INFINITELY MANY NON-RADIAL SOLUTIONS FOR THE HÉNON EQUATION WITH CRITICAL GROWTH

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ABSTRACT. We consider the following Hénon equation with critical growth:

$$(*)\begin{cases}-\Delta u = |y|^\alpha u^{\frac{N+2}{N-2}},\ u>0 & y\in B_1(0),\\ u=0, & \text{on } \partial B_1(0),\end{cases}$$

where $\alpha > 0$ is a positive constant, $B_1(0)$ is the unit ball in $\mathbb{R}^N$, and $N \geq 4$. Ni [9] proved the existence of a radial solution and Serra [12] proved the existence of a nonradial solution for α *large* and $N \geq 4$. In this paper, we show the existence of a nonradial solution for *any* $\alpha > 0$ and $N \geq 4$. Furthermore, we prove that equation (*) has **infinitely many non-radial** solutions, whose energy can be made arbitrarily large.

Keywords: Henon's Equation, Infinitely Many Solutions, Critical Sobolev Exponent, Reduction Method

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1. Introduction

Of concern is the following Hénon equation with critical growth:

$$\begin{cases}-\Delta u = |y|^\alpha u^{\frac{N+2}{N-2}},\ u>0, & y\in B_1(0),\\ u=0, & \text{on } \partial B_1(0),\end{cases} \tag{1.1}$$

where $\alpha > 0$ is a positive constant, $B_1(0)$ is the unit ball in $\mathbb{R}^N$, and $N \geq 3$.

Equation (1.1) arises in the study of astrophysics ([7]). If the exponent $\frac{N+2}{N-2}$ is replaced by p, where $p < \frac{N+2}{N-2}$, a solution can be obtained easily by variational methods. When $p = \frac{N+2}{N-2}$, the loss of compactness from $H_0^1(B_1(0))$ to $L^{\frac{2N}{N-2}}(B_1(0))$ makes problem (1.1) very difficult to study. Ni [9] first proved the existence of a *radial solution* for *any* $\alpha > 0$. On the other hand, it is easy to check that the mountain pass value c corresponding to (1.1) is

$$c = \frac{1}{N} S^{\frac{N}{2}},$$

where S is best Sobolev constant of the embedding from $D^{1,2}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, from which we can deduce that c is not a critical value of the functional corresponding to (1.1). When $N = 2$, Smets-Su-Willem [13] showed that the montain pass solution is non-radial when α is large. When $N \geq 3$, for the Hénon equations with nearly critical growth (replacing $\frac{N+2}{N-2}$ in (1.1) by $\frac{N+2}{N-2}-\varepsilon$ with $\varepsilon>0$ small), Cao-Peng [3] proved that the mountain pass solution is non-radial and blows up as $\varepsilon\to 0$. Thus, it is natural to ask whether (1.1) has a non-radial solution. Using variational method, Serra [12] proved that (1.1) has a non-radial solution when $N\ge 4$ and α *is large.* As far as we know, up to now, there is no existence result of *non-radial solution* for (1.1), and there is no *multiplicity* result for (1.1) either, with arbitrary $\alpha>0$.

The aim of this paper is to prove that (1.1) has infinitely many non-radial solutions if $N\ge 4$. In fact, we will study a more general problem:

$$\begin{cases} -\Delta u = K(|y|)u^{\frac{N+2}{N-2}},\ u>0, & y\in B_1(0),\\ u=0, & \text{on } \partial B_1(0), \end{cases} \tag{1.2}$$

where $K(r)$ is a bounded function defined in $[0,1]$. It is easy to see that a necessary condition for the existence of one solution for (1.2) is that $K(r)$ is positive somewhere. On the other hand, Pohozaev identity implies (1.2) has no solution if $K'(r)\le 0$ in $[0,1]$. Concerning the existence of solutions for (1.2), using the same method as in [15], we can prove the following existence result:

Theorem A. *Suppose that there is a $r_0\in(0,1)$, such that $K(r_0)>0$, and*

$$K(r)=K(r_0)-K_0|r-r_0|^m+O(|r-r_0|^{m+\theta}),\quad \text{as } r\to r_0, \tag{1.3}$$

where $m\in[2,N-2)$, $K_0>0$ and $\theta>0$ are some constants, then, for $N\ge 5$, (1.2) has infinitely many non-radial solutions.

Note that for the Hénon equation, $K(r)=r^\alpha$, which has no critical point in $(0,1)$. So, Theorem A does not apply to the Henon equation (1.1).

Condition (1.3) implies that r_0 is a local maximum point of $K(r)$, and thus a critical point of $K(r)$. The function r^α attains its maximum on $[0,1]$ at $r_0=1$, but $r_0=1$ is not a critical point of r^α.

The aim of this paper is to show that if $K(r)$ is increasing near $r_0=1$ (so it is a maximum point of $K(r)$ on $[1-\delta,1]$ for some small $\delta>0$), the zero Dirichlet boundary condition make it possible to construct infinitely many solutions for (1.2), although $r_0=1$ is not a critical point of $K(r)$. Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that $N\ge 4$. If $K(r)$ satisfies $K(1)>0$ and $K'(1)>0$, then problem (1.2) has infinitely many non-radial solutions. In particular, the Hénon equation (1.1) has infinitely many non-radial solutions.*

Recall that a necessary condition for the existence of at least one solution for (1.2) is that $K'(r)$ is positive somewhere on $[0,1]$. If $K(r)\ge 0$ and

$N \geq 5$, Theorems A and 1.1 show that under a condition which is slightly stronger than this necessary condition, (1.2) has infinitely many solutions.

We think that the condition that $N \geq 4$ is just technical. The reason is that the reduced energy does have a critical point when $N = 3$. The problem lies in the reduction part which should be only technical. (Some partial (negative) results are obtained by O. Druet and Laurain [6].)

The readers can refer to [1, 2, 4, 8, 10, 11, 14] for results on Hénon equations involving sub-critical and near critical exponents.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1.

Let us fix a positive integer $k \geq k_0$, where k_0 is large, which is to be determined later.

Set

$$\mu = k^{\frac{N-1}{N-2}}, \quad N \geq 4$$

to be the scaling parameter.

Let $2^* = \frac{2N}{N-2}$. Using the transformation $u(y) \mapsto \mu^{-\frac{N-2}{2}} u\big(\frac{y}{\mu}\big)$, we find that (1.2) becomes

$$\begin{cases} -\Delta u = K\big(\frac{|y|}{\mu}\big) u^{2^*-1}, u > 0, & y \in B_\mu(0), \\ u = 0, & \text{on } \partial B_\mu(0). \end{cases} \tag{1.4}$$

It is well-known that the functions

$$U_{x,\Lambda}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \left(\frac{\Lambda}{1 + \Lambda^2 |y-x|^2} \right)^{\frac{N-2}{2}}, \ \mu > 0, \ x \in \mathbb{R}^N$$

are the only solutions to the following problem

$$-\Delta u = u^{\frac{N+2}{N-2}}, \ u > 0 \text{ in } \mathbb{R}^N.$$

As the scaling parameter $\Lambda \to +\infty$, $U_{x,\Lambda}$ is called a *single-bubble* centered at the point x. Since there is no *small* parameter in (1.1) (here μ is fixed), we use the scaling parameter Λ as the blow-up parameter. Our main idea is to *place* a large number of bubbles inside Ω. Then the scaling parameter will be determined by the *number of bubbles*. We put many bubbles *along a $k-$polygon inside the domain $B_1(0)$ but near the boundary.* See Figure 1. (The idea of using the number of bubbles as parameter was first introduced in [15].)

Let us remark that the variational method of Serra [12] also uses the dihedral symmetry of $k-$polygons. By using symmetry of $D_k \times O(N-2)$, the problem (1.1) can be reduced to the one in a sector. He then showed that under the dihedral symmetry, the loss of compactness can be recovered if the critical value is below some constant, which holds true when $N \geq 4$. To show that the solution is nonradial, he needed to compare with the energy

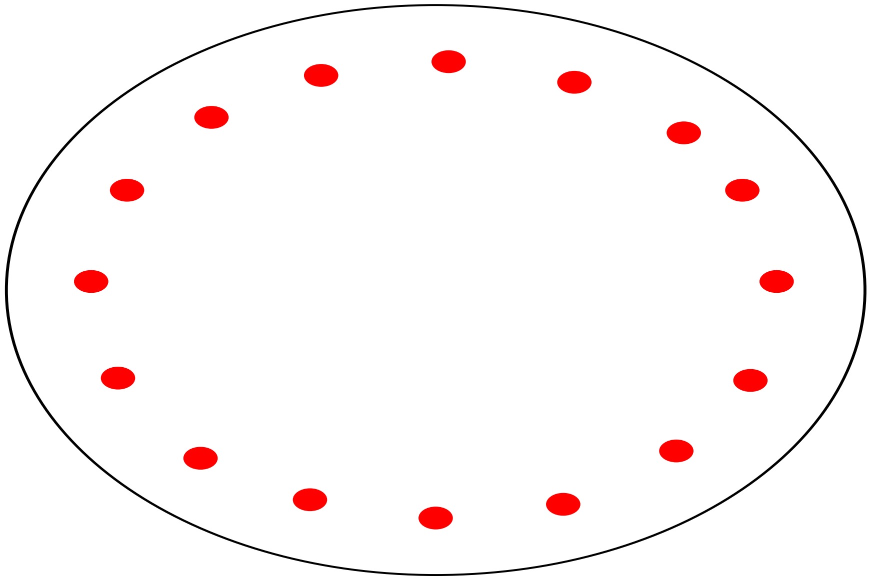

FIGURE 1. The location of the bubbles

level of radial solution. There the condition that α is large is needed. Our method of construction is direct and gives more information.

We continue our construction. Since $U_{x,\Lambda}$ is not zero on $\partial B_\mu(0)$, we define $PU_{x,\Lambda}$ as the solution of the following problem:

$$\Delta PU_{x,\Lambda} = \Delta U_{x,\Lambda}, \text{ in } B_\mu(0), \quad \Delta PU_{x,\Lambda} = 0 \text{ on } \partial B_\mu(0). \tag{1.5}$$

Let $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. Define

$$H_s = \big\{u : u \in H_0^1(B_\mu(0)), u \text{ is even in } y_h, h = 2, \cdots, N,$$
$$u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta + \frac{2\pi j}{k}), r\sin(\theta + \frac{2\pi j}{k}), y'')\big\}.$$

Let

$$x_j = \big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and let

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} PU_{x_j,\Lambda}.$$

In this paper, we always assume that

$$r \in \big[\mu(1 - \frac{r_0}{k}), \mu(1 - \frac{r_1}{k})\big], \quad \text{for some constants } r_1 > r_0 > 0,$$

and

$$L_0 \le \Lambda \le L_1, \quad \text{for some constants } L_1 > L_0 > 0.$$

Theorem 1.1 is a direct consequence of the following result:

Theorem 1.2. *Suppose that $N \ge 4$. If $K(1) > 0$ and $K'(1) > 0$, then there is an integer $k_0 > 0$, such that for any integer $k \ge k_0$, (1.4) has a solution u_k of the form*

$$u_k = W_{r_k,\Lambda_k}(y) + \omega_k,$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $\|\omega_k\|_{L^\infty} \to 0$, $L_0 \le \Lambda_k \le L_1$, and $r_k \in \big(\mu(1-\frac{r_0}{k}), \mu(1-\frac{r_1}{k})\big)$.

Unlike Theorem A, where the result was proved by constructing solutions with many bubbles near the local maximum point $r_0 \in (0,1)$, the solutions constructing in Theorem 1.1 have many bubbles near the boundary of the unit ball $B_1(0)$. In Theorem 1.1, $r_0 = 1$ is not a critical point of $K(r)$ anymore. It is the zero boundary condition that plays a very important role in the construction of solutions with many bubbles near $|y| = 1$.

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2. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction.

Let

$$\|u\|_* = \sup_{y\in B_\mu(0)} \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1} |u(y)|, \tag{2.1}$$

and

$$\|f\|_{**} = \sup_{y\in B_\mu(0)} \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)^{-1} |f(y)|, \tag{2.2}$$

where $\tau = \frac{N-2}{N-1}$ if $N \ge 4$. For this choice of τ, we find that

$$\sum_{j=2}^k \frac{1}{|x_j - x_1|^\tau} \le \frac{Ck^\tau}{\mu^\tau} \sum_{j=2}^k \frac{1}{j^\tau} \le \frac{Ck}{\mu^\tau} \le C'.$$

Let

$$Z_{i,1} = \frac{\partial PU_{x_i,\Lambda}}{\partial r}, \quad Z_{i,2} = \frac{\partial PU_{x_i,\Lambda}}{\partial \Lambda}.$$

Consider

$$
\begin{cases} -\Delta\phi_k-(2^*-1)K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi_k= & h+\sum_{j=1}^2 c_j\sum_{i=1}^k U_{x_i,\Lambda}^{2^*-2}Z_{i,j}, \ \text{ in } B_\mu(0),\\ \phi_k\in H_s,\\ <U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\phi_k>=0 & i=1,\cdots,k,\ l=1,2 \end{cases} \tag{2.3}
$$

for some numbers c_i, where $<u,v>=\int_{B_\mu(0)}uv$.

Lemma 2.1. *Assume that ϕ_k solves (2.3) for $h=h_k$. If $\|h_k\|_{**}$ goes to zero as k goes to infinity, so does $\|\phi_k\|_*$.*

Proof. The proof of this lemma is similar to the proof of Lemma 2.1 in [15]. Thus, we just sketch it.

We argue by contradiction. Suppose that there are $k\to+\infty$, $h=h_k$, $\Lambda_k\in[L_1,L_2]$, $r_k\in\big[\mu(1-\frac{r_0}{k}),\mu(1-\frac{r_1}{k})\big]$, and ϕ_k solving (2.3) for $h=h_k$, $\Lambda=\Lambda_k$, $r=r_k$, with $\|h_k\|_{**}\to 0$, and $\|\phi_k\|_*\ge c'>0$. We may assume that $\|\phi_k\|_*=1$. For simplicity, we drop the subscript k.

We rewrite (2.3) as

$$
\begin{aligned}
\phi(y)=&(2^*-1)\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}K\big(\frac{|z|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi(z)\,dz\\
&+\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}\big(h(z)+\sum_{j=1}^2 c_j\sum_{i=1}^k Z_{i,j}(z)U_{x_i,\Lambda}^{2^*-2}(z)\big)\,dz.
\end{aligned} \tag{2.4}
$$

Using Lemma B.3, we have

$$
\begin{aligned}
&\Big|(2^*-1)\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}K\big(\frac{|z|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi(z)\,dz\Big|\\
\le& C\|\phi\|_*\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}W_{r,\Lambda}^{2^*-2}\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\
\le& C\|\phi\|_*\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.
\end{aligned} \tag{2.5}
$$

It follows from Lemma B.2 that

$$
\Big|\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}h(z)\,dz\le C\|h\|_{**}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}. \tag{2.6}
$$

and

(2.7)
$$\Big|\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}\sum_{i=1}^{k}Z_{i,l}(z)U_{x_i,\Lambda}^{2^*-2}(z)\,dz\Big|\le C\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.$$

Next, we estimate c_l, $l=1,2$. Multiplying (2.3) by $Z_{1,l}$ and integrating, we see that c_t satisfies

(2.8)
$$\sum_{t=1}^{2}\sum_{i=1}^{k}\big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,t},Z_{1,l}\big\rangle c_t=\big\langle -\Delta\phi-(2^*-1)K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi,Z_{1,l}\big\rangle-\big\langle h,Z_{1,l}\big\rangle.$$

It follows from Lemma B.1 that

$$\begin{aligned}\big|\big\langle h,Z_{1,l}\big\rangle\big|&\le C\|h\|_{**}\int_{\mathbb{R}^N}\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz\\ &\le C\|h\|_{**}.\end{aligned}$$

On the other hand, using Lemma B.3, we can prove

$$\begin{aligned}&\big\langle -\Delta\phi-(2^*-1)K\big(\frac{|z|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi,Z_{1,l}\big\rangle\\ =&(2^*-1)\big\langle\big(1-K\big(\frac{|z|}{\mu}\big)W_{r,\Lambda}^{2^*-2}Z_{1,l},\phi\big\rangle=o(\|\phi\|_*).\end{aligned}\tag{2.9}$$

But there is a constant $\bar c>0$,

$$\sum_{i=1}^{k}\big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,t},Z_{1,l}\big\rangle=(\bar c+o(1))\delta_{tl}.$$

Thus we obtain from (2.8) that

$$c_l=o(\|\phi\|_*)+O(\|h\|_{**}).\tag{2.10}$$

So,

$$\|\phi\|_*\le\Big(o(1)+\|h_k\|_{**}+\frac{\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}}\Big).\tag{2.11}$$

Since $\|\phi\|_*=1$, we obtain from (2.11) that there is $R>0$, such that

$$\|\phi(y)\|_{B_R(x_i)}\ge a>0,\tag{2.12}$$

for some i. But $\bar{\phi}(y) = \phi(y - x_i)$ converges uniformly in any compact set to a solution u of

$$-\Delta u - (2^* - 1)U_{0,\Lambda}^{2^*-2} u = 0, \quad \text{in } \mathbb{R}^N, \tag{2.13}$$

for some $\Lambda \in [L_1, L_2]$, and u is perpendicular to the kernel of (2.13). So, $u = 0$. This is a contradiction to (2.12).

□

From Lemma 2.1, using the same argument as in the proof of Proposition 4.1 in [5], we can prove the following result :

Proposition 2.2. *There exists $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \geq k_0$ and all $h \in L^\infty(\mathbb{R}^N)$, problem (2.3) has a unique solution $\phi \equiv L_k(h)$. Besides,*

$$\|L_k(h)\|_* \leq C\|h\|_{**}, \qquad |c_l| \leq C\|h\|_{**}. \tag{2.14}$$

Now, we consider

(2.15)
$$\begin{cases} -\Delta\big(W_{r,\Lambda} + \phi\big) = K\big(\frac{y}{\mu}\big)\big(W_{r,\Lambda} + \phi\big)^{2^*-1} + & \sum_{t=1}^{2} c_t \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^*-2} Z_{i,t}, \text{ in } B_\mu(0), \\ \phi_k \in H_s, \\ < U_{x_i,\Lambda}^{2^*-2} Z_{i,l}, \phi_k >= 0, & i = 1, \cdots, k, \ l = 1, 2. \end{cases}$$

We have

Proposition 2.3. *There is an integer $k_0 > 0$, such that for each $k \geq k_0$, $L_0 \leq \Lambda \leq L_1$, $r \in \big[\mu(1 - \frac{r_0}{k}), \mu(1 - \frac{r_1}{k})\big]$, (2.15) has a unique solution $\phi = \phi(r, \Lambda)$, satisfying*

$$\|\phi\|_* \leq C\big(\frac{1}{\mu}\big)^{\frac{1}{2}+\sigma}, \qquad |c_t| \leq C\big(\frac{1}{\mu}\big)^{\frac{1}{2}+\sigma},$$

if $N \geq 4$, where $\sigma > 0$ is a small constant, $\mu = k^{\frac{N-1}{N-2}}$.

Rewrite (2.15) as

(2.16)
$$\begin{cases} -\Delta\phi - (2^* - 1)K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi = & N(\phi) + l_k + \sum_{t=1}^{2} c_i \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^*-2} Z_{i,t}, \text{ in } B_\mu(0), \\ \phi \in H_s, \\ < U_{x_i,\Lambda}^{2^*-2} Z_{i,l}, \phi >= 0, & i = 1, \cdots, k, \ l = 1, 2, \end{cases}$$

where

$$N(\phi) = K\big(\frac{|y|}{\mu}\big)\Big(\big(W_{r,\Lambda} + \phi\big)^{2^*-1} - W_{r,\Lambda}^{2^*-1} - (2^* - 1)W_r^{2^*-2}\phi\Big),$$

and

$$l_k = K\Big(\frac{|y|}{\mu}\Big)W_{r,\Lambda}^{2^*-1} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{2^*-1}.$$

In order to use the contraction mapping theorem to prove that (2.16) is uniquely solvable in the set that $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

Lemma 2.4. *If $N \geq 4$, then*

$$\|N(\phi)\|_{**} \leq C\|\phi\|_*^{\min(2^*-1,2)}.$$

Proof. We have

$$|N(\phi)| \leq \begin{cases} C|\phi|^{2^*-1}, & N \geq 6; \\ C\big(W_{r,\Lambda}^{\frac{6-N}{N-2}}\phi^2 + |\phi|^{2^*-1}\big), & N = 4, 5. \end{cases}$$

Firstly, we consider $N \geq 6$.

Using

$$\sum_{j=1}^{k} a_j b_j \leq \Big(\sum_{j=1}^{k} a_j^p\Big)^{\frac{1}{p}}\Big(\sum_{j=1}^{k} b_j^q\Big)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \; a_j, \; b_j \geq 0,$$

we obtain

$$\begin{aligned} |N(\phi)| &\leq C\|\phi\|_*^{2^*-1}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} \\ &\leq C\|\phi\|_*^{2^*-1}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\tau}}\Big)^{\frac{4}{N-2}} \qquad (2.17) \\ &\leq C\|\phi\|_*^{2^*-1}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}. \end{aligned}$$

Thus, the result follows.

Suppose that $N = 4, 5$. Noting that $N - 2 \geq \frac{N-2}{2} + \tau$, we find

$$
\begin{aligned}
|N(\phi)| \le& C\|\phi\|_*^2\Big(\sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{N-2}}\Big)^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^k \frac{1}{1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^2\\
&+C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
\le& C\|\phi\|_*^2\Big(\sum_{j=1}^k \frac{1}{1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}+C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
=& C\|\phi\|_*^2\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

So, we have proved that for $N\ge 4$,

$$
\|N(\phi)\|_{**}\le C\|\phi\|_*^{\min(2,2^*-1)}.
$$

□

Next, we estimate l_k.

Lemma 2.5. *Assume that* $r\in\big[\mu(1-\frac{r_0}{k}),\mu(1-\frac{r_1}{k})\big]$. *If* $N\ge 4$, *then*

$$
\|l_k\|_{**}\le C\Big(\frac{1}{\mu}\Big)^{\frac12+\sigma}.
$$

Proof. Define

$$
\Omega_j=\Big\{y:\ y=(y',y'')\in B_\mu(0),\Big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\Big\rangle\ge\cos\frac{\pi}{k}\Big\}.
$$

We have

$$
\begin{aligned}
l_k =& K\Big(\frac{|y|}{\mu}\Big)\Big(W_{r,\Lambda}^{2^*-1}-\sum_{j=1}^k\big(PU_{x_j,\Lambda}\big)^{2^*-1}\Big)+K\Big(\frac{|y|}{\mu}\Big)\Big(\sum_{j=1}^k\big(PU_{x_j,\Lambda}\big)^{2^*-1}-\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-1}\Big)\\
&+\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-1}\Big(K\Big(\frac{|y|}{\mu}\Big)-1\Big)\\
=:& J_0+J_1+J_2.
\end{aligned}
$$

From the symmetry, we can assume that $y\in\Omega_1$. Then,

$$
|y-x_j|\ge|y-x_1|,\quad \forall\, y\in\Omega_1.
$$

Firstly, we claim

$$\frac{1}{1+|y-x_j|} \le \frac{C}{|x_j-x_1|}, \quad \forall\, y \in \Omega_1,\ j \ne 1. \tag{2.18}$$

In fact, if $|y-x_1| \le \frac{1}{2}|x_1-x_j|$, then $|y-x_j| \ge \frac{1}{2}|x_1-x_j|$. If $|y-x_1| \ge \frac{1}{2}|x_1-x_j|$, then $|y-x_j| \ge |y-x_1| \ge \frac{1}{2}|x_1-x_j|$, since $y \in \Omega_1$.

For the estimate of J_0, we have

(2.19)
$$|J_0| \le C\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}} + C\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}.$$

Using (2.18), taking $1<\alpha\le N-2$, we obtain for any $y\in\Omega_1$,

$$\begin{aligned}&\frac{1}{(1+|y-x_1|)^4}\frac{1}{(1+|y-x_j|)^{N-2}}\\ \le& C\frac{1}{(1+|y-x_1|)^{N+2-\alpha}}\frac{1}{|x_j-x_1|^{\alpha}}, \quad j>1.\end{aligned} \tag{2.20}$$

Take $\alpha > \max(\frac{N-1}{2},1)$ satisfying $N+2-\alpha \ge \frac{N+2}{2}+\tau$. Then

$$\begin{aligned}&\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\\ \le& \frac{C}{(1+|y-x_1|)^{N+2-\alpha}}\Big(\frac{k}{\mu}\Big)^{\alpha} = \frac{C}{(1+|y-x_1|)^{N+2-\alpha}}\mu^{-\frac{\alpha}{N-1}}\\ \le& C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{1}{\mu}\Big)^{\frac{1}{2}+\sigma}.\end{aligned} \tag{2.21}$$

Using the Hölder inequality, we obtain

$$\begin{aligned}&\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}\\ \le& \sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})}}\Big)^{\frac{4}{N-2}}.\end{aligned}$$

Noting that $\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})>1$ if $N\ge 4$, we obtain

$$
\begin{aligned}
&\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}\\
\le &C\Big(\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})}}\Big)^{\frac{4}{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
\le &C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})\frac{4}{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \qquad (2.22)\\
= &C\Big(\frac{1}{\mu}\Big)^{\frac{N+2}{N-1}(\frac{1}{2}-\frac{\tau}{N+2})}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
= &C\Big(\frac{1}{\mu}\Big)^{\frac{1}{2}+\sigma}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}},
\end{aligned}
$$

since $\frac{N+2}{N-1}(\frac{1}{2}-\frac{\tau}{N+2})>\frac{1}{2}$. Thus, we have proved that if $N\ge 4$,

$$
\|J_0\|_{**}\le C\Big(\frac{1}{\mu}\Big)^{\frac{1}{2}+\sigma}.
$$

Now, we estimate J_1. Let $H(y,x)$ be the regular part of the Green function for $-\Delta$ in $B_1(0)$ with the zero boundary condition. Let $\bar{x}_j^*$ be the reflection point of $\bar{x}_j$ with respect to $\partial B_1(0)$. Then

$$
\frac{H(\bar{y},\bar{x}_j)}{\mu^{N-2}}=\frac{C}{\mu^{N-2}|\bar{y}-\bar{x}_j^*|^{N-2}}\le\frac{C}{(1+|y-x_j|)^{N-2}}.
$$

Take $t=1-\theta$ with $\theta>0$ small. Then using (A.1), we find

$$
\begin{aligned}
|J_1| \le & \sum_{j=1}^{k} \frac{C}{(1+|y-x_j|)^4} \frac{H(\bar y, \bar x_j)}{\mu^{N-2}} \\
\le & \sum_{j=1}^{k} \frac{C}{(1+|y-x_j|)^{4+t(N-2)}} \Big(\frac{H(\bar y, \bar x_j)}{\mu^{N-2}}\Big)^t \\
\le & C\Big(\frac{1}{\mu d}\Big)^{t(N-2)} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{4+t(N-2)}} \\
\le & C\Big(\frac{1}{\mu}\Big)^{t\frac{N-2}{N-1}} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{4+t(N-2)}} \\
\le & C\Big(\frac{1}{\mu}\Big)^{\frac{1}{2}+\sigma} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}},
\end{aligned}
\tag{2.23}
$$

since $t\frac{N-2}{N-1} > \frac{1}{2}$ for $N \ge 4$, $4 + t(N-2) \ge \frac{N+2}{2} + \tau$, and $d \ge \frac{r_0}{k}$.

Finally, we estimate J_2. For $y \in \Omega_1$, and $j > 1$, using (2.18), we have

$$
U_{x_j,\Lambda}^{2^*-1}(y) \le C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \frac{1}{|x_1-x_j|^{\frac{N+2}{2}-\tau}},
$$

which implies

$$
\begin{aligned}
& \Big|\sum_{j=2}^{k} \Big(K\big(\frac{|y|}{\mu}\big) - 1\Big) U_{x_j,\Lambda}^{2^*-1}\Big| \\
\le & C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \sum_{j=2}^{k} \frac{1}{|x_1-x_j|^{\frac{N+2}{2}-\tau}} \\
\le & C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau} \le C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \Big(\frac{1}{\mu}\Big)^{\frac{1}{2}+\sigma}.
\end{aligned}
\tag{2.24}
$$

For $y \in \Omega_1$ and $||y| - \mu| \ge \delta\mu$, where $\delta > 0$ is a fixed constant, then

$$
||y| - |x_1|| \ge ||y| - \mu| - ||x_1| - \mu| \ge \frac{1}{2}\delta\mu.
$$

As a result,

$$
\begin{aligned}
& \Big|U_{x_1,\Lambda}^{2^*-1}\Big(K\big(\frac{|y|}{\mu}\big) - 1\Big)\Big| \\
\le & C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \frac{1}{\mu^{\frac{N+2}{2}-\tau}}.
\end{aligned}
\tag{2.25}
$$

If $y\in\Omega_1$ and $||y|-\mu|\le\delta\mu$, then

$$
\begin{aligned}
&\Bigl|K\bigl(\frac{|y|}{\mu}\bigr)-1\Bigr|\le C\bigl|\frac{|y|}{\mu}-1\bigr|\\
\le&\frac{C}{\mu}\Bigl((||y|-|x_1||)+||x_1|-\mu|)\Bigr)\\
\le&\frac{C}{\mu}||y|-|x_1||+\frac{C}{k}\\
=&\frac{C}{\mu}||y|-|x_1||+\frac{C}{\mu^{\frac{N-2}{N-1}}}\le\frac{C}{\mu}||y|-|x_1||+\frac{C}{\mu^{\frac12+\sigma}},
\end{aligned}
$$

and

$$
||y|-|x_1||\le||y|-\mu|+|\mu-|x_1||\le2\delta\mu.
$$

But

$$
\begin{aligned}
&\frac{||y|-|x_1||}{\mu}\frac{1}{(1+|y-x_1|)^{N+2}}\\
=&\frac{C}{\mu^{\frac12+\sigma}}\frac{||y|-|x_1||^{\frac12-\sigma}}{(1+|y-x_1|)^{N+2}}\le\frac{C}{\mu^{\frac12+\sigma}}\frac{1}{(1+|y-x_1|)^{N+2-\frac12+\sigma}}\\
\le&\frac{C}{\mu^{\frac12+\sigma}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\Bigl|U_{x_1,\Lambda}^{2^*-1}\Bigl(K\bigl(\frac{|y|}{\mu}\bigr)-1\Bigr)\Bigr|\\
\le&\frac{C}{\mu^{\frac12+\sigma}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}},\quad ||y|-\mu|\le\delta\mu.
\end{aligned}
\tag{2.26}
$$

Combining (2.24), (2.25) and (2.26), we reach

$$
\|J_2\|_{**}\le C\bigl(\frac{1}{\mu}\bigr)^{\frac12+\sigma}.
$$

□

Now, we are ready to prove Proposition 2.3.

Proof of Proposition 2.3. Let us recall that

$$
\mu=k^{\frac{N-1}{N-2}},\quad N\ge4.
$$

Let

$$E=\Big\{u:u\in C(B_\mu(0))\cap H_s, \|u\|_*\le \big(\frac{1}{k}\big)^{\frac12}, \int_{B_\mu(0)} U_{x_i,\Lambda}^{2*-2}Z_{i,l}\phi=0,\ i=1,\cdots,k,\ l=1,2\Big\}.$$

Then, (2.16) is equivalent to

$$\phi=A(\phi)=:L_k(N(\phi))+L_k(l_k),$$

where L_k is defined in Proposition 2.2. We will prove that A is a contraction map from E to E.

We have

$$\begin{aligned}\|A(\phi)\|_*&\le C\|N(\phi)\|_{**}+C\|l_k\|_{**}\\ &\le C\|\phi\|_*^{\min(2^*-1,2)}+C\|l_k\|_{**}\le \frac{C}{k^{\frac12+\sigma}}\le\frac{1}{k^{\frac12}}.\end{aligned}\tag{2.27}$$

Thus, A maps E to E.

On the other hand,

$$\|A(\phi_1)-A(\phi_2)\|_*=\|L_k(N(\phi_1))-L_k(N(\phi_2))\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}.$$

If $N\ge 6$, then

$$|N'(t)|\le C|t|^{2^*-2}.$$

As a result,

$$\begin{aligned}&|N(\phi_1)-N(\phi_2)|\le C\big(|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2}\big)|\phi_1-\phi_2|\\ \le& C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\end{aligned}$$

As before, we have

$$\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\le C\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

So,

$$\begin{aligned}&\|A(\phi_1)-A(\phi_2)\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}\\ \le& C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\le\frac12\|\phi_1-\phi_2\|_*.\end{aligned}$$

Thus, A is a contraction map.

For $N=4,5$,

$$|N'(t)|\le CW_{r,\Lambda}^{\frac{6-N}{N-2}}|t|+C|t|^{2^*-2}.$$

So,

$$
\begin{aligned}
&|N(\phi_1)-N(\phi_2)|\\
\le& C\big(|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2}\big)|\phi_1-\phi_2|+C\big(|\phi_1|+|\phi_2|\big)W_{r,\Lambda}^{\frac{6-N}{N-2}}|\phi_1-\phi_2|\\
\le& C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\
&+C\big(\|\phi_1\|_*+\|\phi_2\|_*\big)\|\phi_1-\phi_2\|_*W_{r,\Lambda}^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2}\\
\le& C\big(\|\phi_1\|_*+\|\phi_2\|_*\big)\|\phi_1-\phi_2\|_*\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
\tag{2.28}
$$

Thus, A is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi\in E$, such that

$$\phi=A(\phi).$$

Moreover, it follows from Proposition 2.2 that

$$\|\phi\|_*\le C\|l_k\|_{**}+C\|N(\phi)\|_{**}\le C\|l_k\|_{**}+C\|\phi\|_*^{\min(2^*-1,2)},$$

which gives

$$\|\phi\|_*\le C\big(\frac{1}{\mu}\big)^{\frac12+\sigma},$$

if $N\ge 4$.

Finally, the estimate of c_t comes from (2.14). See also (2.10).

□

3. Proof of Theorem 1.2

Let

$$F(d,\Lambda)=I\big(W_{r,\Lambda}+\phi\big),$$

where $r=|x_1|$, $d=1-\frac{r}{\mu}$, ϕ is the function obtained in Proposition 2.3, and

$$I(u)=\frac12\int_{B_\mu(0)}|Du|^2-\frac{1}{2^*}\int_{B_\mu(0)}K\big(\frac{|y|}{\mu}\big)|u|^{2^*}.$$

Proposition 3.1. *If $N \geq 4$, then*

$$\begin{aligned} F(d,\Lambda) =& I(W_{r,\Lambda}) + O\Big(\frac{1}{\mu^{1+\sigma}}\Big) \\ =& k\Big(A + \frac{B_1 H(\bar{x}_1,\bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + B_2 K'(1) d - \sum_{i=2}^{k} \frac{B_1 G(\bar{x}_i,\bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big), \end{aligned}$$

where B_1 and B_2 are some positive constants, $A > 0$ is a constant, and $\sigma > 0$ is a small constant.

Proof. Since

$$\big\langle I'\big(W_{r,\Lambda} + \phi\big), \phi\big\rangle = 0, \quad \forall\, \phi \in E,$$

there is $t \in (0,1)$ such that

$$\begin{aligned} F(d,\Lambda) =& I(W_{r,\Lambda}) - \frac{1}{2} D^2 I\big(W_{r,\Lambda} + t\phi\big)(\phi,\phi) \\ =& I(W_{r,\Lambda}) - \frac{1}{2}\int_{B_\mu(0)} \big(|D\phi|^2 - (2^*-1) K\big(\frac{|y|}{\mu}\big)\big(W_{r,\Lambda} + t\phi\big)^{2^*-2}\phi^2\big) \\ =& I(W_{r,\Lambda}) + \frac{2^*-1}{2}\int_{B_\mu(0)} K\big(\frac{|y|}{\mu}\big)\Big(\big(W_{r,\Lambda} + t\phi\big)^{2^*-2} - W_{r,\Lambda}^{2^*-2}\Big)\phi^2 \\ & - \frac{1}{2}\int_{B_\mu(0)} \big(N(\phi) + l_k\big)\phi \\ =& I(W_{r,\Lambda}) + O\Big(\int_{B_\mu(0)} \big(|\phi|^{2^*} + |N(\phi)||\phi| + |l_k||\phi|\big)\Big). \end{aligned}$$

But

$$\begin{aligned} & \int_{B_\mu(0)} \big(|N(\phi)||\phi| + |l_k||\phi|\big) \\ \leq & C\Big(\|N(\phi)\|_{**} + \|l_k\|_{**}\Big)\|\phi\|_* \int_{B_\mu(0)} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}. \end{aligned}$$

Using Lemma B.1, for $N \geq 4$,

$$
\begin{aligned}
&\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\
=&\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N+2\tau}}+\sum_{j=1}^{k}\sum_{i\neq j}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\
\le&\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N+2\tau}}+C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N+\tau}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\tau}}\\
\le&C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N+\tau}}.
\end{aligned}
$$

Thus, we obtain

$$
\int_{B_\mu(0)}\big(|N(\phi)||\phi|+|l_k||\phi|\big)\le Ck\Big(\|N(\phi)\|_{**}+\|l_k\|_{**}\Big)\|\phi\|_*\le Ck\big(\frac{1}{\mu}\big)^{1+\sigma},\quad N\ge 4.
$$

On the other hand,

$$
\int_{B_\mu(0)}|\phi|^{2^*}\le C\|\phi\|_*^{2^*}\int_{B_\mu(0)}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}.
$$

But using (2.18), if $y\in\Omega_1$, and $N\ge 4$,

$$
\begin{aligned}
&\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\\
\le&C\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\tau}}\le C\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}}},
\end{aligned}
$$

Thus,

$$
\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}\le\frac{C}{(1+|y-x_1|)^{N}},\quad y\in\Omega_1,
$$

which gives

$$
\int_{B_\mu(0)}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}\le Ck\ln k.
$$

So, we have proved

$$\int_{B_\mu(0)} |\phi|^{2^*} \le Ck\ln k\|\phi\|_*^{2^*} \le Ck\ln k\big(\frac{1}{\mu}\big)^{2^*(\frac{1}{2}+\sigma)}, \quad N \ge 4.$$

□

Proposition 3.2. *We have*

$$\begin{aligned}&\frac{\partial F(d,\Lambda)}{\partial \Lambda}\\ =&kB_1(N-2)\Big(-\frac{H(\bar x_1,\bar x_1)}{\Lambda^{N-1}\mu^{N-2}}+\sum_{i=2}^{k}\frac{G(\bar x_i,\bar x_1)}{\Lambda^{N-1}\mu^{N-2}}+O\big(\frac{1}{\mu^{1+\sigma}}\big)\Big),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial F(d,\Lambda)}{\partial d}\\ =&k\Big(\frac{B_1\frac{\partial H(\bar x_1,\bar x_1)}{\partial d}}{\Lambda^{N-2}\mu^{N-2}}+B_2K'(1)-\sum_{i=2}^{k}\frac{B_1\frac{\partial G(\bar x_i,\bar x_1)}{\partial d}}{\Lambda^{N-1}\mu^{N-2}}+O\big(\frac{1}{\mu^{\sigma}}\big)\Big),\end{aligned}$$

if $N \ge 4$, where B_1 and B_2 are the same constants as in Proposition 3.1, $\sigma > 0$ is a small constant.

Proof. We estimate $\frac{\partial F(d,\Lambda)}{\partial\Lambda}$ first. We have

$$\begin{aligned}\frac{\partial F(d,\Lambda)}{\partial \Lambda}&=\Big\langle I'(W_{r,\Lambda}+\phi),\frac{\partial W_{r,\Lambda}}{\partial\Lambda}+\frac{\partial\phi}{\partial\Lambda}\Big\rangle\\ =&\Big\langle I'(W_{r,\Lambda}+\phi),\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\Big\rangle+\sum_{l=1}^{2}\sum_{i=1}^{k}c_l\Big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\frac{\partial\phi}{\partial\Lambda}\Big\rangle.\end{aligned}$$

But

$$\Big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\frac{\partial\phi}{\partial\Lambda}\Big\rangle=-\Big\langle\frac{\partial(U_{x_i,\Lambda}^{2^*-2}Z_{i,l})}{\partial\Lambda},\phi\Big\rangle$$

Thus, using Proposition 2.3,

$$\begin{aligned}&\Big|\sum_{i=1}^{k}c_l\Big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\frac{\partial\phi}{\partial\Lambda}\Big\rangle\Big|\\ \le&C|c_l|\|\phi\|_*\int_{\mathbb{R}^N}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{N+2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\\ \le&\frac{C}{\mu^{1+\sigma}}.\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^N} D(W_{r,\Lambda}+\phi)D\frac{\partial W_{r,\Lambda}}{\partial\Lambda}=\int_{\mathbb{R}^N} DW_{r,\Lambda}D\frac{\partial W_{r,\Lambda}}{\partial\Lambda},$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big)(W_{r,\Lambda}+\phi)^{2^*-1}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\\ =&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-1}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}+(2^*-1)\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\phi+O\Big(\int_{\mathbb{R}^N}|\phi|^{2^*}\Big).\end{aligned}$$

Moreover, from $\phi\in E$,

$$\begin{aligned}&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\phi\\ =&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big)\big(W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}-\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial\Lambda}\big)\phi+\sum_{j=1}^k\int_{\mathbb{R}^N}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial\Lambda}\phi\\ =&k\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)\big(W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}-\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial\Lambda}\big)\phi+k\int_{\mathbb{R}^N}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi,\end{aligned}$$

$$\begin{aligned}&\Big|\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)\big(W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}-\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial\Lambda}\big)\phi\Big|\\ \le&C\int_{\Omega_1}\Big(U_{x_1,\Lambda}^{2^*-2}\big(U_{x_1,\Lambda}-PU_{x_1,\Lambda}\big)+U_{x_1,\Lambda}^{2^*-2}\sum_{j=2}^k U_{x_j,\Lambda}+\sum_{j=2}^k U_{x_j,\Lambda}^{2^*-1}\Big)|\phi|\\ \le&\frac{C}{\mu^{1+\sigma}},\end{aligned}$$

and

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi\Big|\\ \le&\Big|\int_{||y|-\mu|\le\sqrt{\mu}}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi\Big|+\Big|\int_{||y|-\mu|\ge\sqrt{\mu}}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi\Big|\\ \le&\frac{C}{\mu^{1+\sigma}}.\end{aligned}$$

Thus, we have proved

$$\frac{\partial F(d,\Lambda)}{\partial\Lambda}=\frac{\partial I(W_{r,\Lambda})}{\partial\Lambda}+O\Big(\frac{1}{\mu^{1+\sigma}}\Big),$$

and the result follows from Proposition A.2.

Finally, noting that $\frac{\partial}{\partial d} = -\mu \frac{\partial}{\partial r}$, we can estimate $\frac{\partial F(d,\Lambda)}{\partial d}$ in a similar way. $\square$

Now, we estimate $H(\bar{x}_1, \bar{x}_1)$ and $G(\bar{x}_i, \bar{x}_1)$, $i \geq 2$. Let $\bar{x}_1^* = (\frac{1}{1-d}, 0, \cdots, 0)$ be the reflection of $\bar{x}_1$ with respect to the unit sphere. Then

$$H(y, \bar{x}_1) = \frac{1}{|y - \bar{x}_1^*|^{N-2}} \big(1 + O(d)\big).$$

So, we obtain

$$H(\bar{x}_1, \bar{x}_1) = \frac{1}{2^{N-2} d^{N-2}} \big(1 + O(d)\big).$$

On the other hand,

$$|\bar{x}_i - \bar{x}_1^*| = \sqrt{|\bar{x}_i - \bar{x}_1|^2 + 4d^2 - 4d|\bar{x}_i - \bar{x}_1| \cos\theta_i},$$

where θ_i is the angle between $\bar{x}_i - \bar{x}_1$ and $(1, 0, \cdots, 0)$. Thus, $\theta_i = \frac{\pi}{2} + \frac{(i-1)\pi}{2}$.

$$\begin{aligned} G(\bar{x}_i, \bar{x}_1) &= \frac{1}{|\bar{x}_i - \bar{x}_1|^{N-2}} - \frac{1}{|\bar{x}_i - \bar{x}_1^*|^{N-2}} \big(1 + O(d)\big) \\ &= \frac{1}{|\bar{x}_i - \bar{x}_1|^{N-2}} \Big(1 - \frac{1 + O(d)}{(1 + \frac{4d^2 + 4d|\bar{x}_i - \bar{x}_1| \sin\frac{(i-1)\pi}{2}}{|\bar{x}_i - \bar{x}_1|^2})^{\frac{N-2}{2}}}\Big) \end{aligned}$$

Since

$$|\bar{x}_i - x_1| = 2|x_1| \sin\frac{(i-1)\pi}{k}, \quad i = 2, \ldots, k,$$

using $dk \to c > 0$ and

$$0 < c' \leq \frac{\sin\frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}} \leq c'', \quad j = 2, \cdots, [\frac{k}{2}],$$

we obtain

$$\frac{a_0}{j^2} \leq \frac{4d^2 + 4d|\bar{x}_i - \bar{x}_1| \sin\frac{(i-1)\pi}{2}}{|\bar{x}_i - \bar{x}_1|^2} \leq \frac{a_1}{j^2}$$

for some constant $a_1 \geq a_0 > 0$, which implies

$$\frac{a_0'}{j^N} + O\big(\frac{d}{j^{N-2}}\big) \leq \frac{1}{k^{N-2}} G(\bar{x}_j, \bar{x}_1) \leq \frac{a_1'}{j^N} + O\big(\frac{d}{j^{N-2}}\big)$$

for some constant $a_1' \geq a_0' > 0$. So there is a constant $B_4 > 0$, such that

$$\sum_{j=2}^{k} G(\bar{x}_j, \bar{x}_1) = k^{N-2}\Big(\frac{B_4}{|\bar{x}_1|^{N-2}} + O\big(\frac{1}{k^{N-1}}\big) + O(d)\Big) = B_4 k^{N-2} + O(k^{N-2}d).$$

Thus, we obtain that there are positive constants A_1, A_2 and A_3, such that

$$F(d,\Lambda) = k\Big(A + \frac{A_1}{\Lambda^{N-2}\mu^{N-2}d^{N-2}} + A_2 d - \frac{A_3 k^{N-2}}{\Lambda^{N-2}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big), \tag{3.29}$$

$$\begin{aligned}&\frac{\partial F(d,\Lambda)}{\partial \Lambda}\\ =&k\Big(-\frac{A_1(N-2)}{\Lambda^{N-1}\mu^{N-2}d^{N-2}} + \frac{A_3(N-2)k^{N-2}}{\Lambda^{N-1}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big),\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}&\frac{\partial F(d,\Lambda)}{\partial d}\\ =&k\Big(-\frac{A_1(N-2)}{\Lambda^{N-2}\mu^{N-2}d^{N-1}} + A_2 + O\Big(\frac{1}{\mu^{\sigma}}\Big)\Big),\end{aligned} \tag{3.31}$$

Note that $d = 1 - \frac{r}{\mu}$, $\mu = k^{\frac{N-1}{N-2}}$. Define

$$D = \frac{d}{k}.$$

Then, from (3.30) and (3.31), $\frac{\partial F(d,\Lambda)}{\partial \Lambda} = 0$ and $\frac{\partial F(d,\Lambda)}{\partial d} = 0$ are equivalent to

$$-\frac{A_1(N-2)}{\Lambda^{N-1}D^{N-2}} + \frac{A_3(N-2)}{\Lambda^{N-1}} + O\Big(\frac{1}{\mu^{\sigma}}\Big) = 0, \tag{3.32}$$

and

$$-\frac{A_1(N-2)}{\Lambda^{N-2}D^{N-1}} + A_2 + O\Big(\frac{1}{\mu^{\sigma}}\Big) = 0, \tag{3.33}$$

respectively.

Proof of Theorem 1.2. Let

$$f_1(D,\Lambda) = -\frac{A_1(N-2)}{\Lambda^{N-1}D^{N-2}} + \frac{A_3(N-2)}{\Lambda^{N-1}},$$

and

$$f_2(D,\Lambda) = -\frac{A_1(N-2)}{\Lambda^{N-2}D^{N-1}} + A_2.$$

Then, $f_1 = 0$ and $f_2 = 0$ have a unique solution

$$D_0 = \Big(\frac{A_1}{A_3}\Big)^{\frac{1}{N-2}}, \quad \Lambda_0 = \Big(\frac{A_1(N-2)}{A_2 D_0^{N-1}}\Big)^{\frac{1}{N-2}}.$$

On the other hand, it is easy to see that

$$\frac{\partial f_1(D_0,\Lambda_0)}{\partial \Lambda} = 0, \quad \frac{\partial f_2(D_0,\Lambda_0)}{\partial D} > 0,$$

and

$$\frac{\partial f_1(D_0,\Lambda_0)}{\partial D} = \frac{\partial f_2(D_0,\Lambda_0)}{\partial \Lambda} > 0.$$

Thus the linear operator of $f_1 = 0$ and $f_2 = 0$ at (D_0, Λ_0) is invertible. As a result, (3.32) and (3.33) have a solution near (D_0, Λ_0). □

Appendix A. Energy Expansion

In all of the appendixes, we always assume that

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and $r \in \big[\mu(1-\frac{r_0}{k}), \mu(1-\frac{r_1}{k})\big]$.

Let

$$\bar{x}_j = \frac{1}{\mu} x_j.$$

Let $G(y,z)$ be the Green function of $-\Delta$ in $B_1(0)$ with the Dirichlet boundary condition. Let $H(y,z)$ be the regular part of the Green function.

Let recall that

$$\mu = k^{\frac{N-1}{N-2}},$$

$$I(u) = \frac{1}{2}\int_{B_\mu(0)} |Du|^2 - \frac{1}{2^*}\int_{B_\mu(0)} K\Big(\frac{|y|}{\mu}\Big)|u|^{2^*},$$

$$U_{x_j,\Lambda}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-x_j|^2)^{\frac{N-2}{2}}},$$

and

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} PU_{x_j,\Lambda}(y),$$

where $PU_{x,\Lambda}$ is the solution of (1.5). It is well known that

$$U_{x_j,\Lambda}(y) - PU_{x_j,\Lambda}(y) = \frac{H(\bar{y}, \bar{x})}{\mu^{N-2}} + O\Big(\frac{1}{d^N\mu^N}\Big), \tag{A.1}$$

where $d = 1 - |\bar{x}| = 1 - \frac{|x|}{\mu}$.

In this section, we will calculate $I(W_{r,\Lambda})$.

Proposition A.1. *We have*

$$I(W_{r,\Lambda}) = k\Big(A + \frac{B_1 H(\bar{x}_1, \bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + B_2 K'(1)d - \sum_{i=2}^{k} \frac{B_1 G(\bar{x}_i, \bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + O\big(\frac{1}{\mu^{1+\sigma}}\big)\Big),$$

where B_1 and B_2 are some positive constants, $A > 0$ is a constant.

Proof. By using the symmetry, we have

$$\begin{aligned}
&\int_{B_\mu(0)} |DW_{r,\Lambda}|^2 = \sum_{j=1}^{k}\sum_{i=1}^{k}\int_{B_\mu(0)} U_{x_j,\Lambda}^{2^*-1} PU_{x_i,\Lambda} \\
=&k\Big(\int_{B_\mu(0)} U_{0,1}^{2^*} - \int_{B_\mu(0)} U_{x_1,\Lambda}^{2^*-1}\big(U_{x_1,\Lambda} - PU_{x_1,\Lambda}\big) + \sum_{i=2}^{k}\int_{B_\mu(0)} U_{x_1,\Lambda}^{2^*-1} PU_{x_i,\Lambda}\Big) \\
=&k\Big(\int_{\mathbb{R}^N} U^{2^*} - \frac{\bar{B}_1 H(\bar{x}_1, \bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + \frac{\bar{B}_1 \sum_{i=2}^{k} G(\bar{x}_i, \bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + O\big(\frac{1}{\mu^{\frac{N}{N-1}}}\big)\Big),
\end{aligned}$$

where $\bar{B}_1 = \int_{\mathbb{R}^N} U^{2^*-1}$.

Let

$$\Omega_j = \Big\{y:\ y = (y', y'') \in B_\mu(0),\ \big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\big\rangle \ge \cos\frac{\pi}{k}\Big\}.$$

Then,

$$\begin{aligned}
&\int_{B_\mu(0)} K\big(\frac{|y|}{\mu}\big)|W_{r,\Lambda}|^{2^*} = k\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)|W_{r,\Lambda}|^{2^*} \\
=&k\Big(\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)(PU_{x_1,\Lambda})^{2^*} - 2^*\int_{\Omega_1}\sum_{i=2}^{k}(PU_{x_1,\Lambda})^{2^*-1}PU_{x_i,\Lambda} \\
&+ O\Big(\int_{\Omega_1}\big|K\big(\frac{|y|}{\mu}\big) - 1\big|\sum_{i=2}^{k} U_{x_1,\Lambda}^{2^*-1}U_{x_i,\Lambda} + \int_{\Omega_1} U_{x_1,\Lambda}^{2^*/2}\big(\sum_{i=2}^{k} U_{x_i,\Lambda}\big)^{2^*/2}\Big)\Big).
\end{aligned}$$

Note that for $y \in \Omega_1$, $|y - x_i| \ge |y - x_1|$. Using (2.18), we find that for any $t \in (1, N-2)$,

$$\sum_{i=2}^{k} U_{x_i,\Lambda} \le \frac{C}{(1+|y-x_1|)^{N-2-t}} \sum_{i=2}^{k} \frac{1}{|x_i-x_1|^t}.$$

If we take the constant t close to $N-2$, then

$$\int_{\Omega_1} U_{x_1,\Lambda}^{2^*/2} \big(\sum_{i=2}^{k} U_{x_i,\Lambda}\big)^{2^*/2} = O\Big(\big(\frac{k}{\mu}\big)^{t\frac{N}{N-2}}\Big) = O\big(\frac{1}{\mu^{1+\sigma}}\big).$$

On the other hand, it is easy to show

$$\int_{\Omega_1} \sum_{i=2}^{k} (PU_{x_1,\Lambda})^{2^*-1} PU_{x_i,\Lambda} = \frac{\bar B_2 G(\bar x_i, \bar x_1)}{\Lambda^{N-2}\mu^{N-2}} + O\big(\frac{k^N}{\mu^N}\big) = \frac{\bar B_2 G(\bar x_i, \bar x_1)}{\Lambda^{N-2}\mu^{N-2}} + O\big(\frac{1}{\mu^{1+\sigma}}\big),$$

and

$$\int_{\Omega_1} \big|K\big(\frac{|y|}{\mu}\big) - 1\big| \sum_{i=2}^{k} U_{x_1,\Lambda}^{2^*-1} U_{x_i,\Lambda} = O\big(\frac{1}{\mu^{1+\sigma}}\big).$$

Moreover,

$$\begin{aligned}
&\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)(PU_{x_1,\Lambda})^{2^*} \\
=&\int_{\Omega_1} (PU_{x_1,\Lambda})^{2^*} + \int_{\Omega_1} \Big(K\big(\frac{|y|}{\mu}\big) - 1\Big) U_{x_1,\Lambda}^{2^*} + O\Big(\int_{\Omega_1} \big|K\big(\frac{|y|}{\mu}\big) - 1\big| U_{x_1,\Lambda}^{2^*-1} \frac{H(y,x_1)}{\mu^{N-2}}\Big) \\
=&\int_{\mathbb{R}^N} U^{2^*} - 2^* \frac{\bar B_1 H(\bar x_1, \bar x_1)}{\Lambda^{N-2}\mu^{N-2}} + \int_{\Omega_1} \Big(K\big(\frac{|y|}{\mu}\big) - 1\Big) U_{x_1,\Lambda}^{2^*} + O\big(\frac{1}{\mu^{1+\sigma}}\big).
\end{aligned}$$

But

$$\begin{aligned}
&\int_{\Omega_1} \Big(K\big(\frac{|y|}{\mu}\big) - 1\Big) U_{x_1,\Lambda}^{2^*} = \big(K(|\bar x_1|) - 1\big) \int_{\mathbb{R}^N} U^{2^*} + O\big(\frac{1}{\mu^2}\big) \\
&= - K'(1) d \int_{\mathbb{R}^N} U^{2^*} + O(d^2) = -K'(1) d \int_{\mathbb{R}^N} U^{2^*} + O\big(\frac{1}{\mu^{1+\sigma}}\big).
\end{aligned}$$

Thus, we have proved

$$\begin{aligned}
&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big) |W_{r,\Lambda}|^{2^*} \\
=&k\Big(\int_{\mathbb{R}^N} U^{2^*} - K'(1) d \int_{\mathbb{R}^N} U^{2^*} - 2^* \frac{\bar B_1 H(\bar x_1, \bar x_1)}{\Lambda^{N-2}\mu^{N-2}} \\
&\quad + 2^* \sum_{i=2}^{k} \frac{\bar B_1 G(\bar x_i, \bar x_1)}{\Lambda^{N-2}\mu^{N-2}} + O\big(\frac{1}{\mu^{1+\sigma}}\big)\Big).
\end{aligned}$$

□

We also need to calculate $\frac{\partial I(W_{r,\Lambda})}{\partial \Lambda}$ and $\frac{\partial I(W_{r,\Lambda})}{\partial r}$.

Proposition A.2. *We have*

$$\frac{\partial I(W_{r,\Lambda})}{\partial \Lambda} = k(N-2)B_1\Big(-\frac{H(\bar{x}_1,\bar{x}_1)}{\Lambda^{N-1}\mu^{N-2}} + \sum_{i=2}^{k}\frac{G(\bar{x}_1,\bar{x}_i)}{\Lambda^{N-1}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big),$$

and

$$\frac{\partial I(W_{r,\Lambda})}{\partial r} = k\Big(B_1\frac{\frac{\partial H(\bar{x}_1,\bar{x}_1)}{\partial r}}{\Lambda^{N-2}\mu^{N-2}} - B_2K'(1)\frac{1}{\mu} - \sum_{i=2}^{k}\frac{B_1\frac{\partial G(\bar{x}_1,\bar{x}_i)}{\partial r}}{\Lambda^{N-1}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big),$$

where B_1 is same positive constant in Proposition A.1

Proof. We use ∂ to denote either $\frac{\partial}{\partial\Lambda}$ or $\frac{\partial}{\partial r}$.

Using the symmetry, we have

$$\begin{aligned}\partial I(W_{r,\Lambda}) =& k\Big((2^*-1)\sum_{i=2}^{k}\int_{\mathbb{R}^N} U_{x_1,\Lambda}^{2^*-2}\partial(U_{x_1,\Lambda})PU_{x_i,\Lambda} \\ &- \int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-1}\partial W_{r,\Lambda}\Big).\end{aligned}$$

Then the proof of this proposition is similar to the proof of Proposition A.1. So we just omit it.

□

Appendix B. Basic Estimates

In this section, we list some lemmas, whose proof can be found in [15].

For each fixed i and j, $i \neq j$, consider the following function

$$g_{ij}(y) = \frac{1}{(1+|y-x_j|)^{\alpha}}\frac{1}{(1+|y-x_i|)^{\beta}}, \tag{B.2}$$

where $\alpha \geq 1$ and $\beta \geq 1$ are two constants.

Lemma B.1. *For any constant $0 < \sigma \leq \min(\alpha,\beta)$, there is a constant $C > 0$, such that*

$$g_{ij}(y) \leq \frac{C}{|x_i-x_j|^{\sigma}}\Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\Big).$$

Lemma B.2. *For any constant $0 < \sigma < N-2$, there is a constant $C > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}}\, dz \le \frac{C}{(1+|y|)^{\sigma}}.$$

Let recall that

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} PU_{x_j,\Lambda}.$$

Lemma B.3. *Suppose that $N \ge 4$. Then there is a small $\theta > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\, dz$$
$$\le C \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.$$

Proof. The proof can be found in [15]. We just need to use

$$W_{r,\Lambda}(y) \le C \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{N-2}}.$$

□

References

[1] J. Byeon and Z.Q. Wang, On the Hénon equation: asymptotic profile of ground states. I. *Ann. Inst. H. Poincar Anal. Non Linaire* 23 (2006), 803–828.
[2] J. Byeon and Z.Q. Wang, On the Hénon equation: asymptotic profile of ground states. II. *J. Differential Equations* 216 (2005), 78–108.
[3] D. Cao and S.Peng, The asymptotic behaviour of the ground state solutions for Hénon equation. *J. Math. Anal. Appl.* 278 (2003), 1–17.
[4] D. Cao, S.Peng and S.Yan, Asymptotic behaviour of ground state solutions for the Hénon equation. *IMA J. Appl. Math.* 74 (2009), 468–480.
[5] M. del Pino, P. Felmer and M.Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem. *Calc. Var. Partial Differential Equations* 16 (2003), 113–145.
[6] O. Druet and P. Laurain, private communications.
[7] M. Henon, Numerical experiments on the stability of spherical stellar systems, *Astronom. Astrophys.* 34(1973), 229-237.
[8] N. Hirano, Existence of positive solutions for the Henon equation involving critical Sobolev exponent, *J. Diff. Eqns.* 247(2009), 1311-1333.
[9] W.M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, *Indiana Univ. Math. J.* 31(1982), 801-807.
[10] S. Peng, Multiple boundary concentrating solutions to Dirichlet problem of Hénon equation. *Acta Math. Appl. Sin. Engl. Ser.* 22 (2006), 137–162.

[11] A. Pistoia and E. Serra, Multi-peak solutions for the Hénon equation with slightly subcritical growth. *Math. Z.* 256 (2007), 75–97.

[12] E. Serra, Non-radial positive solutions for the Henon equation with critical growth, *Cal. Var. PDE* 23(3) (2005), 301-326.

[13] D. Smets, J. Su and M. Willem, Non-radial ground states for the Henon equation, *Comm. Contemp. Math.* 4(3)(2002) 467-480.

[14] D. Smets and M. Willem, Partial symmetry and asymptotic behavior for some elliptic variational problems. *Calc. Var. Partial Differential Equations* 18 (2003), 57–75.

[15] J. Wei and S. Yan, Infinitely many solutions for the prescribed scalar curvature problem on S^N. *J. Funct. Anal.* 258(2010), no.9, 3048-3081.

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