of E_1 in the sequel. The first term is the following

$$\begin{aligned} &\phi^*_{xx} + \phi^*_{zz} + F^{'}(u_0)\phi^* \\ &\quad = \varepsilon \sum_{j=1}^{N} \Big[\phi^*_{j,xx} + \phi^*_{j,zz} + F^{'}(H_j)\phi^*_j \Big] - 3\,\varepsilon \sum_{j=1}^{N} \Big[\left(u_0\right)^2 - H_j^2 \Big] \phi^*_j \\ &\quad = \sqrt{2}\,\varepsilon\, a_t(0,\varepsilon z) \sum_{j=1}^{N} (x-e_j)\,(-1)^{j+1}\, H_{j,x} - 3\,\varepsilon \sum_{j=1}^{N} \Big[u_0^2 - H_j^2 \Big] \phi^*_j + O(\varepsilon^3). \end{aligned}$$

The other term is in the sequel

$$\begin{aligned} &B_4(u_0+\phi^*) - B_4(u_0) \\ &\quad = 2\sqrt{2}\,\varepsilon^2\, a_t^2 \sum_{j=1}^{N} (x-e_j)\, H(x-e_j)\Psi(x-e_j) + 2\sqrt{2}\,\varepsilon^2\, a_t^2 \sum_{j=1}^{N} x\,(u_0 - H_j)\phi^*_j \\ &\qquad + 2\sqrt{2}\,\varepsilon^2\, a_t^2 \sum_{j=1}^{N} e_j\, H(x-e_j)\Psi(x-e_j) + O(\varepsilon^3). \end{aligned}$$

Therefore, the following lemma is readily checked, which implies that ϕ^* is the right correction layer as we just stated in previous subsection.

Lemma 2.3. *With the notation of the previous subsection we have*

$$\begin{aligned} E_1 &\equiv \Upsilon(u_1) \\ &= E_{02} - \sqrt{2}\,\varepsilon a_t \sum_{j=1}^{N} f_j\,(-1)^{j+1} H_{j,x} + 2\sqrt{2}\,\varepsilon^2\, a_t^2 \sum_{j=1}^{N} (x-e_j)\, H(x-e_j)\Psi(x-e_j) \\ &\quad - 3\,\varepsilon \sum_{j=1}^{N} \Big(u_0^2 - H_j^2 \Big) \phi^*_j + 2\sqrt{2}\,\varepsilon^2\, a_t^2 \sum_{j=1}^{N} x\,(u_0 - H_j)\phi^*_j - 3\, u_0 \left(\phi^*\right)^2 - \left(\phi^*\right)^3 \\ &\quad + 2\sqrt{2}\,\varepsilon^2\, a_t^2 \sum_{j=1}^{N} e_j\, H(x-e_j)\Psi(x-e_j) + \sqrt{2}\,\varepsilon^2\, k\, a_t \sum_{j=1}^{N} (-1)^{j+1}\, \Psi_x \\ &\equiv E_{02} + E_{11} + E_{12}, \end{aligned}$$

where we have denoted

$$E_{11} \,=\, -\sqrt{2}\,\varepsilon\, a_t \sum_{j=1}^{N} f_j(-1)^{j+1} H_{j,x}, \quad E_{12} \,=\, E_1 \,-\, E_{02} \,-\, E_{11}.$$

Moreover, we have the following estimate

$$\big\|E_{11}\big\|_{L^2(\mathfrak{S})} \le C\,\varepsilon^{1/2}|\log\varepsilon|^2, \quad \big\|E_{02}+E_{12}\big\|_{L^2(\mathfrak{S})} \le C\,\varepsilon^{3/2}|\log\varepsilon|^2. \tag{2.27}$$

□

Locally and formally, we set up the full problem in the form $\Upsilon(u_1+\phi)=0$, which takes the form near the curve Γ_ε,

$$\Upsilon(u_1+\phi) \,=\, L_1(\phi) \,+\, B_8(\phi) \,+\, E_1 \,+\, N_1(\phi) = 0, \tag{2.28}$$

where

$$L_1(\phi) \;=\; \phi_{xx} \,+\, \phi_{zz} \,+\, \Big(1-3(u_1)^2\Big)\phi, \tag{2.29}$$

$$B_8(\phi) \;=\; B_3(\phi) \,+\, B_0(u) \,+\, \Big[\varepsilon\, 2a_t(0,\varepsilon z)\,x + \varepsilon^2\, a_{tt}(0,\varepsilon z)\,x^2 + \varepsilon^3\, 2a_4(0,\varepsilon z)\,x^3\Big] u_1\,\phi, \tag{2.30}$$

$$N_1(\phi) \;=\; \phi^3 \,+\, 3u_1\phi^2 \,+\, \Big[\varepsilon 2a_t(0,\varepsilon z)\,x + \varepsilon^2\, a_{tt}(0,\varepsilon z)\,x^2 + \varepsilon^3 2a_4(0,\varepsilon z)x^3\Big]\,\phi^2. \tag{2.31}$$

3 The gluing procedure

In this section, we use a gluing technique (as in [14]) to reduce the global problem in Ω_ε to a nonlinear projected problem in the infinite strip $\mathfrak{S}$ defined in (2.6).

Let $\delta < \delta_0/100$ be a fixed number, where δ_0 is a constant defined in (2.3). We consider a smooth cut-off function $\eta_\delta(t)$ where $t\in\mathbb{R}_+$ such that $\eta_\delta(t)=1$ for $0\le t\le\delta$ and $\eta(t)=0$ for $t>2\delta$. Set $\eta_\delta^\varepsilon(x)=\eta_\delta(\varepsilon|x|)$, where x is the normal coordinate to Γ_ε. Let $u_1(x,z)$ denote the approximate solution constructed near the curve Γ_ε in the coordinate (x,z), which was introduced in (2.5). We define our global approximation on Ω_ε to be simply

$$W(y) \,=\, \begin{cases} \eta_{3\delta}^\varepsilon(x)\,\big[\,u_1+1\,\big] \,-\, 1, & \text{for } x<0,\\ \eta_{3\delta}^\varepsilon(x)\,\big[\,u_1-1\,\big] \,+\, 1, & \text{for } x>0.\end{cases}$$

For $u=W+\hat\phi$ where $\hat\phi$ globally defined in Ω_ε, denote

$$S(u) \,=\, \triangle_y u \,+\, \big(u-a(\varepsilon y)\big)\,(1-u^2) \quad \text{in } \Omega_\varepsilon.$$

Then u satisfies (2.4) if and only if

$$\widetilde{\mathcal{L}}(\hat\phi) \,=\, -\widetilde{E} \,+\, \widetilde{N}(\hat\phi) \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial}{\partial\nu_y}\,\hat\phi \,=\, -\frac{\partial}{\partial\nu_y}\,W \,=\, 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{3.1}$$

where

$$\widetilde{\mathcal{L}}(\hat{\phi}) \,=\, \triangle_y\hat{\phi} \,+\, \Big[1 - 3W^2 + 2a(\varepsilon y)W\Big]\hat{\phi},$$
$$\widetilde{N}(\hat{\phi}) \,=\, (\hat{\phi})^3 \,+\, 3W(\hat{\phi})^2 \,-\, a(\varepsilon y)(\hat{\phi})^2, \qquad \widetilde{E} \,=\, S(W).$$

We further separate $\hat{\phi}$ in the following form

$$\hat{\phi} = \eta^{\varepsilon}_{3\delta}\phi + \psi$$

where, in the coordinate (x, z) in (2.5), we assume that ϕ is defined in the whole strip $\mathfrak{S}$. Obviously, (3.1) is equivalent to the following problem

$$\eta^{\varepsilon}_{3\delta}\Big[\,\triangle_y\phi \,+\, \big(1 - 3W^2\big)\phi \,+\, 2a(\varepsilon y)W\,\phi\,\Big] \;=\; \eta^{\varepsilon}_{\delta}\Big[\,\widetilde{N}(\eta^{\varepsilon}_{3\delta}\phi + \psi) - \widetilde{E} - 3(1 - W^2)\psi\,\Big], \tag{3.2}$$
$$\triangle_y\psi - 2(1 - aW)\psi + 3(1 - \eta^{\varepsilon}_{\delta})(1 - W^2)\psi \;=\; -\,\varepsilon^2(\triangle_y\eta^{\varepsilon}_{3\delta})\phi \,-\, 2\varepsilon(\nabla_y\eta^{\varepsilon}_{3\delta})(\nabla_y\phi)$$
$$+\,(1 - \eta^{\varepsilon}_{\delta})\widetilde{N}(\eta^{\varepsilon}_{3\delta}\phi + \psi) \,-\, (1 - \eta^{\varepsilon}_{\delta})\widetilde{E}, \tag{3.3}$$

where ψ, defined in Ω_ε, is required to satisfy the homogeneous Neumann boundary condition. For further references, we write the following estimates. From Lemma 2.3

$$\eta^{\varepsilon}_{\delta}\widetilde{E} \;=\; \eta^{\varepsilon}_{\delta}E_{11} \,+\, \eta^{\varepsilon}_{\delta}(E_1 - E_{11}) \tag{3.4}$$

with the validity of estimates

$$\big\|\eta^{\varepsilon}_{\delta}E_{11}\big\|_{L^2(\Omega_\varepsilon)} \,\le\, C\,\varepsilon^{1/2}|\log\varepsilon|^2, \quad \big\|\eta^{\varepsilon}_{\delta}(E_1 - E_{11})\big\|_{L^2(\Omega_\varepsilon)} \,\le\, C\,\varepsilon^{3/2}|\log\varepsilon|^2. \tag{3.5}$$

A trivial computation gives that

$$\|(1 - \eta^{\varepsilon}_{\delta})\widetilde{E}\|_{L^\infty(\Omega_\varepsilon)} \;=\; \|\widetilde{E}\|_{L^\infty(\Omega_\varepsilon\cap\{|x|>\delta/\varepsilon\})} \,\le\, Ce^{-\frac{\sqrt{2}}{2}\delta/\varepsilon}. \tag{3.6}$$

The key observation is that, after solving (3.3), the problem can be transformed to the following nonlinear problem involving the parameter ψ

$$\widetilde{\mathcal{L}}(\phi) = \eta^{\varepsilon}_{\delta}\Big[\,\widetilde{N}(\eta^{\varepsilon}_{3\delta}\phi + \psi) - \widetilde{E} - 3(1 - W^2)\psi\,\Big]. \tag{3.7}$$

Observe that the operator $\widetilde{\mathcal{L}}$ in Ω_ε may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of Γ_ε in the strip $\mathfrak{S}$.

First, we solve, given a small ϕ, problem (3.3) for ψ. Assume now that ϕ satisfies the following decay property

$$\big|\nabla\phi(y)\big| + \big|\phi(y)\big| \le e^{-\gamma/\varepsilon} \;\text{ if }\; |x| > \delta/\varepsilon, \tag{3.8}$$

for certain constant $\gamma > 0$. The solvability can be done in the following way. Let us observe that $1 - W^2$ is exponentially small for $|x| > \delta/\varepsilon$, where x is the normal coordinate to Γ_ε. From the expression of W and the fact that $|a(\varepsilon y)| < 1$, we get

$$\min_{y\in\bar{\Omega}_\varepsilon} 2\big[\,1 - a(\varepsilon y)W\,\big] > 0.$$

Then the problem

$$\triangle_y \psi - 2(1-aW)\psi + 3(1-\eta^{\varepsilon}_{\delta})(1-W^2)\psi = h \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

has a unique bounded solution ψ whenever $||h||_\infty \leq +\infty$. Moreover,

$$||\psi||_\infty \leq C||h||_\infty.$$

Since $\widetilde{N}$ is power-like with power greater than one, a direct application of contraction mapping principle yields that (3.3) has a unique (small) solution $\psi = \psi(\phi)$ with

$$||\psi(\phi)||_{L^\infty} \leq C\varepsilon\big[\ ||\phi||_{L^\infty(|x|>\delta/\varepsilon)} + ||\nabla\phi||_{L^\infty(|x|>\delta/\varepsilon)} + e^{-\delta/\varepsilon}\ \big], \tag{3.9}$$

where $|x| > \delta/\varepsilon$ denotes the complement in Ω_ε of δ/ε-neighborhood of Γ_ε. Moreover, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$||\psi(\phi_1) - \psi(\phi_2)||_{L^\infty} \leq C\varepsilon\big[\ ||\phi_1 - \phi_2||_{L^\infty(|x|>\delta/\varepsilon)} + ||\nabla\phi_1 - \nabla\phi_2||_{L^\infty(|x|>\delta/\varepsilon)}\ \big]. \tag{3.10}$$

Therefore, from the above discussion, the full problem has been reduced to solving the following (nonlocal) problem in the infinite strip $\mathfrak{S}$

$$\mathcal{L}(\phi) = \eta^{\varepsilon}_{\delta}\Big[\,\widetilde{N}\big(\eta^{\varepsilon}_{3\delta}\phi + \psi(\phi)\big) - \widetilde{E} - 3(1-W^2)\psi(\phi)\,\Big], \tag{3.11}$$

for $\phi \in H^2(\mathfrak{S})$ satisfying condition (3.8). Here $\mathcal{L}$ denotes a linear operator that coincides with $\widetilde{\mathcal{L}}$ on the region $|x| < 8\delta/\varepsilon$. The definitions of this operator can be showed as follows. The operator $\widetilde{\mathcal{L}}$ for $|x| < 8\delta/\varepsilon$ is given in coordinate (y_1, y_2) by the formula

$$\triangle_y\phi\, +\, \Big(1 - 3(u_1)^2\Big)\phi\, +\, 2a(\varepsilon y)u_1\,\phi. \tag{3.12}$$

We extend it for functions ϕ defined in the strip $\mathfrak{S}$ in terms of (x, z) as the following

$$\mathcal{L}(\phi)\, =\, \phi_{xx}\, +\, \phi_{zz}\, +\, \Big(1 - 3(u_1)^2\Big)\phi\, +\, \eta^{\varepsilon}_{10\delta}B_8(\phi), \tag{3.13}$$

where B_8 is the operator defined in (2.30), in which all derivatives and variables are expressed in the coordinate (x, z).

Rather than solving problem (3.11), we deal with the following projected problem: given $\mathbf{f} = (f_1, \ldots, f_N)$ satisfying (2.22)-(2.24), finding functions $\phi \in H^2(\mathfrak{S})$, $\mathbf{c} = (c_1, \ldots, c_N)$ with $c_j \in L^2(0, \ell/\varepsilon)$ such that

$$\mathcal{L}(\phi) = N_2(\phi)\, -\, E_2\, +\, \sum_{j=1}^{N} c_j(z)\chi_j(x,z)\, H_{j,x} \quad \text{in } \mathfrak{S}, \tag{3.14}$$

$$\phi(x,0) = \phi(x,\ell/\varepsilon), \quad \phi_z(x,0) = \phi_z(x,\ell/\varepsilon), \quad x \in \mathbb{R}, \tag{3.15}$$

$$\int_{\mathbb{R}} \phi(x,z)\chi_j(x,z)H_{j,x}\,\mathrm{d}x = 0, \quad 0 < z < \ell/\varepsilon,\ j = 1, \ldots, N, \tag{3.16}$$

where $N_2(\phi) = \eta_\delta^\varepsilon \widetilde{N}\big(\phi + \psi(\phi)\big) - 3\eta_\delta^\varepsilon(1 - W^2)\psi(\phi),\ E_2 = \eta_\delta^\varepsilon \widetilde{E}$. The smooth cut-off functions are defined by

$$\chi_j(x,z) = \eta_a^b\Big(\frac{x - f_j(\varepsilon z)}{\log\frac{1}{\varepsilon}}\Big), \text{ where } a = \sqrt{2}\frac{2^6-1}{2^7},\ b = \sqrt{2}\frac{2^7-1}{2^8},\ \eta_a^b(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > b, \end{cases} \tag{3.17}$$

We notice that with this choice $\chi_j\chi_i \equiv 0$, for $i \neq j$, provided that ε is taken sufficiently small.

In Proposition 5.1, we will prove that this problem has a unique solution ϕ whose norm is controlled by the L^2-norm of E_{12}. Moreover, ϕ will satisfies (3.8). After this has been done, our task is to choose suitable parameters f_j's, possessing all properties in (2.22)-(2.24), such that the function $\mathbf{c}$ is identically zero. It is equivalent to solving a nonlocal, nonlinear second order differential equation for the unknown $\mathbf{f}$ under periodic boundary conditions, which will be shown in section 7.

4 Linear Theory

This section will be devoted to the resolution of the basic linear theory for the operator $\mathcal{L}$ defined in (3.13). Given functions $h \in L^2(\mathfrak{S})$, we consider the problem of finding $\phi \in H^2(\mathfrak{S})$ such that for certain functions $c_j \in L^2(0,1)$, $j = 1, \dots, N$, we have

$$\mathcal{L}(\phi) = h + \sum_{j=1}^{N} c_j(z)\,\chi_j(x,z) H_{j,x} \quad \text{in } \mathfrak{S}, \tag{4.18}$$

$$\phi(x,0) = \phi(x,\ell/\varepsilon), \quad \phi_z(x,0) = \phi_z(x,\ell/\varepsilon), \quad x \in \mathbb{R}, \tag{4.19}$$

$$\int_{-\infty}^{\infty} \phi(x,z)\, H_{j,x}(x,z)\,\chi_j(x,z)\mathrm{d}x = 0, \quad 0 < z < \ell/\varepsilon, \quad j = 1, \dots, N. \tag{4.20}$$

Our main result in this section is the following.

Proposition 4.1. *There exists a constant $C > 0$, independent of ε and uniform for the parameters $\mathbf{f}$ in (2.22)-(2.24) such that for all small ε problem (4.18)-(4.20) has a solution $(\mathbf{c}, \phi) = T_{\mathbf{f}}(h)$, which defines a linear operator of its arguments and satisfies the estimate*

$$\|\phi\|_{H^2(\mathfrak{S})} \leq C\,\|h\|_{L^2(\mathfrak{S})}.$$

□

For the proof of Proposition 4.1, we need the validity of the existence result for a simpler problem. Let us define the linear operator

$$L_0(\phi) = \phi_{xx} + \phi_{zz} + (1 - 3H^2)\phi,$$

and consider the problem: given $h \in L^2(\mathfrak{S})$, finding functions $\phi \in H^2(\mathfrak{S})$ and $c \in L^2(0, \ell/\varepsilon)$ to

$$L_0(\phi) \;=\; h \,+\, c(z)\chi(x)\, H_x \quad \text{in } \mathfrak{S}, \tag{4.21}$$

$$\phi(x,0) \;=\; \phi(x,\ell/\varepsilon), \quad \phi_z(x,0) \;=\; \phi_z(x,\ell/\varepsilon), \quad x \in \mathbb{R}, \tag{4.22}$$

$$\int_{\mathbb{R}} \phi(x,z)\, H_x(x)\chi(x)\,\mathrm{d}x \;=\; 0, \quad 0 < z < \frac{\ell}{\varepsilon}, \tag{4.23}$$

where $\chi(x) = \eta_a^b(x)$ and η_a^b is the function in (3.17).

Lemma 4.2. *Problem (4.21)-(4.23) possesses a unique solution, denoted by $(c, \phi) = T_0(h)$. Moreover, we have*

$$||c\, H_x||_{L^2(\mathfrak{S})} \;\;\le\;\; C\,\|h\|_{L^2(\mathfrak{S})},$$

$$\|\phi\|_{H^2(\mathfrak{S})} \;\;\le\;\; C\,\|h\|_{L^2(\mathfrak{S})}.$$

Proof. We will first prove an apriori estimate for (4.21)-(4.23). To this end let ϕ be a solution of (4.21)-(4.23). We observe that for the purpose of the a priori estimate we can assume that $c \equiv 0$ in (4.21). Let us consider Fourier series decompositions for h and ϕ of the form

$$\phi(x,z) = \sum_{k=0}^{\infty} \phi_k(x) e^{ik\varepsilon z}, \quad h(x,z) = \sum_{k=0}^{\infty} h_k(x) e^{ik\varepsilon z}.$$

Then we have the validity of the equations

$$-\, k^2\varepsilon^2\phi_k + \mathcal{L}_0(\phi_k) = h_k, \quad x \in \mathbb{R}, \tag{4.24}$$

and conditions

$$\int_{\mathbb{R}} \phi_k\, H_x \chi(x)\,\mathrm{d}x = 0, \tag{4.25}$$

for all k. We have denoted here

$$\mathcal{L}_0(\phi_k) = \phi_{k,xx} + F'(H(x))\phi_k.$$

Let us consider the bilinear form in $H^1(\mathbb{R})$ associated to the operator $\mathcal{L}_0$, namely

$$B(\psi,\psi) = \int_{\mathbb{R}} \left[|\psi_x|^2 - F'(H)|\psi|^2 \right] dx \ .$$

Since (4.25) holds uniformly in k we conclude that

$$C[\|\phi_k\|^2_{L^2(\mathbb{R})} + \|\phi_{k,x}\|^2_{L^2(\mathbb{R})}] \;\le\; B(\phi_k, \phi_k) \tag{4.26}$$

for a constant $C > 0$ independent of k. Using this fact and equation (4.24) we find the estimate

$$(1 + k^4\varepsilon^4)\|\phi_k\|^2_{L^2(\mathbb{R})} + \|\phi_{k,x}\|^2_{L^2(\mathbb{R})} \;\le\; C\|h_k\|^2_{L^2(\mathbb{R})}.$$

In particular, we see from (4.24) that ϕ_k satisfies an equation of the form

$$\phi_{k,xx} - 2\phi_k = \tilde{h}_k, \quad x \in \mathbb{R}.$$

where $\|\tilde{h}_k\|_{L^2(\mathbb{R})} \le C\|h_k\|_{L^2(\mathbb{R})}$. Hence it follows that additionally we have the estimate

$$\|\phi_{k,xx}\|^2_{L^2(\mathbb{R})} \le C\|h_k\|^2_{L^2(\mathbb{R})}. \tag{4.27}$$

Adding up estimates (4.26), (4.27) in k we conclude that

$$\|D^2\phi\|^2_{L^2(\mathfrak{S})} + \|D\phi\|^2_{L^2(\mathfrak{S})} + \|\phi\|^2_{L^2(\mathfrak{S})} \le C\|h\|^2_{L^2(\mathfrak{S})}, \tag{4.28}$$

which ends the proof in the case $c \equiv 0$. To prove the general case we multiply equation (4.21) by $H_x\chi(x)$ and use (4.23). This yields:

$$\begin{aligned} c(z)\int_{\mathbb{R}} H_x^2\chi^2(x)\,\mathrm{d}x &= \int_{\mathbb{R}} \mathcal{L}_0(\phi)H_x\chi\,\mathrm{d}x - \int_{\mathbb{R}} hH_x\chi\,\mathrm{d}x \\ &= \int_{\mathbb{R}} (H_x\chi_{xx} + 2H_{xx}\chi_x)\phi\,\mathrm{d}x - \int_{\mathbb{R}} hH_x\chi\,\mathrm{d}x, \end{aligned}$$

hence

$$\|cH_x\|_{L^2(\mathfrak{S})} \le C\varepsilon^{\mu}\|\phi\|_{L^2(\mathfrak{S})} + C\|h\|_{L^2(\mathfrak{S})}, \tag{4.29}$$

where $\mu \in (0,1)$. Taking ε sufficiently small and using (4.28) we get the required a priori estimates in the general case.

The existence part of the Lemma follows from standard Fredholm alternative argument. The proof is completed. □

In order to apply the previous result to the resolution of the full problem (4.18)-(4.20), we define first the operator (c.f. (3.13)) for a fixed number j

$$\mathcal{L}^j(\phi) = \mathcal{L}(\phi) = \phi_{xx} + \phi_{zz} + \Big(1 - 3(H_j)^2\Big)\phi + \eta^{\varepsilon}_{6\delta}B_8(\phi),$$

and consider the following problem

$$\mathcal{L}^j(\phi) = h + c_j(z)\chi_j(x,z)\,H_{j,x} \quad \text{in } \mathfrak{S}, \tag{4.30}$$

$$\phi(x,0) = \phi(x,\ell/\varepsilon), \quad \phi_z(x,0) = \phi_z(x,\ell/\varepsilon), \quad x \in \mathbb{R}, \tag{4.31}$$

$$\int_{\mathbb{R}} \phi(x,z)\chi_j(x,z)\,H_{j,x}\,\mathrm{d}x = 0, \quad 0 < z < \frac{\ell}{\varepsilon}. \tag{4.32}$$

We have

Lemma 4.3. *Problem (4.30)-(4.32) possesses a unique solution (c_j,ϕ). Moreover,*

$$\begin{aligned} ||c_j\,H_{j,x}||_{L^2(\mathfrak{S})} &\le C\,\|h\|_{L^2(\mathfrak{S})}, \\ \|\phi\|_{H^2(\mathfrak{S})} &\le C\,\|h\|_{L^2(\mathfrak{S})}. \end{aligned}$$

Proof. We recall that $H_j = (-1)^{j+1} H\big(x - e_j(\varepsilon z)\big)$ defined in $\mathfrak{S}$ and denote below

$$\tilde\xi(x,z) \,=\, \xi\big(x + e_j(\varepsilon z), z\big).$$

Direct computation gives that problem (4.30)-(4.32) is equivalent to

$$\begin{aligned}
&\tilde\phi_{xx} + \tilde\phi_{zz} + \big(1 - 3H^2\big)\tilde\phi + \tilde B_0(\tilde\phi) + \tilde B_1(\tilde\phi) \,=\, \tilde h \,+\, c_j(z)\chi(x)\, H_x \quad \text{ in } \mathfrak{S},\\
&\tilde\phi(x,0) \,=\, \tilde\phi(x,\ell/\varepsilon), \quad \tilde\phi_z(x,0) \,=\, \tilde\phi_z(x,\ell/\varepsilon), \quad x\in\mathbb{R},\\
&\int_{\mathbb{R}} \chi(x)\tilde\phi\, H_x \,\mathrm{d}x \,=\, 0, \quad 0 < z < \frac{\ell}{\varepsilon},
\end{aligned}\tag{4.33}$$

where

$$\tilde B_0(\tilde\phi) \quad = \quad \varepsilon^2 \Big(e_j'(\varepsilon z)\Big)^2 \tilde\phi_{xx} \,-\, \varepsilon^2\, e_j''(\varepsilon z)\,\tilde\phi_x \,-\, 2\varepsilon e'(\varepsilon z)\,\tilde\phi_{xz}.$$

In the above, $\tilde B_1$ is the operator transformed from $\eta^{\varepsilon}_{6\delta} B_8$ in (2.30) under the changing of variable. Direct computations will show that the linear operators $\tilde B_0$ and $\tilde B_1$ are small in the sense that

$$\|\tilde B_1(\tilde\phi) + \tilde B_0(\tilde\phi)\|_{L^2(\mathfrak{S})} \,\le\, o(1)\, \|\tilde\phi\|_{H^2(\mathfrak{S})},$$

with $o(1) \to 0$ as $\varepsilon \to 0$. This problem is then equivalent to the fixed point linear problem

$$\tilde\phi \,=\, T_0\big(\tilde h + \tilde B_0(\phi) + \tilde B_1(\tilde\phi)\big),$$

where T_0 is the linear operator defined by Lemma 4.2. From this, unique solvability of the problem and the desired estimate immediately follow. □

Proof of Proposition 4.1. We will first define some cut-off functions that will be important in the sequel. As before, let $\eta_a^b(s)$ be a smooth function with $\eta_a^b(s) = 1$ for $|s| < a$ and $= 0$ for $|s| > b$, where $0 < a < b < 1$. Recall that $e_j = (-1)^{j+1} f + f_j$. Then, with $R = \log\frac{1}{\varepsilon}$, and $\mathtt{X}_j = x - e_j(\varepsilon z)$ we set

$$\eta_j(x,z) = \eta_a^b\Big(\frac{|\mathtt{X}_j|}{R}\Big), \quad a = \sqrt{2}\,\frac{2^7 - 1}{2^8}, \quad b = \sqrt{2}\,\frac{2^8 - 1}{2^9}, \tag{4.34}$$

(c.f. (3.17)). We search for a solution of $\phi = T(h)$ to problem (4.18)-(4.20) in the form

$$\phi \,=\, \psi \,+\, \sum_{j=1}^{N} \eta_j\,\bar\phi_j. \tag{4.35}$$

From the definition of the functions η_j and χ_j, we have

$$\eta_j\chi_j = \chi_j, \quad \chi_j\nabla\eta_j = \chi_j\Delta\eta_j = 0. \tag{4.36}$$

We will denote

$$\bar\chi \,= 1 - \sum_{j=1}^{N} \eta_j.$$

It is readily checked that ϕ given by (4.35) solves problem (4.18)-(4.20) if the functions $\bar{\phi}_j, j = 1, \cdots, N$, satisfy the following linear system of equations, for $j = 1, \cdots, N$,

$$\mathcal{L}^j(\bar{\phi}_j) = \eta_j\big(h - \psi + 3u_1^2\psi\big) + \chi_j\, c_j(\varepsilon z)\, H_{j,x} + 3\eta_j\,\big(u_1^2 - H_j^2\big)\bar{\phi}_j \quad \text{in } \mathfrak{S}, \tag{4.37}$$

$$\bar{\phi}_j(x, 0) = \bar{\phi}_j(x, \ell/\varepsilon), \quad \bar{\phi}_{j,z}(x, 0) = \bar{\phi}_{j,z}(x, \ell/\varepsilon), \quad x \in \mathbb{R}, \tag{4.38}$$

$$\int_{\mathbb{R}} \big(\bar{\phi}_j + \chi_j\psi\big)\, H_{j,x}\chi_j \,\mathrm{d}x = 0, \quad 0 < z < \frac{\ell}{\varepsilon}, \tag{4.39}$$

and the function ψ satisfies

$$\psi_{xx} + \psi_{zz} + \bar{\chi}\big(1 - 3u_1^2\big)\psi = \bar{\chi}h + \sum_{j=1}^{N}(1 - \eta_j)\, c_j(\varepsilon z)\, H_{j,x} - \sum_{j=1}^{N}\big(2\nabla\eta_j \cdot \nabla\bar{\phi}_j + \bar{\phi}_j\Delta\eta_j\big), \tag{4.40}$$

$$\psi(x, 0) = \psi(x, \ell/\varepsilon), \quad \psi_z(x, 0) = \psi_z(x, \ell/\varepsilon), \quad -\infty < x < \infty. \tag{4.41}$$

In order to solve this system we will set up a fixed point argument. Observe that the orthogonality condition in (4.39) is satisfied for $\bar{\phi}_j + \chi_j\psi$ rather than $\bar{\phi}_j$, hence it is convenient to introduce new variable $\tilde{\phi}_j = \bar{\phi}_j + \chi_j\psi$. Then combining (4.37) and (4.40) we get the following system for $\tilde{\phi}_j$

$$\begin{aligned} \mathcal{L}^j(\tilde{\phi}_j) &= \eta_j\big(h - \psi + 3W^2\psi\big) + \chi_j\, c_j(\varepsilon z)\, H_{j,x} + 3\eta_j\,\big(W^2 - H_j^2\big)\bar{\phi}_j \\ &\quad + \chi_j\mathcal{L}^j(\psi) + \psi\Delta\chi_j + 2\nabla\psi\cdot\nabla_g\chi_j, \quad \text{in } \mathfrak{S}, \\ \tilde{\phi}_j(x, 0) &= \tilde{\phi}_j(x, \ell/\varepsilon), \quad \tilde{\phi}_{j,z}(x, 0) = \tilde{\phi}_{j,z}(x, \ell/\varepsilon), \quad x \in \mathbb{R}, \\ &\int_{\mathbb{R}} \tilde{\phi}_j H_{j,x}\chi_j \,\mathrm{d}x = 0, \quad 0 < z < \frac{\ell}{\varepsilon}, \end{aligned} \tag{4.42}$$

To solve (4.42) we assume that functions $\bar{\Phi}_j, j = 1, \cdots, N$, and $\tilde{\Psi}$ are given. First we replace $\bar{\phi}_j$, ψ by $\bar{\Phi}_j$, $\tilde{\Psi}$ on the right hand sides of (4.42) and solve (4.42) for each $\bar{\phi}_j$, $j = 1, \ldots, N$, using Lemma 4.3. We get the following estimates, for all $j = 1, \cdots, N$

$$\|\bar{\phi}_j\|_{H^2(\mathfrak{S})} \leq C\Big[\,\|h\|_{L^2(\mathfrak{S})} + \|\tilde{\Psi}\|_{H^2(\mathfrak{S})}\Big] + o(1)\sum_{j=1}^{N}\|\bar{\Phi}_j\|_{H^2(\mathfrak{S})}, \tag{4.43}$$

as $\varepsilon \to 0$. Given $\tilde{\Psi}$ we can now find functions $\bar{\phi}_j = \bar{\phi}_j(\tilde{\Psi})$ which solve (4.42) by a fixed point argument. Next, we can now solve (4.40) for ψ which in addition satisfies

$$\|\psi\|_{H^2(\mathfrak{S})} \leq C\|h\|_{L^2(\mathfrak{S})} + \frac{C}{R}\sum_{j=1}^{N}\|\bar{\Phi}_j(\tilde{\Psi})\|_{H^2(\mathfrak{S})}, \quad R = \log\frac{1}{\varepsilon}. \tag{4.44}$$

Combining this with (4.43), taking ε small, and applying a fixed point argument again we get finally a solution to (4.40). This ends the proof. □

5 Solving the Nonlinear Intermediate Problem

In this section we have the following theorem for the resolution of problem (3.14)-(3.16).

Proposition 5.1. *There exist numbers $D>0$, $\tau>0$ such that for all sufficiently small ε and all $\mathbf{f}$ satisfying (2.22)-(2.24) problem (3.14)-(3.16) has a unique solution $\phi=\phi(\mathbf{f})$ which satisfies*

$$\|\phi\|_{H^2(\mathfrak{S})} \,\le\, D\varepsilon^{\frac{3}{2}}|\log\varepsilon|^2,$$
$$\|\phi\|_{L^\infty(|x|>\delta/\varepsilon)}+\|\nabla\phi\|_{L^\infty(|x|>\delta/\varepsilon)} \,\le\, e^{-\tau\delta/\varepsilon}.$$

Besides ϕ is a Lipschitz function of $\mathbf{f}$, and for given $\mathbf{f}_1,\mathbf{f}_2:(0,\ell)\to\mathbb{R}^N$ such that:

$$\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,\ell)}\le\frac{|\log\varepsilon|}{2^{12}}, \tag{5.1}$$

it holds

$$\|\phi(\mathbf{f}_1)-\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\le C\varepsilon|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,\ell)}. \tag{5.2}$$

Proof. A first elementary, but crucial observation is the following: there holds $E_1=\eta_\delta^\varepsilon\widetilde{E}=E_2$; Moreover, the term

$$\eta_\delta^\varepsilon E_{11} \,=\, -\sqrt{2}\,\varepsilon\,\eta_\delta^\varepsilon a_t\sum_{j=1}^N f_j\,(-1)^{j+1}\,H_{j,x},$$

in the decomposition of E_2 can be absorbed in that term $\sum_{j=1}^N\chi_j\,c_j\,H_{j,x}$. Let $T_{\mathbf{f}}$ be the operator defined by Proposition 4.1. Given $\mathbf{f}$ in (2.22)-(2.24), the equation (3.14)-(3.16) is equivalent to the fixed point problem for $\phi(\mathbf{f})$:

$$\phi(\mathbf{f}) \,=\, T_{\mathbf{f}}\Big(\,h\,\Big) \,\equiv\, \mathcal{A}(\phi,\mathbf{f}), \tag{5.3}$$

with

$$h \,=\, -\eta_\delta^\varepsilon\Big(E_{02}(\mathbf{f})+E_{12}(\mathbf{f})\Big) \,+\, N_2\big(\phi(\mathbf{f})\big). \tag{5.4}$$

In the sequel we will not emphasize the dependence on $\mathbf{f}$ whenever it is not necessary.

We will define now the region where contraction mapping principle applies. From the estimates in Lemma 2.3, the terms E_{02} and E_{12} are of order $O(\varepsilon^{3/2}|\log\varepsilon|^2)$. Hence, we consider the following closed, bounded subset of $H^2(\mathfrak{S})$:

$$\mathcal{B}=\left\{\phi\in H^2(\mathfrak{S})\;\middle|\;\begin{array}{l}\|\phi\|_{H^2(\mathfrak{S})}\le D\varepsilon^{\frac{3}{2}}|\log\varepsilon|^2,\\[2mm] \|\phi\|_{L^\infty(|x|>\delta/\varepsilon)}+\|\nabla\phi\|_{L^\infty(|x|>\delta/\varepsilon)}\le e^{-\tau\delta/\varepsilon}\end{array}\right\},$$

and claim that there are constants $D,\tau>0$ such that the map $\mathcal{A}$ defined in (5.3) is a contraction from $\mathcal{B}$ into itself, uniform with respect to $\mathbf{f}$. Given $\tilde\phi\in\mathcal{B}$ we denote $\phi=\mathcal{A}(\tilde\phi,\mathbf{f})$ and then have the following estimates. Firstly, (3.9) and Lemma 2.3 imply that for $\tilde\phi\in\mathcal{B}$

$$\begin{aligned}\Big\|-\eta_\delta^\varepsilon\Big(E_{02}(\mathbf{f})+E_{12}(\mathbf{f})\Big)\,+\,N_2(\tilde\phi)\Big\|_{L^2(\mathfrak{S})} \;\le\;& C_0\varepsilon^{3/2}|\log\varepsilon|^2\,+\,C\|\tilde\phi\|^2_{H^2(\mathfrak{S})}\,+\,e^{-\gamma\delta/\varepsilon}\\ &+C\varepsilon^{1/4}\Big[\,\|\tilde\phi\|_{L^\infty(|x|>\delta/\varepsilon)}\,+\,\|\nabla\tilde\phi\|_{L^\infty(|x|>\delta/\varepsilon)}\,\Big],\end{aligned} \tag{5.5}$$

with some $\gamma > 0$. Secondly, the exponential decay of the basic approximate solution u_1 outside the region: $\{|x| > \delta_0/\varepsilon\}$ and the fact that $F'(W) = -1 + O(e^{-\gamma|x|})$ for some constant $\gamma > 0$ imply

$$\begin{aligned} \|\mathcal{A}(\tilde{\phi}, \mathbf{f})\|_{L^\infty(|x|>\delta/\varepsilon)} &+ \|\nabla \mathcal{A}(\tilde{\phi}, \mathbf{f})\|_{L^\infty(|x|>\delta/\varepsilon)} \\ &\leq Ce^{-\gamma\delta/\varepsilon} + C\delta\Big[\|\tilde{\phi}\|_{L^\infty(|x|>\delta/\varepsilon)} + \|\nabla\tilde{\phi}\|_{L^\infty(|x|>\delta/\varepsilon)} \Big]. \end{aligned} \tag{5.6}$$

From (5.5)–(5.6) we get that $\mathcal{A}$ indeed applies $\mathcal{B}$ into itself provided that D is chosen sufficiently large and τ sufficiently small.

Let us analyze the Lipschitz character of the nonlinear operator involved in $\mathcal{A}$ for functions in the subset $\mathcal{B}$, namely $\eta_\delta^\varepsilon \tilde{N}\big(\eta_{3\delta}^\varepsilon \phi + \psi(\phi)\big)$. For $\phi_1,\ \phi_2 \in \mathcal{B}$ we have, using (3.9) and (3.10):

$$\begin{aligned} \Big\| \eta_\delta^\varepsilon \tilde{N}\big(\eta_{3\delta}^\varepsilon \phi_1 + \psi(\phi_1)\big) - \eta_\delta^\varepsilon \tilde{N}\big(\eta_{3\delta}^\varepsilon \phi_2 + \psi(\phi_2)\big) \Big\|_{L^2(\mathfrak{S})} & \\ \leq C\Big[\varepsilon^{3/2}|\log\varepsilon|^2 + e^{-\gamma_0\delta/\varepsilon}\Big] \Bigg\{ \|\phi_1 - \phi_2\|_{L^2(\mathfrak{S})} & \\ + \varepsilon^{1/4}\|\phi_1 - \phi_2\|_{L^\infty(|x|>\delta/\varepsilon)} + \varepsilon^{1/4}\|\nabla(\phi_1 - \phi_2)\|_{L^\infty(|x|>\delta/\varepsilon)} \Bigg\}. & \end{aligned} \tag{5.7}$$

Using this one can show that $\mathcal{A}$ is a contraction map in $\mathcal{B}$ and thus show the existence of the fixed point.

We will now analyze the dependence of the solution ϕ found above as a fixed point of the mapping $T_{\mathbf{f}}$ on the parameter $\mathbf{f}$. We will denote $\phi = \phi(\mathbf{f})$ whenever convenient. We will consider periodic functions $\mathbf{f}_1, \mathbf{f}_2 : (0, \ell) \to \mathbb{R}^N$, such that (5.1) holds. A tedious but straightforward analysis of all terms involved in the differential operator and in the error yield that the operator $T_{\mathbf{f}}(\phi)$ is continuous with respect to $\mathbf{f}$. Indeed, indicating now the dependence on $\mathbf{f}$, let us make the following decomposition:

$$\mathcal{L}_{\mathbf{f}_1}(\,\phi(\mathbf{f}_1)\,) - \mathcal{L}_{\mathbf{f}_2}(\,\phi(\mathbf{f}_2)\,) \;=\; \mathcal{L}_{\mathbf{f}_1}\Big(\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)\Big) \;+\; \Big[F'(u_1(\mathbf{f}_1)) - F'(u_1(\mathbf{f}_2))\Big]\phi(\mathbf{f}_2).$$

Above, and in what follows $\mathbf{f}_n = \mathbf{f}_n(\varepsilon z)$. We will denote

$$\bar{\phi} = \phi(\mathbf{f}_1) \;-\; \phi(\mathbf{f}_2) \;+\; \sum_{j=1}^{N} \frac{H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)}{\displaystyle\int_{\mathbb{R}} H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)\,\mathrm{d}x} \int_{\mathbb{R}} \phi(\mathbf{f}_2) H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)\,\mathrm{d}x.$$

With these notation we have that $\bar{\phi}$ satisfies:

$$\begin{aligned} \mathcal{L}_{\mathbf{f}_1}\bar{\phi} &= \mathcal{A}\Big(\mathbf{f}_1, \mathbf{f}_2, \phi(\mathbf{f}_1), \phi(\mathbf{f}_2)\Big) + \sum_{j=1}^{N} \bar{c}_j H_{j,x}(\mathbf{f}_1), \\ \int_{\mathbb{R}} &\bar{\phi} H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)\,\mathrm{d}x = 0, \end{aligned} \tag{5.8}$$

where

$$\bar{c}_j = c_j(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - c_j(\mathbf{f}_2)\chi_j(\mathbf{f}_2),$$

and

$$\begin{aligned}
&\mathcal{A}\big(\mathbf{f}_1, \mathbf{f}_2, \phi(\mathbf{f}_1), \phi(\mathbf{f}_2)\big) \\
&\quad = -\tilde{\mathcal{L}}_{\mathbf{f}_1}\left\{ \sum_{j=1}^{N} \frac{H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)}{\displaystyle\int_{\mathbb{R}} H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)\,\mathrm{d}x} \int_{\mathbb{R}} \phi(\mathbf{f}_2)\Big[H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - H_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)\Big]\mathrm{d}x \right\} \qquad (5.9)\\
&\qquad + \Big[F'(W(\mathbf{f}_1)) - F'(W(\mathbf{f}_2))\Big]\phi(\mathbf{f}_2) - \sum_{j=1}^{N} c_j(\mathbf{f}_2)\chi_j(\mathbf{f}_2)\Big[H_{j,x}(\mathbf{f}_2) - H_{j,x}(\mathbf{f}_1)\Big] + h(\mathbf{f}_1) - h(\mathbf{f}_2),
\end{aligned}$$

with $h(\mathbf{f}_n)$ defined in (5.4). Using these decompositions one can estimate $\|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}$ by employing the theory developed in the previous section.

In fact, observe that by Proposition 4.1 we only need to estimate $\|\mathcal{A}\|_{L^2(\mathfrak{S})}$. For instance we have:

$$\Big\|\big[F'(W(\mathbf{f}_1)) - F'(W(\mathbf{f}_2))\big]\phi(\mathbf{f}_2)\Big\|_{L^2(\mathfrak{S})} \le C\varepsilon^{-1/2}\|\mathbf{f}_1 - \mathbf{f}_2\|_{L^2(\mathfrak{S})}\|\phi(\mathbf{f}_2)\|_{L^2(\mathfrak{S})}. \qquad (5.10)$$

To estimate the L^2 norm of the first term in (5.9) we fix a j and denote:

$$\mathtt{h} = \frac{H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)}{\displaystyle\int_{\mathbb{R}} H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1)\,\mathrm{d}x}, \quad \mathtt{g} = \int_{\mathbb{R}} \phi(\mathbf{f}_2)\Big[H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - H_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)\Big]\mathrm{d}x.$$

Then,

$$\begin{aligned}
\|\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{hg})\|_{L^2(\mathfrak{S})} \le\ & C\sup_{z\in(0,\ell/\varepsilon)}|\mathtt{g}| \cdot \|\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{h})\|_{H^2(\mathfrak{S})} + C\sup_{z\in(0,\ell/\varepsilon)}|\mathtt{g}_z| \cdot \|\nabla \mathtt{h}\|_{H^2(\mathfrak{S})} \\
& + C\|\mathtt{h}\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{g})\|_{L^2(\mathfrak{S})}. \qquad (5.11)
\end{aligned}$$

We have:

$$\begin{aligned}
|\mathtt{g}| &= \int_{|x|\le C|\log\varepsilon|} |\phi(\mathbf{f}_2)||H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - H_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)|\mathrm{d}x \\
&\le C\varepsilon^{-1/2}|\log\varepsilon|^{1/2}\|\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\|\mathbf{f}_1 - \mathbf{f}_2\|_{H^2(0,\ell)}.
\end{aligned}$$

Using the fact that $H_{xxx} + F'(H)H_x = 0$ and that $\operatorname{supp}\chi_j'(\mathbf{f}_1) \subset \{|x - \mathbf{f}_{1j}| > \frac{\sqrt{2}}{4}|\log|\varepsilon|\}$ we can estimate:

$$\|\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{h})\|_{H^2(\mathfrak{S})} \le C.$$

Furthermore,

$$\begin{aligned}
|\mathtt{g}_{zz}|^2 \le\ & C|\log\varepsilon|\int_{|x|\le C|\log\varepsilon|} |\phi_{zz}(\mathbf{f}_2)|^2|H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - H_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)|^2\mathrm{d}x \\
& + C|\log\varepsilon|\int_{|x|\le C|\log\varepsilon|} |\phi_z(\mathbf{f}_2)|^2|[H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - H_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)]_z|^2\mathrm{d}x \\
& + C|\log\varepsilon|\int_{|x|\le C|\log\varepsilon|} |\phi(\mathbf{f}_2)|^2|[H_{j,x}(\mathbf{f}_1)\chi_j(\mathbf{f}_1) - H_{j,x}(\mathbf{f}_2)\chi_j(\mathbf{f}_2)]_{zz}|^2\mathrm{d}x \\
\le\ & C|\log\varepsilon|(I + II + III).
\end{aligned}$$

We have:

$$
\begin{aligned}
I+II \;&\le\; C\|\mathbf{f}_1-\mathbf{f}_2\|^2_{H^2(0,\ell)}\int_{|x|\le C|\log\varepsilon|}(|\phi_{zz}(\mathbf{f}_2)|^2+|\phi_z(\mathbf{f}_2)|^2)\mathrm{d}x,\\
III \;&\le\; C(|\mathbf{f}_{1,zz}-\mathbf{f}_{2,zz}|^2+|\mathbf{f}_{1,z}-\mathbf{f}_{2,z}|^2)\sup_{z\in(0,\ell/\varepsilon)}\int_{|x|\le C|\log\varepsilon|}|\phi(\mathbf{f}_2)|^2\mathrm{d}x\\
&\quad+C(|\mathbf{f}_{2,zz}|^2+|\mathbf{f}_{2,z}|^2)\|\mathbf{f}_1-\mathbf{f}_2\|^2_{H^2(0,\ell)}\sup_{z\in(0,\ell/\varepsilon)}\int_{|x|\le C|\log\varepsilon|}|\phi(\mathbf{f}_2)|^2\mathrm{d}x\\
&\le\; C(|\mathbf{f}_{1,zz}-\mathbf{f}_{2,zz}|^2+(|\mathbf{f}_{1,z}-\mathbf{f}_{2,z}|^2)\|\phi(\mathbf{f}_2)\|^2_{H^2(\mathfrak{S})}\\
&\quad+C(|\mathbf{f}_{2,zz}|^2+|\mathbf{f}_{2,z}|^2)\|\mathbf{f}_1-\mathbf{f}_2\|^2_{H^2(0,\ell)}\|\phi(\mathbf{f}_2)\|^2_{H^2(\mathfrak{S})}.
\end{aligned}
$$

Similar estimate, but depending only on the first derivatives in z of $\mathbf{f}_n$, $\phi(\mathbf{f}_2)$, holds for $\mathbf{g}_z$. Using these estimates we conclude from (5.11) that

$$
\|\tilde{\mathcal{L}}_{\mathbf{f}_1}(\mathtt{hg})\|_{L^2(\mathfrak{S})}\;\le\; C\varepsilon^{-1/2}|\log\varepsilon|^q\|\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,\ell)}. \tag{5.12}
$$

We will now estimate:

$$
\begin{aligned}
\|h(\mathbf{f}_1)-h(\mathbf{f}_2)\|_{L^2(\mathfrak{S})}\;&\le\;\|\eta^\varepsilon_\delta(\widetilde{E}(\mathbf{f}_1)-\widetilde{E}(\mathbf{f}_2))\|_{L^2(\mathfrak{S})}\\
&\quad+\|\eta^\varepsilon_\delta[\widetilde{N}(\eta^\varepsilon_{3\delta}\phi(\mathbf{f}_1)+\psi(\mathbf{f}_1))-\widetilde{N}(\eta^\varepsilon_{3\delta}\phi(\mathbf{f}_2)+\psi(\mathbf{f}_2))\|_{L^2(\mathfrak{S})}\\
&\quad+3\|\eta^\varepsilon_\delta[(1-W^2(\mathbf{f}_1))\psi(\mathbf{f}_1)-(1-W^2(\mathbf{f}_2))\psi(\mathbf{f}_2)]\|_{L^2(\mathfrak{S})}.
\end{aligned}\tag{5.13}
$$

Using the equation satisfied by $\psi(\mathbf{f}_n)$, $n=1,2$ we find:

$$
\begin{aligned}
\|\eta^\varepsilon_\delta(\widetilde{E}(\mathbf{f}_1)-\widetilde{E}(\mathbf{f}_2))\|_{L^2(\mathfrak{S})}\;&\le\; C\varepsilon^{3/2}|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(\mathfrak{S})},\\
\|\psi(\mathbf{f}_1)-\psi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\;&\le\; C\varepsilon\|\phi(\mathbf{f}_1)-\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}+C\varepsilon^{3/2}|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(\mathfrak{S})}.
\end{aligned}
$$

Then we get that:

$$
\|h(\mathbf{f}_1)-h(\mathbf{f}_2)\|_{L^2(\mathfrak{S})}\;\le\; C\varepsilon\|\phi(\mathbf{f}_1)-\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}+C\varepsilon^{3/2}|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(\mathfrak{S})}. \tag{5.14}
$$

Term involving $c_j(\mathbf{f}_2)$ in (5.9) can be estimated in a similar way. In summary we obtain:

$$
\begin{aligned}
\|\bar\phi\|_{H^2(\mathfrak{S})}\;&\le\; C\varepsilon|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,\ell)}+C\varepsilon\|\phi(\mathbf{f}_1)-\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\\
&\le\; C\varepsilon|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,\ell)}+C\varepsilon\|\bar\phi\|_{H^2(\mathfrak{S})}+C\varepsilon\|\bar\phi-\phi(\mathbf{f}_1)+\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}.
\end{aligned}\tag{5.15}
$$

Since

$$
\|\bar\phi-\phi(\mathbf{f}_1)+\phi(\mathbf{f}_2)\|_{H^2(\mathfrak{S})}\;\le\; C\varepsilon|\log\varepsilon|^q\|\mathbf{f}_1-\mathbf{f}_2\|_{H^2(0,\ell)},
$$

estimate (5.2) follows from (5.15). This ends the proof. □

Clearly Proposition 5.1 and the gluing procedure yield a solution to our original problem (1.3) if we can find $\mathbf{f}$ in (2.22)-(2.24) such that

$$
\mathbf{c}(\mathbf{f})\;=\;0. \tag{5.16}
$$

As we will see this leads to a system of N nonlinear ODE's. We carry out this argument and solve the nonlinear system in the next two sections.

6 The projection on $H_{n,x}$ and the Toda System

It is easy to see that the identities (5.16) is equivalent to the following equations

$$\int_{\infty}^{\infty} \big[\, \mathcal{L}(\phi) \,+\, \eta_\delta^\varepsilon \tilde{E} \,-\, N_2(\phi) \,\big]\, H_{n,x}\, \mathrm{d}x \,=\, 0, \quad n = 1, ..., N. \tag{6.1}$$

Hence, it is crucial to get estimates of all terms in (6.1), which will be carried out in the sequel.

Make notations

$$\mathcal{S} \,=\, \big\{ x \in \mathbb{R} :\, (x,z) \in \mathfrak{S} \big\}, \quad \mathcal{S}_n \,=\, \big\{ x \in \mathbb{R} :\, (x,z) \in A_n \big\},$$

and consider for each n, $(n = 1, \cdots, N)$, the following integrals

$$\begin{aligned} \int_{\mathcal{S}} \eta_\delta^\varepsilon\, \tilde{E}(x,z)\, H'(x - e_n(\varepsilon z))\, \mathrm{d}x \;&=\; \left\{ \int_{\mathcal{S}_n} + \int_{\mathcal{S}\setminus\mathcal{S}_n} \right\} \eta_\delta^\varepsilon\, \tilde{E}(x,z)\, H'(x - e_n(\varepsilon z))\, \mathrm{d}x \\ &\equiv\; \mathcal{E}_{n1}(\varepsilon z) \,+\, \mathcal{E}_{n2}(\varepsilon z). \end{aligned}$$

Note that in A_n, $\eta_\delta^\varepsilon \tilde{E}(x,z) = E_1(x,z)$, we begin with

$$\begin{aligned} \mathcal{E}_{n1}(\varepsilon z) \;&=\; \int_{\mathcal{S}_n} \Big[E_{02} + (E_{11} + E_{12}) \Big]\, H'(x - e_n)\, \mathrm{d}x \\ &\equiv\; I_0 + I_1. \end{aligned}$$

From the computations of section 6 in [16], we get the estimates for some terms in I_0 as follows

$$\begin{aligned} I_{01} \;&\equiv\; \int_{\mathcal{S}_n} \Big[-\varepsilon^2 \sum_{j=1}^{N} e_j^{''} H_{j,x} \,+\, \varepsilon^2 \sum_{j=1}^{N} (e_j^{'})^2 H_{j,xx} \,+\, 3\big(1 - H_n^2\big)(u_0 - H_n) \Big]\, H'(x - e_n)\, \mathrm{d}x \\ &\qquad + \int_{\mathcal{S}_n} \Big[\frac{1}{2} F^{''}(H_n)(u_0 - H_n)^2 \,-\, \frac{1}{2} F^{''}(\sigma_{nj})(\sigma_{nj} - H_j)^2 \Big]\, H'(x - e_n)\, \mathrm{d}x \\ &=\; (-1)^n \Big[\varepsilon^2 \gamma_0\, e_n^{''} \,-\, \gamma_1\, e^{-\sqrt{2}\,(e_n - e_{n-1})} \,+\, \gamma_1\, e^{-\sqrt{2}\,(e_{n+1} - e_n)} \Big] \\ &\qquad + O(\varepsilon^3) \sum_{j=1}^{N} \big[\, e^{'} + (e_j^{\prime})^2 + e^{''} \,\big] \,+\, \varepsilon^{1/2} \max_{j \neq n} O(e^{-\sqrt{2}\,|e_j - e_n|}), \end{aligned}$$

where we define

$$\gamma_0 = \int_{\mathbb{R}} H_x^2\, \mathrm{d}x = \frac{2\sqrt{2}}{3}, \quad \gamma_1 = 3A_0 \int_{\mathbb{R}} (1 - H^2) H_x\, e^{-\sqrt{2}\,x}\, \mathrm{d}x.$$

Obviously, we obtain

$$
\begin{aligned}
I_{02} \;\equiv\;& -\frac{\sqrt{2}}{2}\,\varepsilon^2\, a_{tt} \int_{\mathcal{S}_n} \sum_{j=1}^{N} e_j^2\, H^{'}(x-e_j)\, H'(x-e_n)\,\mathrm{d}x \;-\; \varepsilon^2 k^2 \int_{\mathcal{S}_n} \sum_{j=1}^{N} e_j\, H_{j,x}\, H'(x-e_n)\,\mathrm{d}x \\
& -\frac{\sqrt{2}}{2}\,\varepsilon^2\, a_{tt} \int_{\mathcal{S}_n} \sum_{j=1}^{N} (x-e_j)^2\, H^{'}(x-e_j)\, H'(x-e_n)\,\mathrm{d}x \\
=\;& -\frac{2}{3}\,\varepsilon^2\, a_{tt}(0,\varepsilon z)\, e_n^2 \;+\; (-1)^n \varepsilon^2\, \gamma_0\, k^2(\varepsilon z)\, e_n \;+\; \varepsilon^2\, a_{tt}(0,\varepsilon z)\, \gamma_2 \;+\; \varepsilon^{1/2} \max_{j\neq n} O(e^{-\sqrt{2}\,|e_j-e_n|}),
\end{aligned}
$$

where

$$
\gamma_2 = -\frac{\sqrt{2}}{2} \int_{\mathbb{R}} x^2\, H_x^2\,\mathrm{d}x.
$$

Using the facts that the function $H^{'}(x)$ is even, the conditions in (2.15)-(2.17) and the asymptotic expansion of H in (2.11), one carries out the estimates for other terms in I_0 and gets

$$
\begin{aligned}
I_{03} \;\equiv\;& I_0 - I_{01} - I_{02} \\
=\;& -\varepsilon^2\, a_{tt}\, e_j \int_{\mathcal{S}_n} \sum_{j=1}^{N} (x-e_j)(1-H_j^2)\, H'(x-e_n)\,\mathrm{d}x \\
& + \varepsilon a_t \int_{\mathcal{S}_n} x \sum_{j\neq n} \Big(1 \,-\, \big(H_j\big)^2\Big)\, H'(x-e_n)\,\mathrm{d}x \;+\; \varepsilon a_t \int_{\mathcal{S}_n} x \Big[\, \big(u_0\big)^2 - H_n^2 \,\Big]\, H'(x-e_n)\,\mathrm{d}x \\
& - \frac{1}{2}\,\varepsilon^2\, a_{tt} \int_{\mathcal{S}_n} \sum_{j\neq n} x^2 (1-H_j^2)\, H'(x-e_n)\,\mathrm{d}x \\
& + \frac{1}{2}\,\varepsilon^2 a_{tt} \int_{\mathcal{S}_n} x^2 \Big[\, \big(u_0\big)^2 - H_n^2 \,\Big]\, H'(x-e_n)\,\mathrm{d}x - \varepsilon^2 k^2 \int_{\mathcal{S}_n} \sum_{j=1}^{N} (x-e_j)\, H_{j,x}\, H'(x-e_n)\,\mathrm{d}x \\
=\;& O(\varepsilon^3) \sum_{j=1}^{N} \big[\, e^{'} + (e_j')^2 + e^{''} \,\big] \;+\; \varepsilon^{1/2} \max_{j\neq n} O(e^{-\sqrt{2}\,|e_j-e_n|}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_0 \;=\;& (-1)^n \Big[\, \varepsilon^2\, \gamma_0\, e_n^{''} \;-\; \gamma_1\, e^{-\sqrt{2}\,(e_n-e_{n-1})} \;+\; \gamma_1\, e^{-\sqrt{2}\,(e_{n+1}-e_n)} \;+\; \varepsilon^2\, \gamma_0\, k^2(\varepsilon z)\, e_n \,\Big] \\
& -\frac{2}{3}\,\varepsilon^2\, a_{tt}(0,\varepsilon z)\, e_n^2 \;+\; \varepsilon^2\, a_{tt}(0,\varepsilon z)\, \gamma_2 \;+\; O(\varepsilon^3) \sum_{j=1}^{N} \big[\, e^{'} + (e_j')^2 + e^{''} \,\big] \\
& + \varepsilon^{1/2} \max_{j\neq n} O(e^{-\sqrt{2}\,|e_j-e_n|}).
\end{aligned}
$$

Similarly, we obtain the estimates for the components in I_1

$$
\begin{aligned}
I_{11} \quad \equiv \quad & -\sqrt{2}\,\varepsilon\, a_t \int_{\mathcal{S}_n} \sum_{j=1}^{N} H'(x-e_j)\, f_j\, H'(x-e_n)\,\mathrm{d}x \\
& + \sqrt{2}\,\varepsilon^2 k a_t \int_{\mathcal{S}_n} \sum_{j=1}^{N} (-1)^{j+1}\, \Psi_x(x-e_j)\, H'(x-e_n)\,\mathrm{d}x \\
& + 2\sqrt{2}\,\varepsilon^2 a_t^2 \int_{\mathcal{S}_n} \sum_{j=1}^{N} e_j\, H(x-e_j)\Psi(x-e_j)\, H'(x-e_n)\,\mathrm{d}x \\
= \quad & -\frac{4}{3}\,\varepsilon\, a_t(0,\varepsilon z) f_n \,+\, \varepsilon^2\, a_t^2(0,\varepsilon z)\,\gamma_3\, e_n \,+\, (-1)^{n+1}\varepsilon^2\, k(\varepsilon z)\, a_t(0,\varepsilon z)\,\gamma_4 \,+\, O(\varepsilon^2) \sum_{j=1}^{N} f_j,
\end{aligned}
$$

where

$$
\gamma_3 = 2\sqrt{2} \int_{\mathbb{R}} H\,\Psi\, H^{'}\,\mathrm{d}x, \quad \gamma_4 = \sqrt{2} \int_{\mathbb{R}} \Psi_x\, H^{'}\,\mathrm{d}x.
$$

Using the facts that the functions $H^{'}(x)$ and $\phi^*(x,z)$ are even, we can derive

$$
\begin{aligned}
I_{12} \quad \equiv \quad & -3\,\varepsilon \int_{\mathcal{S}_n} \sum_{j=1}^{N} \Big[\big(u_0\big)^2 - H_j^2 \Big]\, \phi_j^*\, H'(x-e_n)\,\mathrm{d}x \\
& + 2\sqrt{2}\,\varepsilon^2\, a_t^2 \int_{\mathcal{S}_n} \sum_{j=1}^{N} (x-e_j)\, H(x-e_j)\, \Psi(x-e_j)\, H'(x-e_n)\,\mathrm{d}x \\
& + 2\sqrt{2}\,\varepsilon^2 a_t \int_{\mathcal{S}_n} \sum_{j=1}^{N} x(u_0-H_j)\,\phi_j^*\, H'(x-e_n)\,\mathrm{d}x \\
& - \int_{\mathcal{S}_n} \Big[3\,u_0\, \big(\phi^*\big)^2 \,+\, \big(\phi^*\big)^3 \Big]\, H'(x-e_n)\,\mathrm{d}x \\
= \quad & O(\varepsilon^3) a_t(0,\varepsilon z) \,+\, O(\varepsilon^2) \sum_{j=1}^{N} f_j,
\end{aligned}
$$

Thus, we finish the estimate of I_1.

To compute $\mathcal{E}_{n2}(\varepsilon z)$, recall that for $(x,z)\in S_{\delta/\varepsilon}\setminus A_n$,

$$
H'(x-e_n) = \max_{j\neq n} O(e^{-\frac{1}{2}|e_j-e_n|}),
$$

and that $\eta_\delta^\varepsilon \tilde{E}(x,z) = \eta_\delta^\varepsilon E_1(x,z)$. Thus we can estimate

$$
\mathcal{E}_{n2}(\varepsilon z) = \varepsilon \max_{j\neq n} O(e^{-|e_j-e_n|}) + O(\varepsilon^{1/2}) \sum_{i=1}^{2} I_i.
$$

Gathering the above estimates, we get the following, for $n=1,\ldots,N$

$$
\begin{aligned}
\int_{\mathcal{S}} \eta_\delta^\varepsilon\, \tilde{E}(x,z)\, H'(x-e_n(\varepsilon z))\,\mathrm{d}x \quad = \quad & (-1)^n \Big[\varepsilon^2\, \gamma_0\, e_n^{''} \,-\, \gamma_1\, e^{-\sqrt{2}\,(e_n-e_{n-1})} \,+\, \gamma_1\, e^{-\sqrt{2}\,(e_{n+1}-e_n)} \Big] \\
& + (-1)^n \varepsilon^2\, \gamma_0\, k^2(\varepsilon z)\, e_n \,-\, \frac{2}{3}\,\varepsilon^2\, a_{tt}(0,\varepsilon z)\, e_n^2 \,+\, \varepsilon^2\, a_{tt}(0,\varepsilon z)\,\gamma_2 \\
& - \frac{4}{3}\,\varepsilon\, a_t(0,\varepsilon z) f_n \,+\, \varepsilon^2\, a_t^2(0,\varepsilon z)\,\gamma_3\, e_n \\
& + (-1)^{n+1}\varepsilon^2\, k(\varepsilon z) a_t(0,\varepsilon z)\gamma_4 \,+\, \mathcal{P}_n(\varepsilon z). \qquad (6.2)
\end{aligned}
$$

For further references we observe that

$$||\mathcal{P}_n||_{L^2(0,1)} \leq C\varepsilon^{3/2},\ n = 1, \cdots, N. \tag{6.3}$$

Continuing with other two terms involved in (6.1), using the quadratic nature of the nonlinear term $N_2(\phi)$ and Proposition 5.1, we get for

$$\mathcal{Q}_n(\varepsilon z) \ \equiv \int_{\mathcal{S}} N_2(\phi)\, H'(x - e_n(\varepsilon z))\, \mathrm{d}x$$

a similar estimate

$$||\mathcal{Q}_n||_{L^2(0,1)} \leq C\varepsilon^{3/2}, \quad n = 1, \cdots, N. \tag{6.4}$$

We point out that, by Proposition 5.1, $\mathcal{Q}_n$ is a continuous function of the parameters $\mathbf{f}$.

The last term in (6.1) can be written as

$$\begin{aligned}
\mathfrak{V}_n(\varepsilon z) &\equiv \int_{\mathcal{S}} \mathcal{L}(\phi)\, H'(x - e_n(\varepsilon z))\, \mathrm{d}x \\
&= \int_{\mathcal{S}} \phi_{zz}\, H'(x - e_n(\varepsilon z))\, \mathrm{d}x + \int_{\mathcal{S}} \eta^{\varepsilon}_{10\delta}\, B_8(\phi)\, H'(x - e_n(\varepsilon z))\, \mathrm{d}x \\
&\quad + \int_{\mathcal{S}} \phi\, \big[\, H'''(x - e_n(\varepsilon z)) + F^{'}(W)\, H'(x - e_n(\varepsilon z))\, \big]\, \mathrm{d}x.
\end{aligned}$$

A similar estimate holds

$$||\mathfrak{V}_n||_{L^2(0,1)} \leq C\varepsilon^{3/2}, \quad n = 1, \cdots, N. \tag{6.5}$$

In fact, the proof is very straightforward. For example, using the orthogonality conditions we can get

$$\begin{aligned}
\mathfrak{V}^1_n(\varepsilon z) &\equiv \int_{\mathcal{S}} \phi_{zz}\, H'(x - e_n(\varepsilon z))\, \mathrm{d}x \\
&= \varepsilon^2 \int_{\mathcal{S}} \phi\, \big[\, e^{''}_n H''(x - e_n(\varepsilon z)) - (e^{'}_n)^2 H'''(x - e_n(\varepsilon z))\, \big]\, \mathrm{d}x \\
&\quad + 2\varepsilon e^{'}_n \int_{\mathcal{S}} \phi_z\, H''(x - e_n(\varepsilon z))\, \mathrm{d}x
\end{aligned}$$

The estimate for $\mathfrak{V}^1_n$ follows from Proposition 5.1. Moreover, it also depends continuously on the parameters $\mathbf{f}$.

Now, from above discussion, for $\theta = \varepsilon z$, by the notations of

$$\mathcal{M}_n(\theta, \mathbf{f}, \mathbf{f}^{'}, \mathbf{f}^{''}) \equiv \mathcal{P}_n + \mathcal{Q}_n + \mathfrak{V}_n, \quad \alpha_1(\theta) = \frac{4}{3}\, a_t(0,\theta), \quad \alpha_3(\theta) = \frac{2}{3}\, a_{tt}(0,\theta), \tag{6.6}$$

$$\alpha_{2,n}(\theta) = -\,\gamma_0\, k^2(\theta) - (-1)^n a_t^2(0,\varepsilon z)\gamma_3, \quad \alpha_{4,n}(\theta) = \,-\,(-1)^n\gamma_2\, a_{tt}(0,\theta) - \gamma_4\, k(\theta)\, a_t(0,\theta), \tag{6.7}$$

we draw a conclusion as the following proposition:

Proposition 6.1. *For the validity of (5.16), there should hold the following equations,*

$$\varepsilon^2\,\gamma_0\,e_n'' - \gamma_1\,e^{-\sqrt{2}\,(e_n - e_{n-1})} + \gamma_1\,e^{-\sqrt{2}\,(e_{n+1}-e_n)} + (-1)^{n+1}\,\varepsilon\,\alpha_1(\theta)\,f_n$$
$$-\,\varepsilon^2\,\alpha_{2,n}(\theta)\,e_n - (-1)^n\,\varepsilon^2\,\alpha_3(\theta)\,e_n^2 = \varepsilon^2\,\alpha_{4,n}(\theta) + \mathcal{M}_n, \quad n = 1, \ldots, N.$$

Moreover, $\mathcal{M}_n$ can be decomposed in the following way

$$\mathcal{M}_n(\theta, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{f}^{''}) = \mathcal{M}_{n1}(\theta, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{f}^{''}) + \mathcal{M}_{n2}(\theta, \boldsymbol{f}, \boldsymbol{f}^{'}).$$

where $\mathcal{M}_{n1}$ and $\mathcal{M}_{n2}$ are continuous of their arguments. Function $\mathcal{M}_{n1}$ satisfies the following properties, for $n = 1, \cdots, N$

$$\begin{aligned} \big|\big|\mathcal{M}_{n1}(\theta, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{f}^{''})\big|\big|_{L^2(0,1)} &\leq C\varepsilon^{3/2}, \\ \big|\big|\mathcal{M}_{n1}(\theta, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{f}^{''}) - \mathcal{M}_{n1}(\theta, \boldsymbol{f}_1, \boldsymbol{f}_1^{'}, \boldsymbol{f}_1^{''})\big|\big|_{L^2(0,1)} &\leq C\varepsilon^{3/2}\,|\log\varepsilon|^q\,||\boldsymbol{f} - \boldsymbol{f}_1||_{H^2(0,1)}. \end{aligned}$$

Function $\mathcal{M}_{n2}$ satisfies the following estimates for $n = 1, \cdots, N$

$$\big|\big|\mathcal{M}_{n2}(\theta, \boldsymbol{f}, \boldsymbol{f}^{'})\big|\big|_{L^2(0,1)} \leq C\varepsilon^{3/2}.$$

We omit the proof of this proposition. In fact, careful examining of all terms will lead the decomposition of the operator $\mathcal{M}_n$ and the properties of its components $\mathcal{M}_{n1}$ and $\mathcal{M}_{n2}$. □

7 Location and interaction of clustered Layers

Recall that $N = 2m + 1$ is a fixed odd integer as in Theorem 1.2. Define the operators by

$$\begin{aligned} \mathcal{L}_n^*(f_n) \;\equiv\; & \varepsilon^2\,\gamma_0\,f_n'' + \varepsilon\,(-1)^{n+1}\,\alpha_1(\theta)\,f_n - \gamma_1\,e^{-\sqrt{2}\left(f_n - f_{n-1} + \beta_n\right)} + \gamma_1\,e^{-\sqrt{2}\left(f_{n+1} - f_n + \beta_{n+1}\right)} \\ & -\varepsilon^2\,\alpha_{5,n}(\theta)\,f_n - \varepsilon^2\,(-1)^n\,\alpha_3(\theta)\,f_n^2, \end{aligned}$$

where $\alpha_{5,n}$ and β_n are given by

$$\alpha_{5,n}(\theta) = \alpha_{2,n}(\theta) + 2\alpha_3(\theta) f(\theta), \quad \beta_n = 2(-1)^{n+1} f,$$

with f is the function defined in (2.21). α_1 is a positive function defined in (6.6). Using the fact $e_n = f_n + (-1)^{n+1} f$ and Proposition 6.1, after obvious algebra, we have to deal with the following system, with n running from 1 to N

$$\mathcal{L}_n^*(f_n) = \varepsilon^2\,\alpha_{6,n} + \mathcal{M}_n, \quad f_n'(\ell) = f_n'(0), \quad f_n(\ell) = f_n(0), \tag{7.1}$$

where $f_0 = -\infty$, $f_{N+1} = \infty$ and the function $\alpha_{6,n}$ is defined by

$$\alpha_{6,n}(\theta) = \alpha_{4,n}(\theta) - \gamma_0(-1)^{n+1} f^{''}(\theta) + (-1)^{n+1}\alpha_{2,n}(\theta) f(\theta) + (-1)^n\alpha_3(\theta) f^2(\theta).$$

7.1 Solvability of Toda System

Before solving the above system (7.1), we study a simpler problem in this subsection. Consider the following Toda system for $n = 1, \cdots, N$

$$\varepsilon^2 \gamma_0 f_n'' + \varepsilon (-1)^{n+1} \alpha_1(\theta) f_n - \gamma_1 e^{-\sqrt{2}\left(f_n - f_{n-1} + \beta_n \right)} + \gamma_1 e^{-\sqrt{2}\left(f_{n+1} - f_n + \beta_{n+1} \right)} = \varepsilon^{3/2} h_n, \quad (7.2)$$

$$f_n'(\ell) = f_n'(0), \quad f_n(\ell) = f_n(0), \quad (7.3)$$

where $f_0 = -\infty$, $f_{N+1} = \infty$. We have the following solvability theory.

Proposition 7.1. *For given $\mathbf{h} = (h_1, \cdots, h_N) \in L^2(0, \ell)$, there exists a sequence $(\varepsilon_l)_l$ approaching 0 such that problem (7.2)-(7.3) admits a solution $\mathbf{f} = (f_1, \cdots, f_N)$ with the form:*

$$f_n(\theta) = \big(n - (m-1)\big)\rho_{\varepsilon_l}(\theta) + \bar{f}_n(\theta) + \check{f}_n(\theta), \quad n = 1, \ldots, N, \quad (7.4)$$

where $\rho_{\varepsilon_l}(\theta)$ satisfies

$$e^{-\sqrt{2}\,\rho_{\varepsilon_l}} = \varepsilon_l \alpha_1(\theta) \rho_{\varepsilon_l}, \quad (7.5)$$

and in particular

$$\rho_{\varepsilon_l}(\theta) = \frac{1}{\sqrt{2}} \log \frac{1}{\varepsilon_l} - \frac{1}{\sqrt{2}} \log\log \frac{1}{\varepsilon_l} - \frac{1}{\sqrt{2}} \log \frac{\alpha_1(\theta)}{2} + O\Big(\frac{\log\log \frac{1}{\varepsilon_l}}{\log \frac{1}{\varepsilon_l}} \Big).$$

Functions $\bar{f}_1, \cdots, \bar{f}_N$, defined by Lemma 7.2, do not depend on $\mathbf{h}$ and satisfy

$$\bar{f}_n(\theta) = O(1), \ n = 1, 2, \cdots, N.$$

Finally, for functions $\check{f}_1, \cdots, \check{f}_N$, we have

$$\|\check{f}_n\|_{L^2(0,\ell)} \le C \|\mathbf{h}\|_{L^2(0,\ell)}, \ n = 1, 2, \cdots, N.$$

Moreover, if $\mathbf{h} \in H^2(0, \ell)$, then

$$\frac{\varepsilon_l}{|\log \varepsilon_l|} ||\check{f}''||_{L^2(0,\ell)} + \sqrt{\frac{\varepsilon_l}{|\log \varepsilon_l|}} ||\check{f}'||_{L^2(0,\ell)} + ||\check{f}||_{L^\infty(0,\ell)} \le C \left[\log \frac{1}{\varepsilon_l} \right]^{-1/2} ||\mathbf{h}||_{H^2(0,\ell)}.$$

Proof. We divide the proof into three steps.

Step 1: Recall that $\alpha_1(\theta) > 0$. Let us define positive functions $\rho_\varepsilon(\theta)$ and $\delta(\theta)$ by

$$e^{-\sqrt{2}\,\rho_\varepsilon} = \varepsilon \alpha_1(\theta) \rho_\varepsilon, \quad (7.6)$$

$$\delta^{-2}(\theta) = \alpha_1(\theta)\rho_\varepsilon(\theta) = \frac{\alpha_1(\theta)}{\sqrt{2}} \Big[\log \frac{1}{\varepsilon} - \log\log \frac{1}{\varepsilon} + O(1) \Big]. \quad (7.7)$$

Then multiplying equation (7.2) by δ^2 and setting

$$f_n = \big(n - (m+1)\big)\rho_\varepsilon + \hat{f}_n, \quad n = 1, \cdots, N,$$

we get an equivalent system, for $n = 1, 2, \cdots, N$

$$\delta^2\Big(\varepsilon\,\gamma_0\,\hat{f}_n^{''} \,+\, (-1)^{n+1}\,\alpha_1(\theta)\,\hat{f}_n\Big) \,-\, \gamma_1\,e^{-\sqrt{2}\,\left(\hat{f}_n-\hat{f}_{n-1}+\beta_n\right)} \,+\, \gamma_1\,e^{-\sqrt{2}\,\left(\hat{f}_{n+1}-\hat{f}_n+\beta_{n+1}\right)}$$
$$= \delta^2\,\varepsilon^{1/2}h_n \,+\, \varepsilon\,\delta^2\,\gamma_0\big(n-(m+1)\big)\rho_\varepsilon^{''} \,+\, (-1)^n\big(n-(m+1)\big), \tag{7.8}$$
$$\hat{f}_n'(\ell) \,=\, \hat{f}_n'(0), \quad \hat{f}_n(\ell) \,=\, \hat{f}_n(0), \tag{7.9}$$

where $\hat{f}_0 = -\infty$, $\hat{f}_{N+1} = \infty$.

Step 2: For the full resolution of the system (7.8)-(7.9), we now want to cancel the terms of $O(1)$ in right hand side of (7.8) at first. To this end, we give a lemma as follows

Lemma 7.2. *There exists a solution* $\bar{\mathbf{f}} = (\bar{f}_1, \cdots, \bar{f}_N)$ *to the following nonlinear algebraic system*

$$\delta^2(-1)^{n+1}\alpha_1(\theta)\bar{f}_n - \gamma_1 e^{-\sqrt{2}\left(\bar{f}_n-\bar{f}_{n-1}+\beta_n\right)} + \gamma_1 e^{-\sqrt{2}\left(\bar{f}_{n+1}-\bar{f}_n+\beta_{n+1}\right)} = (-1)^n\big(n-(m+1)\big), \tag{7.10}$$

with n running from 1 to N, where $\bar{f}_0 = -\infty$, $\bar{f}_{N+1} = \infty$.

Proof. By setting

$$a_0 \,=\, a_N \,=\, 0, \quad a_n \,=\, \gamma_1\,e^{-\sqrt{2}\,\left(f^0_{n+1}-f^0_n+\beta_{n+1}\right)}, \quad n = 1, \cdots, N-1, \tag{7.11}$$

we look for a solution of (7.10) in the form $\bar{f}_n \,=\, f_n^0 + \tilde{f}_n$, where a_n satisfy the following system of equations:

$$\mathbf{MX} \,=\, \mathbf{C} \tag{7.12}$$

where we have defined

$$\mathbf{M}_{(N-1)\times N} \,=\, \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}, \; \mathbf{X} \,=\, \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2m-1} \\ a_{2m} \end{pmatrix}, \; \mathbf{C} \,=\, \begin{pmatrix} m \\ -(m-1) \\ \vdots \\ 0 \\ \vdots \\ m-1 \\ -m \end{pmatrix}.$$

Obviously, the system (7.12) can be uniquely solved for the unknown variables a_n. In fact, we have that $a_n, n = 1, 2, \cdots, 2m$, are positive constants with expression as

$$a_n \,=\, a_{N-n} \,=\, \sum_{j=1}^{n}(-1)^{j+1}(m+1-j) > 0 \quad \text{for all } n = 1, \cdots, m. \tag{7.13}$$

By (7.11), for $n = 1, \cdots, m, m+2, \cdots, N$, functions f_n^0 can be written as

$$
\begin{aligned}
f_n^0(\theta) &= \frac{\sqrt{2}}{2} \log \prod_{i=n}^{m} a_i - \frac{\sqrt{2}}{2} (m+1-n) \log \gamma_1 \\
&\quad + (-1)^m \big[(-1)^{m+n-1} + 1\big] f(\theta) + f_{m+1}^0(\theta), \quad n = 1, 2, \cdots, m, \\
f_n^0(\theta) &= -\frac{\sqrt{2}}{2} \log \prod_{i=m+1}^{n-1} a_i - \frac{\sqrt{2}}{2} (n-m-1) \log \gamma_1 \\
&\quad + (-1)^m \big[(-1)^{m+n} + 1\big] f(\theta) + f_{m+1}^0(\theta), \quad n = m+2, m+3, \cdots, N.
\end{aligned}
$$

Hence all functions $f_n^0, n = 1, \cdots, N$, can be uniquely determined provided that we set

$$f_{m+1}^0(\theta) = 4(-1)^m f(\theta).$$

Moreover, there holds

$$\sum_{n=1}^{N} (-1)^{n+1} f_n^0 = 0. \tag{7.14}$$

Then functions $\tilde{f}_n, n = 1, 2, \cdots, N$, satisfy:

$$
\begin{aligned}
\delta^2 (-1)^{n+1} \alpha_1(\theta) \tilde{f}_n - a_{n-1}\Big[e^{-\sqrt{2}\left(\tilde{f}_n - \tilde{f}_{n-1}\right)} - 1\Big] + a_n\Big[e^{-\sqrt{2}\left(\tilde{f}_{n+1} - \tilde{f}_n\right)} - 1\Big] \\
= -\delta^2 (-1)^{n+1} \alpha_1(\theta) f_n^0,
\end{aligned} \tag{7.15}
$$

where $\tilde{f}_0 = -\infty$, $\tilde{f}_{N+1} = \infty$.

Notice that the right hand of (7.15) is of order $O(\delta^2)$ now. This however is not enough to solve our nonlinear problem since there is a term of the same order in front of the linear part of the operator. Thus we need to find one more term in the expansion of $\tilde{f}_n$ to cancel the terms $\delta^2 (-1)^{n+1} \alpha_1 f_n^0$, $n = 1, 2, \cdots, N$. To this end let $\tilde{f}_n = \dot{f}_n + f_n^1$ where f_n^1, $n = 1, \ldots, N$, solve the following system of equations:

$$\sqrt{2}\, \mathbf{A} \mathbf{f}^1 = \alpha\, \tilde{\mathbf{h}} \quad \text{with } \alpha = \delta^2 \alpha_1(\theta), \tag{7.16}$$

where

$$
\mathbf{A} = \begin{pmatrix}
a_1 & -a_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-a_1 & (a_1 + a_2) & -a_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & & & & \ddots & & & & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & -a_{2m-1} & (a_{2m-1} + a_{2m}) & -a_{2m} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -a_{2m} & a_{2m}
\end{pmatrix}, \tag{7.17}
$$

and

$$\mathbf{J}(\alpha) = \begin{pmatrix} \alpha & 0 & \cdots & 0 & 0 \\ 0 & -\alpha & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & -\alpha & 0 \\ 0 & 0 & \cdots & 0 & \alpha \end{pmatrix}, \qquad \mathbf{f}^1 = \begin{pmatrix} f_1^1 \\ \vdots \\ f_N^1 \end{pmatrix}, \qquad \tilde{\mathbf{h}} = \begin{pmatrix} f_1^0 \\ -f_2^0 \\ \vdots \\ -f_{2m}^0 \\ f_N^0 \end{pmatrix}. \tag{7.18}$$

The matrix $\mathbf{A}$ has only one zero eigenvalue with corresponding eigenvector $(1, 1, \cdots, 1)^T$. The system (7.16) can be solved due to the formula (7.14). It is easy to derive that the term $\mathbf{f}^1$ is small in the order of $O(|\log \varepsilon|^{-1})$.

Now, $\dot{f}_n$, $n = 1, \cdots, N$, solve the following nonlinear system of equations,

$$\begin{aligned} \delta^2(-1)^{n+1}\,\alpha_1(\theta)\,\dot{f}_n \,+\, \sqrt{2}\,a_{n-1}\left(\dot{f}_n - \dot{f}_{n-1}\right) \,-\, \sqrt{2}\,a_n\left(\dot{f}_{n+1} - \dot{f}_n\right) \\ = \,-\,\delta^2\,(-1)^{n+1}\,\alpha_1\,f_n^1 \,+\, \bar{N}_n(\dot{\mathbf{f}}, \mathbf{f^1}), \end{aligned} \tag{7.19}$$

where $\bar{N}_n$, $n = 1, \cdots, N$, are given by

$$\begin{aligned} \bar{N}_n(\dot{\mathbf{f}}, \mathbf{f^1}) \,=\, & a_{n-1}\Big[e^{-\sqrt{2}\,(\dot{f}_n - \dot{f}_{n-1} + f_n^1 - f_{n-1}^1)} - 1 + \sqrt{2}\,(\dot{f}_n - \dot{f}_{n-1}) + \sqrt{2}\,(f_n^1 - f_{n-1}^1)\Big] \\ & -\, a_n\Big[e^{-\sqrt{2}\,(\dot{f}_{n+1} - \dot{f}_n + f_{n+1}^1 - f_n^1)} - 1 + \sqrt{2}\,(\dot{f}_{n+1} - \dot{f}_n) + \sqrt{2}\,(f_{n+1}^1 - f_n^1)\Big], \end{aligned}$$

and $\dot{f}_0 = -\infty$, $\dot{f}_{N+1} = \infty$. In order to use fixed point argument to solve (7.19), we consider the following system of equations:

$$\big(\sqrt{2}\mathbf{A} + \mathbf{J}\big)\mathbf{f} \,=\, \alpha\,\mathbf{h} \quad \text{with } \alpha = \delta^2\alpha_1(\theta), \tag{7.20}$$

where the matrix $\mathbf{J}$ is defined in (7.18). The system (7.20) has a uniformly (with respect to ε) bounded solution due to the following fact:

$$\det\big(\sqrt{2}\mathbf{A} + \mathbf{J}(\alpha)\big) \,=\, (-1)^m\,\alpha^N \,+\, \cdots \,+\, \Big(\Pi_{j=1}^N \sqrt{2}\,a_j\Big)\,\alpha.$$

We claim that problem (7.19) can be solved by a contraction mapping principle in the set

$$\mathfrak{X} \,=\, \left\{ \|\dot{\mathbf{f}}\|_{L^2(0,\ell)} \le \frac{1}{\big(\log\frac{1}{\varepsilon}\big)^{1+\sigma}} \right\} \quad \text{with small constant } \sigma > 0.$$

In fact, from the smallness of $\mathbf{f}^1$ in $\mathfrak{X}$, we have

$$\|\bar{\mathbf{N}}(\dot{\mathbf{f}}, \mathbf{f^1})\|_{L^2(0,\ell)} \,\le\, \frac{C}{|\log\varepsilon|^{1+2\sigma}}.$$

The result of Lemma 7.2 follows by a straightforward argument. $\square$

Step 3: We turn to the solvability of the system (7.8)-(7.9). By setting

$$\hat{f}_n = \bar{f}_n + \check{f}_n, \qquad b_n(\theta) \,=\, \gamma_1\, e^{-\sqrt{2}\,\left(\bar{f}_{n+1}-\bar{f}_n+\beta_{n+1}\right)}, \quad n=1,2,\cdots,N, \tag{7.21}$$

where $\bar{f}_n = f_n^0+f_n^1+\dot{f}_n, n=1,\cdots,N$, satisfy the system (7.10), then all functions $\check{f}_n$, $n=1,\cdots,N$, solve the following nonlinear system of equations,

$$\begin{aligned} \delta^2\left(\varepsilon\,\gamma_0\,\check{f}_n^{''} \,+\, (-1)^{n+1}\,\alpha_1(\theta)\,\check{f}_n\right) \,+\, \sqrt{2}\,b_{n-1}\left(\check{f}_n-\check{f}_{n-1}\right) \,-\, \sqrt{2}\,b_n\left(\check{f}_{n+1}-\check{f}_n\right) \\ =\, \delta^2\,\varepsilon^{1/2}h_n \,+\, \varepsilon\,\delta^2\,\gamma_0\big(n-(m+1)\big)\rho_\varepsilon^{''} \,-\, \delta^2\,\varepsilon\,\gamma_0\,(\bar{f}_n)^{''} \,+\, N_n\big(\check{\mathbf{f}},\bar{\mathbf{f}}\big), \end{aligned} \tag{7.22}$$

$$\check{f}_n'(\ell) \,=\, \check{f}_n'(0), \quad \check{f}_n(\ell) \,=\, \check{f}_n(0), \tag{7.23}$$

where N_n, $n=1,\cdots,N$, are given by

$$\begin{aligned} N_n\big(\check{\mathbf{f}},\bar{\mathbf{f}}\big) \,=\, b_{n-1}\Big[e^{-\sqrt{2}\,(\check{f}_n-\check{f}_{n-1})}-1+\sqrt{2}\,(\check{f}_n-\check{f}_{n-1})\Big] \\ -\, b_n\Big[e^{-\sqrt{2}\,(\check{f}_{n+1}-\check{f}_n)}-1+\sqrt{2}\,(\check{f}_{n+1}-\check{f}_n)\Big], \end{aligned} \tag{7.24}$$

and $\check{f}_0=-\infty$, $\check{f}_{N+1}=\infty$.

For the purpose of using a fixed point argument to solve (7.22)-(7.23) we also need the following solvability theory for a linear system of differential equations

Lemma 7.3. *Consider the following system*

$$\delta^2\left(\varepsilon\,\gamma_0\,v_n^{''}+(-1)^{n+1}\,\alpha_1(\theta)\,v_n\right)+\sqrt{2}\,b_{n-1}\big(v_n-v_{n-1}\big)-\sqrt{2}\,b_n\big(v_{n+1}-v_n\big)=g_n, \tag{7.25}$$

$$v_n'(\ell) \,=\, v_n'(0), \quad v_n(\ell) \,=\, v_n(0). \tag{7.26}$$

There exists a sequence $(\varepsilon_l)_l$ such that problem (7.25)-(7.26) has a unique solution $\mathbf{v}=\mathbf{v}(\mathbf{g})$ and

$$\|\mathbf{v}\|_{L^2(0,\ell)} \le C\sqrt{1/\varepsilon_l}\log\frac{1}{\varepsilon_l}\,\|\mathbf{g}\|_{L^2(0,\ell)}, \tag{7.27}$$

where $\mathbf{v}=(v_1,\cdots,v_N)^T$ and $\mathbf{g}=(g_1,\cdots,g_N)^T$. Moreover, if $\mathbf{g}\in H^2(0,\ell)$ then

$$\frac{\varepsilon_l}{|\log\varepsilon_l|}\,||\mathbf{v}^{''}||_{L^2(0,\ell)}+\sqrt{\frac{\varepsilon_l}{|\log\varepsilon_l|}}\,||\mathbf{v}^{'}||_{L^2(0,\ell)}+||\mathbf{v}||_{L^\infty(0,\ell)} \le C\sqrt{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon_l}}\,||\mathbf{g}||_{H^2(0,\ell)}. \tag{7.28}$$

Proof. We will denote:

$$\frac{\delta^{-2}(\theta)}{\frac{\sqrt{2}}{2}\left(\log\frac{1}{\varepsilon}-\log\log\frac{1}{\varepsilon}\right)} \,=\, \alpha_1(\theta) \,+\, \sigma^\varepsilon(\theta). \tag{7.29}$$

By (7.7) we have

$$\sigma^\varepsilon(\theta) \,=\, O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big). \tag{7.30}$$

Multiplying (7.25) by $\alpha_1+\sigma^\varepsilon$, we need to solve the following system

$$\kappa\left(\varepsilon\,\gamma_0\,\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}+\alpha_1\right)\mathbf{v} \,+\, \big(\alpha_1+\sigma^\varepsilon\big)\Big[\sqrt{2}\mathbf{B}-2\mathbf{K}\delta^2\alpha_1\Big]\mathbf{v} \,=\, \big(\alpha_1+\sigma^\varepsilon\big)\,\mathbf{g}, \tag{7.31}$$

where we have denoted κ as the form

$$\kappa = \frac{1}{\frac{\sqrt{2}}{2}\left(\log\frac{1}{\varepsilon} - \log\log\frac{1}{\varepsilon}\right)}, \tag{7.32}$$

and $\mathbf{K}$ is a diagonal $N \times N-$matrix with explicit form as

$$\mathbf{K} \ = \ \mathrm{diag}(0, 1, 0, 1, \cdots, 0, 1, 0). \tag{7.33}$$

In the above, we also defined the matrix $\mathbf{B}$ as

$$\mathbf{B} \ = \begin{pmatrix} b_1 & -b_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -b_1 & (b_1+b_2) & -b_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -b_{2m-1} & (b_{2m-1}+b_{2m}) & -b_{2m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -b_{2m} & b_{2m} \end{pmatrix}.$$

Note that for the entries in $\mathbf{A}$ and $\mathbf{B}$, we have the relations (c.f. (7.11) and (7.21))

$$b_n \ = \ \gamma_1\, e^{-\sqrt{2}\left(\bar{f}_{n+1}-\bar{f}_n+\beta_{n+1}\right)} \ = \ a_n\Big(1+O\big(\frac{1}{\log\frac{1}{\varepsilon}}\big)\Big). \tag{7.34}$$

For decomposition of system (7.31), we analyze the properties of the matrix $\sqrt{2}\mathbf{B} - 2\mathbf{K}\delta^2\alpha_1$ at first. For the symmetric matrix $\mathbf{A}$ defined in (7.17), using elementary matrix operations it is easy to prove that there exists an invertible matrix $\mathbf{Q}$ such that

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^T = \mathrm{diag}(a_1, a_2, \cdots, a_{N-1}, 0).$$

Since $a_1, \cdots, a_{N-1}$ are positive constants defined in (7.13), we assume that all eigenvalues of the matrix $\mathbf{A}$ are

$$\big\{\lambda_1, \cdots, \lambda_{N-1}, \lambda_N\big\} \quad \text{ with } \lambda_1, \cdots, \lambda_{N-1} > 0,\ \lambda_N = 0.$$

Similarly, using elementary matrix operations it is easy to prove that there exists an invertible matrix $\hat{\mathbf{Q}}$ such that

$$\hat{\mathbf{Q}}\Big[\sqrt{2}\mathbf{B} - 2\mathbf{K}\delta^2\alpha_1\Big]\hat{\mathbf{Q}}^T \ = \ \mathrm{diag}\Big(\sqrt{2}\,a_1 + O(\delta^2),\ \sqrt{2}\,a_2 + O(\delta^2),\ \cdots,\ \sqrt{2}\,a_{N-1} + O(\delta^2),\ O(\delta^2)\Big).$$

We also assume that all eigenvalues of the matrix $\sqrt{2}\mathbf{B} - 2\mathbf{K}\delta^2\alpha_1$ are $\big\{\lambda_1^\varepsilon, \cdots, \lambda_{N-1}^\varepsilon, \lambda_N^\varepsilon\big\}$. From the formulas (7.7) and (7.34), we have

$$\lambda_i^\varepsilon \ = \ \lambda_i + O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big), \quad i = 1, 2, \cdots, N-1. \tag{7.35}$$

Since the N-th eigenvalue, with associated eigenvector $(1, \cdots, 1)$, of $\mathbf{B}$ is zero, we can prove that

$$\lambda_N^\varepsilon = -\frac{2m}{2m+1}\delta^2\,\alpha_1. \tag{7.36}$$

Moreover, since $\sqrt{2}\mathbf{B} - 2\mathbf{K}\delta^2\alpha_1$ is a symmetric matrix, there exists another invertible matrix $\mathbf{P}$ such that

$$\mathbf{P}\Big(\sqrt{2}\mathbf{B} - 2\mathbf{K}\delta^2\alpha_1\Big)\mathbf{P}^{-1} = \mathrm{diag}(\lambda_1^\varepsilon, \cdots, \lambda_N^\varepsilon).$$

Now, after setting two new vectors

$$\mathfrak{u} = (\mathfrak{u}_1, \cdots, \mathfrak{u}_N)^T = \mathbf{P}\mathbf{v}, \quad \mathfrak{g} = (\mathfrak{g}_1, \cdots, \mathfrak{g}_N)^T = \big(\alpha_1 + \sigma^\varepsilon\big)\,\mathbf{P}\mathbf{g},$$

system (7.31) is equivalent to the following system with suitable periodic boundary conditions,

$$\begin{aligned}\kappa\left(\varepsilon\,\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\mathfrak{u} + \alpha_1\mathfrak{u}\right) \,+\, \kappa\,\varepsilon\,\gamma_0\mathbf{P}\frac{\mathrm{d}^2\mathbf{P^{-1}}}{\mathrm{d}\theta}\,\mathfrak{u} \,+\, 2\kappa\,\varepsilon\,\gamma_0\mathbf{P}\frac{\mathrm{d}\mathbf{P^{-1}}}{\mathrm{d}\theta}\,\mathfrak{u}&\\ +\,\mathrm{diag}(\lambda_1^\varepsilon, \cdots, \lambda_N^\varepsilon)\Big(\alpha_1 + \sigma^\varepsilon\Big)\mathfrak{u} \,=\, \mathfrak{g}.&\end{aligned} \tag{7.37}$$

Hence, we need to consider the systems, as follows, with suitable periodic boundary conditions. One describes the location of the center of mass of $N-$interfaces as

$$\kappa\left(\varepsilon\,\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\mathfrak{u}_N + \alpha_1\mathfrak{u}_N\right) \,+\, \lambda_N^\varepsilon\Big(\alpha_1 + \sigma^\varepsilon\Big)\mathfrak{u}_N \,=\, \mathfrak{g}_N, \tag{7.38}$$

and the other relate to the balance of mutual interaction of N-interfaces as

$$\kappa\left(\varepsilon\,\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\mathfrak{u}_i + \alpha_1\mathfrak{u}_i\right) \,+\, \lambda_i^\varepsilon\Big(\alpha_1 + \sigma^\varepsilon\Big)\mathfrak{u}_i \,=\, \mathfrak{g}_i, \quad i = 1, 2, \cdots, N-1. \tag{7.39}$$

Firstly, using formulas (7.29), (7.32) and (7.36), problem (7.38) read as

$$\varepsilon\,\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\mathfrak{u}_N + \frac{\alpha_1}{2m+1}\mathfrak{u}_N \,=\, \frac{1}{\kappa}\mathfrak{g}_N, \tag{7.40}$$

with suitable periodic boundary conditions, which can be solved by the following claim.

Claim 1: If $g \in L^2(0,\ell)$ then for all small ε satisfying the gap condition: given $c > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition

$$\left|j^2\varepsilon - \frac{\lambda_*}{N}\right| \geq c\sqrt{\varepsilon} \text{ for all } j \in \mathbb{N}, \tag{7.41}$$

there is a unique solution $\mathfrak{u}_N \in H^2(0,\ell)$ to problem (7.40) which satisfies

$$\varepsilon||{\mathfrak{u}_N}''||_{L^2(0,\ell)} + ||\mathfrak{u}_N||_{L^\infty(0,\ell)} \leq C\varepsilon^{-1/2}\log\frac{1}{\varepsilon}\,||\mathfrak{g}_N||_{H^2(0,\ell)}.$$

Moreover, if $\mathfrak{g}_N \in H^2(0,\ell)$ then

$$\varepsilon||{\mathfrak{u}_N}''||_{L^2(0,\ell)} + ||\mathfrak{u}_N||_{L^\infty(0,\ell)} \leq C\,\log\frac{1}{\varepsilon}\,||\mathfrak{g}_N||_{H^2(0,\ell)}.$$

For proof of this Claim, the reader can refer to Lemma 8.1 in [15].

Secondly, for the solvability theory of equations in (7.39), define linear operators by

$$L_\varepsilon^i = \kappa\Big(\varepsilon\,\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \,+\, \alpha_1\Big) \,+\, \lambda_i^\varepsilon\Big(\alpha_1 + \sigma^\varepsilon\Big), \quad i = 1, 2, \cdots, N-1.$$

Claim 2: There exists a sequence $(\varepsilon_l)_l$ such that $L^i_{\varepsilon_l}, i = 1, 2, \cdots, N-1$ is invertible and

$$\left|\left|\left(L^i_{\varepsilon_l}\right)^{-1}\right|\right|_{L^2(0,\ell)\to H^1(0,\ell)} \leq \frac{C}{\sqrt{\varepsilon_l}\,\sqrt{\kappa_l}},\ i = 1, 2, \cdots, N. \tag{7.42}$$

To characterize Morse indices of above operators, first of all, for any fixed $i \in \{1, 2, \cdots, N-1\}$ we give an asymptotic estimate on the number $\mathcal{N}^i_\varepsilon$ of negative eigenvalues of the operator L^i_ε. Denote its eigenvalues $(\tau^i_{\varepsilon,j})_j$ with corresponding eigenfunctions $(\psi^i_{\varepsilon,j})_j$, in non-decreasing order and counting them with multiplicity. From the Courant-Fisher characterization we can write $\tau^i_{\varepsilon,j}$ in two different ways:

$$-\tau^i_{\varepsilon,j} = \sup_{M\in M_j}\ \inf_{u\in M, u\neq 0} \frac{\int uL^i_\varepsilon u}{\int \alpha_1 u^2}, \tag{7.43}$$

$$-\tau^i_{\varepsilon,j} = \inf_{M\in M_{j-1}}\ \sup_{u\perp M, u\neq 0} \frac{\int uL^i_\varepsilon u}{\int \alpha_1 u^2}. \tag{7.44}$$

Here M_j (resp. M_{j-1}) represents the family of j dimensional (resp. $j-1$ dimensional) subspaces of $H^2(0,\ell)$, and the symbol $\perp$ denotes orthogonality with respect to the L^2 scalar product with positive weight function α_1.

By $(\Lambda^\varepsilon_j)_j, (\Theta^\varepsilon_j)_j$ we will denote, respectively, the set of eigenvalues and eigenfunctions of the following eigenvalue problem: (c.f. (2.1))

$$\varepsilon\,\gamma_0 \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Theta + \alpha_1\Theta = -\Lambda\,\alpha_1\,\Theta \quad \text{in } (0,\ell), \quad \Theta(0) = \Theta(\ell), \quad \Theta'(0) = \Theta'(\ell).$$

As we proved in (2.2), if ε is small then we have, as $j\to\infty$,

$$\Lambda^\varepsilon_j = \frac{4\pi^2}{\ell_0^2}\,(j^2\varepsilon - \lambda_*) + O(\frac{\varepsilon}{j^2}). \tag{7.45}$$

From the asymptotic formula in (7.45) and the forma in (7.43), one derives

$$\mathcal{N}^i_\varepsilon \geq \big(C + o(1)\big)(\varepsilon\kappa)^{-1/2}, \tag{7.46}$$

where C is a fixed constant independent of ε. To prove a similar upper bound, we choose j to be the first index such that $\kappa\Lambda^\varepsilon_j - \lambda^\varepsilon_i > |\sigma^\varepsilon|$. Then from the formula in (7.45) we find that

$$j = \big(C + o(1)\big)(\varepsilon\kappa)^{-1/2},$$

with the same C as in (7.46). Define $M_{j-1} = \mathrm{span}\{\Theta^\varepsilon_l : l = 1, 2, \cdots, j-1\}$. For an arbitrary function $u \in H^2$ and $u \perp M_j$, we can write

$$u = \sum_{l\geq j} \beta_l \Theta^\varepsilon_l.$$

Plugging this u into (7.44) and using the formula (7.45), we also have

$$\mathcal{N}^i_\varepsilon \leq \big(C + o(1)\big)(\varepsilon\kappa)^{-1/2}. \tag{7.47}$$

Hence we get that

$$\mathcal{N}_\varepsilon^i \sim C(\varepsilon\kappa)^{-1/2} \text{ as } \varepsilon \to 0, \quad i = 1, 2, \cdots, N-1. \tag{7.48}$$

For further arguments, we now compute the derivatives of $\tau_{\varepsilon,j}^i$ (order $o(\sqrt{\varepsilon\kappa})$) with respect to ε. Differentiating with respect to ε the equation $||\psi_{\varepsilon,j}^i|| = 1$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}||\psi_{\varepsilon,j}^i|| = 0 \Rightarrow \Big(\frac{\mathrm{d}\psi_{\varepsilon,j}^i}{\mathrm{d}\varepsilon},\, \psi_{\varepsilon,j}^i\Big) = 0, \tag{7.49}$$

where $(\cdot,\,\cdot)$ denotes the scalar product with weight function α_1. On the other hand, by T. Kato's Theorem 2.2 differentiating the following equation with respect to ε

$$\kappa\big(\varepsilon\,\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \alpha_1\big)\psi_{\varepsilon,j}^i + \lambda_i^\varepsilon\Big(\alpha_1 + \sigma^\varepsilon\Big)\psi_{\varepsilon,j}^i = -\tau_{\varepsilon,j}^i\,\alpha_1\,\psi_{\varepsilon,j}^i, \tag{7.50}$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}\varepsilon}\big(\varepsilon\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \alpha_1\big)\psi_{\varepsilon,j}^i + \kappa\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\psi_{\varepsilon,j}^i + \kappa\big(\varepsilon\gamma_0\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \alpha_1\big)\frac{\mathrm{d}\psi_{\varepsilon,j}^i}{\mathrm{d}\varepsilon} + \frac{\mathrm{d}\lambda_i^\varepsilon}{\mathrm{d}\varepsilon}\big(\alpha_1 + \sigma^\varepsilon\big)\psi_{\varepsilon,j}^i \\ + \lambda_i^\varepsilon\frac{\mathrm{d}\sigma^\varepsilon}{\mathrm{d}\varepsilon}\,\psi_{\varepsilon,j}^i + \lambda_i^\varepsilon\big(\alpha_1 + \sigma^\varepsilon\big)\frac{\mathrm{d}\psi_{\varepsilon,j}^i}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\tau_{\varepsilon,j}^i}{\mathrm{d}\varepsilon}\alpha_1\psi_{\varepsilon,j}^i + \tau_{\varepsilon,j}^i\alpha_1\frac{\mathrm{d}\psi_{\varepsilon,j}^i}{\mathrm{d}\varepsilon}.\end{aligned}$$

Multiplying above equation by $\psi_{\varepsilon,j}^i$, integrating by parts and using (7.49), one gets

$$\frac{\mathrm{d}\tau_{\varepsilon,j}^i}{\mathrm{d}\varepsilon}\Big|_{\tau_{\varepsilon,j}=0} = \frac{\lambda_i}{\varepsilon}\big(1 + o(1)\big) > 0. \tag{7.51}$$

Next, for $l \in \mathbb{N}$, we choose ε_l in such a way that $\varepsilon_l\kappa_l = 2^{-l}$ with κ_l defined as in (7.32). Then from (7.48) it follows that

$$N_{\varepsilon_{l+1}}^i - N_{\varepsilon_l}^i \sim C\big[2^{(l+1)/2} - 2^{l/2}\big] = C\big(\sqrt{2} - 1\big)\big(\varepsilon_l\kappa_l\big)^{-1/2}. \tag{7.52}$$

By the formula (7.51), the eigenvalue of L_ε^i bounded in absolute value by $o(\sqrt{\varepsilon}\kappa)$ are decreasing in ε. Equivalently, by the last equation, the number of eigenvalues which become negative, when ε decreases from ε_l to ε_{l+1}, is of order $\big(\varepsilon_l\kappa_l\big)^{-1/2}$. We define

$$F_l^1 = \big\{\varepsilon \in (\varepsilon_{l+1}, \varepsilon_l) : \ \ker L_\varepsilon^i \neq \varnothing,\, i = 1, 2, \cdots, N-1\big\}, \tag{7.53}$$

$$F_l^2 = \big\{\varepsilon \in (\varepsilon_{l+1}, \varepsilon_l) : \ \varepsilon \text{ does not satisfies } (7.41)\big\}, \tag{7.54}$$

$$\hat{F}_l = (\varepsilon_{l+1}, \varepsilon_l) \setminus (F_l^1 \cup F_l^2). \tag{7.55}$$

By (7.52) and the monotonicity (in ε) of the small eigenvalues, it follows that $\mathrm{card}(F_l) < C\big(\varepsilon_l\kappa_l\big)^{-1/2}$. Whence there exists an interval (a_l, b_l) such that

$$(a_l, b_l) \subset \hat{F}_l, \quad |b_l - a_l| \geq \frac{\mathrm{meas}(\hat{F}_l)}{\mathrm{card}(F_l^1 \cup F_l^2)} \geq C^{-1}(\varepsilon_l)^{3/2}\sqrt{\kappa_l}. \tag{7.56}$$

Hence, by setting $\varepsilon_l \kappa_l = (a_l + b_l)/2$ and using (7.51), we conclude that $L^i_{\varepsilon_l}, i = 1, 2, \cdots, N-1$ is invertible and

$$\left|\left|\left(L^i_{\varepsilon_l}\right)^{-1}\right|\right|_{L^2(0,\ell)\to H^1(0,\ell)} \leq \frac{C}{\sqrt{\varepsilon_l}\,\sqrt{\kappa_l}}, \; i = 1, 2, \cdots, N-1. \tag{7.57}$$

This completes the proof of Claim 2.

As a consequence of the last two claims and an obvious contraction mapping theory, the solution to (7.37) exists and satisfies

$$\|\mathfrak{u}\|_{L^2(0,\ell)} \leq C\sqrt{1/\varepsilon_l}\log\frac{1}{\varepsilon_l}\,\|\mathfrak{g}\|_{L^2(0,\ell)}. \tag{7.58}$$

From (7.58) by a standard argument one can show

$$\frac{1}{\frac{1}{\varepsilon_l}\log\frac{1}{\varepsilon_l}}\,\|\mathfrak{u}^{''}\|_{L^2(0,\ell)} + \frac{1}{\sqrt{\frac{1}{\varepsilon_l}\log\frac{1}{\varepsilon_l}}}\,\|\mathfrak{u}^{'}\|_{L^2(0,\ell)} + \|\mathfrak{u}\|_{L^2(0,\ell)} \leq C\sqrt{1/\varepsilon_l}\log\frac{1}{\varepsilon_l}\,\|\mathfrak{g}\|_{L^2(0,\ell)}.$$

The estimates (7.27) and (7.28) are also a direct consequence. □

By using Lemma 7.3, let $\mathbf{v} = (v_1, \cdots, v_n)$ be the solution to the problem (7.25)-(7.26) with right hand side replaced by the following term

$$\delta^2\,\varepsilon^{1/2}h_n + \varepsilon\,\delta^2\,\gamma_0\big(n-(m+1)\big)\rho_\varepsilon^{''} - \delta^2\,\varepsilon\,\gamma_0\,(\bar{f}_n)^{''}.$$

Then problem (7.22)-(7.23) can be solved by a contraction mapping principle in the set

$$\mathcal{X} = \left\{ \check{\mathbf{f}} \in H^2(0,1) \;\middle|\; \frac{1}{\sqrt{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}}}\,\|\check{\mathbf{f}}^{'}\|_{L^2(0,\ell)} + \|\check{\mathbf{f}}\|_{L^2(0,\ell)} \leq \varepsilon^{1/2+\sigma} \right\}.$$

In fact, in $\mathcal{X}$ we have

$$\|\mathbf{N}(\hat{\mathbf{f}}, \bar{\mathbf{f}})\|_{L^2(0,\ell)} \leq \varepsilon^{1+2\sigma}.$$

Now, the result follows by a straightforward argument using Lemma 7.3. The proof of the Proposition 7.1 is complete. □

7.2 Proof of Theorem 1.2

In the final part of this section, we solve the system (7.1), which will give a proof of Theorem 1.2.

Proof of Theorem 1.1: Define

$$\mathcal{D} = \left\{ \mathbf{f} \in H^2(0,1) \,\middle|\, \|\mathbf{f}\|_{H^2(0,1)} \leq D|\log\varepsilon|^{\varrho} \right\}$$

with small constant $0 < \varrho < 1$ and for given $\ddot{\mathbf{f}} \in \mathcal{D}$, we can set for $n = 1, \cdots, N$

$$\varepsilon^{3/2}h_n(\mathbf{f}) \equiv \varepsilon^2\,\alpha_{6,n}(\theta) + \mathcal{M}_{n1}(\mathbf{f}, \mathbf{f}^{'}, \mathbf{f}^{''}) + \varepsilon^2\,\alpha_{5,n}(\theta)\,f_n + \varepsilon^2\,(-1)^n\,\alpha_3(\theta)\,f_n^2 + \mathcal{M}_{n2}(\ddot{\mathbf{f}}, \ddot{\mathbf{f}}^{'}).$$

For any $\mathbf{f}^3$, we can use Proposition 7.1 to get

$$\mathbf{f}^4 = \widetilde{T}_1\big(\,\varepsilon^{3/2}h_1(\mathbf{f}^3),\,\cdots,\,\varepsilon^{3/2}h_N(\mathbf{f}^3)\,\big).$$

Whence, by the fact that $\mathcal{M}_{n1}$ are contractions on $\mathcal{D}$, making use of the theory developed in Proposition 7.1 and the Contraction Mapping theorem, we find $\mathbf{f}$ for a fixed $\ddot{\mathbf{f}}$ in $\mathcal{D}$. This way we define a mapping as

$$\mathcal{Z}(\ddot{\mathbf{f}}) = (\mathbf{f}),$$

and the solution of our problem is simply a fixed point of $\mathcal{Z}$. Continuity of $\mathcal{M}_{n2}$ with respect to its parameters and a standard regularity arguments allows us to conclude that $\mathcal{Z}$ is compact as mapping from $H^1(0,1)$ into itself. The Schauder Theorem applies to yield the existence of a fixed point of $\mathcal{Z}$ as required. This ends the proof of Theorem 1.2. □

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