Lazer-McKenna conjecture: the critical case

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Dedicated to Professor Dancer on the occasion of his 60^{th} birthday

Abstract
We consider an elliptic problem of Ambrosetti-Prodi type involving critical Sobolev exponent on a bounded smooth domain of dimension six or higher. By constructing solutions with many sharp peaks near the boundary of the domain, but not on the boundary, we prove that the number of solutions for this problem is unbounded as the parameter tends to infinity, thereby proving the Lazer-McKenna conjecture in the critical case.

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1. Introduction

In this paper, we consider the following elliptic problem involving critical Sobolev exponent:

$$\begin{cases} -\Delta u = u_+^{2^*-1} + \lambda u - \bar{s}\varphi_1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with C^2 boundary, $N \geq 3$, $\bar{s}$ and $\lambda > 0$ are positive parameters, $\varphi_1 > 0$ is the eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition corresponding to the first eigenvalue λ_1, $u_+ = \max(u, 0)$, and $2^* = 2N/(N-2)$.

Problem (1.1) belongs to the following elliptic problem of Ambrosetti-Prodi type

$$\begin{cases} -\Delta u = g(u) - \bar{s}\varphi_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $g(t)$ satisfies $\lim_{t\to-\infty} \frac{g(t)}{t} = \nu < \lambda_1$, $\lim_{t\to+\infty} \frac{g(t)}{t} = \mu > \lambda_1$. Here $\mu = +\infty$ and $\nu = -\infty$ are allowed. It is well-known that the location of μ, ν with respect to the spectrum

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of $(-\Delta, H_0^1(\Omega))$ plays an important role in the multiplicity of solutions for problem (1.1). See for example [1, 6, 7, 14, 15, 16, 20, 21, 22, 23, 28, 29, 30, 31]. In the early 1980s, Lazer and McKenna conjectured that if $\mu = +\infty$ and $g(t)$ does not grow too fast at infinity, (1.2) has an unbounded number of solutions as $\bar{s} \to +\infty$. See [21].

There is no result relating to the Lazer-McKenna conjecture in the partial differential equation setting until recently. Firstly, by using a partially numerical method, Breuer, McKenna and Plum showed in [5] that (1.2) has at least four solutions if $g(t) = t^2$ and Ω is the unit square in R^2. Secondly, Dancer and the second author proved in [8] that the Lazer-McKenna conjecture is true if $g(t) = |t|^p$, where $p \in (1, +\infty)$ for $N = 2$, $p \in (1, (N+2)/(N-2))$ for $N \geq 3$. We remark that for the nonlinearity $g(t) = |t|^p$, $\nu = -\infty$ and $\mu = +\infty$. In the case that ν is finite, it is shown in [9] that the Lazer-McKenna conjecture is also true if $g(t) = t_+^p + \lambda t$, $\lambda \in (-\infty, \lambda_1)$, $N \geq 3$ and $p \in (1, (N+2)/(N-2))$. In two dimensional case, Del Pino and Munoz [13] showed that the Lazer-McKenna conjecture holds if $g(t)$ is an exponential nonlinearity.

In this paper, we treat the critical case and prove the Lazer-McKenna conjecture for dimensions $N \geq 6$. The nonlinearity $t_+^{2^*-1}$ is of critical growth in view of Sobolev embedding. We assume that λ and $\bar{s}$ satisfy one of the following conditions:

(Λ_1) $\lambda \in (0, \lambda_1)$ and $\bar{s} > 0$;
(Λ_2) $\lambda \in (\lambda_i, \lambda_{i+1})$ for some $i \geq 1$, and $\bar{s} < 0$.

The main result of this paper is the following:

Theorem 1.1. *Assume that $N \geq 6$, λ and $\bar{s}$ satisfy either (Λ_1) or (Λ_2). Then, the number of the solutions for (1.1) is unbounded as $|\bar{s}| \to +\infty$.*

It is easy to see that (1.1) has a negative solution

$$\underline{u}_{\bar{s}} = -\frac{\bar{s}}{\lambda_1 - \lambda}\varphi_1,$$

if (Λ_1) or (Λ_2) holds. Moreover, if $\underline{u}_{\bar{s}} + u$ is a solution of (1.1), then u satisfies

$$\begin{cases} -\Delta u = \left(u - s\varphi_1\right)_+^{2^*-1} + \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $s = \frac{\bar{s}}{\lambda_1 - \lambda} > 0$. The functional corresponding to (1.3) is

$$I_s(u) = \frac{1}{2}\int_\Omega \left(|Du|^2 - \lambda u^2\right) dy - \frac{1}{2^*}\int_\Omega \left(u - s\varphi_1\right)_+^{2^*} dy, \quad u \in H_0^1(\Omega).$$

To prove Theorem 1.1, we only need to prove that the number of solutions for (1.3) is unbounded as $s \to +\infty$.

For any $\bar{x} \in R^N$, $\bar{\mu} > 0$, denote

$$U_{\bar{x},\bar{\mu}}(y) = \frac{c_0 \bar{\mu}^{(N-2)/2}}{(1 + \bar{\mu}^2|y - \bar{x}|^2)^{(N-2)/2}}, \tag{1.4}$$

where $c_0 > 0$ is the constant such that $U_{\bar{x},\bar{\mu}}$ satisfies $-\Delta U_{\bar{x},\bar{\mu}} = U_{\bar{x},\bar{\mu}}^{2^*-1}$. In this paper, we will use the following notation: $U = U_{0,1}$.

Let $PU_{\bar{x},\bar{\mu}}$ be the solution of

$$\begin{cases} \Delta PU_{\bar{x},\bar{\mu}} = \Delta U_{\bar{x},\bar{\mu}}, & \text{in } \Omega, \\ PU_{\bar{x},\bar{\mu}} = 0, & \text{on } \partial\Omega. \end{cases}$$

For any $u, v \in H_0^1(\Omega)$, we define

$$\langle u, v \rangle = \int_\Omega DuDv \, dy, \quad \|u\| = \langle u, u \rangle^{1/2}.$$

We have the following existence result for problem (1.3):

Theorem 1.2. *Suppose that $N \geq 6$, λ and $\bar{s}$ satisfy either (Λ_1) or (Λ_2). Let Γ be any connected component of $\partial\Omega$. Then, for any integer $k \geq 1$, there is an $s_k > 0$, depending on k, such that for any $s \geq s_k$, (1.3) has a solution of the form*

$$u_s = \sum_{j=1}^{k} PU_{x_{s,j},\mu_{s,j}} + \omega_{s,k},$$

satisfying that as $s \to +\infty$,

(i) $x_{s,j} \to x_j \in \Gamma$ *with* $-\frac{\partial \varphi_1(x_j)}{\partial n} = \max_{y\in\Gamma}\big(-\frac{\partial \varphi_1(y)}{\partial n}\big)$, $j = 1, \cdots, k$;
(ii) $s^{2(N-2)/(N^2-6N+4)}|x_{s,i} - x_{s,j}| \to +\infty$, $i \neq j$;
(iii) $\mu_{s,j} s^{-2(N-2)/(N^2-6N+4)} \to \bar{t} > 0$, $j = 1, \cdots, k$;
(iv) $d(x_{s,j}, \partial\Omega) s^{2(N-4)/(N^2-6N+4)} \to \bar{b} > 0$, $j = 1, \cdots, k$;
(v) $\omega_{s,k} \in H_0^1(\Omega)$, $\|\omega_{s,k}\| \to 0$,

where n is the outward unit normal of $\partial\Omega$ at $y \in \partial\Omega$, $\bar{t}$ and $\bar{b}$ are defined in (1.6) and (1.7) respectively.

It is easy to see that Theorem 1.1 follows directly from Theorem 1.2.

De Figueiredo and Yang [17] proved that (1.3) has at least two solutions if $N \geq 7$ and either (Λ_1) or (Λ_2) holds. In [24], it is proved that if $N \geq 7$ and (Λ_1) or (Λ_2) holds, then for any integer $k > 0$, (1.3) has a k-peak solution, which blows up near the maximum points of the function $\varphi_1(y)$ in Ω. On the other hand, it is proved in [25] that if $N \geq 6$ and (Λ_1) holds, the mountain pass solution exists and the mountain pass solution is a single peak solution, which blows up near a point $x_0 \in \partial\Omega$ with $-\frac{\partial \varphi_1(x_0)}{\partial n} = \min_{z\in\partial\Omega}\big(-\frac{\partial \varphi_1(z)}{\partial n}\big)$. The solutions obtained in Theorem 1.2 have several peaks clustering near a boundary point which is a maximum of the function $-\frac{\partial \varphi_1(z)}{\partial n}$. Therefore, the solutions constructed in this paper are different from those in [24, 25]. As far as the authors know, no such type of solutions have been obtained for Dirichlet elliptic problems involving critical nonlinearities. The readers can refer to [2, 4, 10, 11, 12, 18, 19, 27, 32] for results on the existence of multipeak solutions for other problems involving critical Sobolev exponent.

Our calculations also show that in lower dimensions $N = 3, 4, 5$, (1.3) has no solution concentrating at some points of the domain Ω. Indeed, if $N = 3, 4, 5$, μ_i^{-3} is smaller than the other two terms in the right hand side of (B.5). Therefore, there is no balance for the concentration rate μ_i.

Before we close this section, let us outline the proof of Theorem 1.2.

We first reduce the proof of Theorem 1.2 to a finite dimensional problem. To achieve this goal, for any integer $k > 0$, $x_i \in \Omega$, $i = 1, \cdots, k$, $\mu_i \in R^1_+$, $i = 1, \cdots, k$, we define

$$E_{x,\mu,k} = \Big\{\omega : \ \omega \in H^1_0(\Omega), \big\langle \omega, \frac{\partial (PU_{x_j,\mu_j})}{\partial \mu_j}\big\rangle = \big\langle \omega, \frac{\partial (PU_{x_j,\mu_j})}{\partial x_{j,h}}\big\rangle = 0,$$
$$h = 1, \cdots, N, \ j = 1, \cdots, k.\Big\}$$

Denote

$$\bar{a} = \max_{y \in \Gamma}\big(-\frac{\partial \varphi_1(y)}{\partial n}\big), \tag{1.5}$$

$$\bar{t} = \Big(\frac{(N-2)c_0\tilde{c}}{2}\big(\frac{NB_2}{4\lambda B_1}\big)^{N-1}\Big)^{2/(N^2-6N+4)} \bar{a}^{2(N-2)/(N^2-6N+4)}, \tag{1.6}$$

and

$$\bar{b} = \frac{4\lambda B_1}{NB_2 \bar{a}} \bar{t}^{(N-6)/2}, \tag{1.7}$$

where $B_1 = \frac{1}{2}\int_{R^N} U^2 \, dy$, $B_2 = \int_{R^N} U^{2^*-1} \, dy$, c_0 and $\tilde{c}$ are the constants given by (1.4) and (A.3) respectively.

For any $x_i \in \Omega$, which is close to $\partial\Omega$, there is a unique $\bar{x}_i \in \partial\Omega$, such that

$$x_i = \bar{x}_i - d(x_i, \partial\Omega) n(\bar{x}_i).$$

Let

$$\begin{aligned} V_k = \Big\{\bar{x} : \bar{x}_i \in \Gamma, \ -\frac{\partial \varphi_1(\bar{x}_i)}{\partial n} \ge \bar{a} - s^{-\theta}, \\ |\bar{x}_i - \bar{x}_j| \ge s^{-2(N-4)/(N^2-6N+4)+\theta/N}, \ i, j = 1, \cdots, k, \ i \ne j\Big\}, \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} W_k = \Big\{(d, \mu) : \mu_i \in [(\bar{t} - Ls^{-\theta}) s^{2(N-2)/(N^2-6N+4)}, (\bar{t} + Ls^{-\theta}) s^{2(N-2)/(N^2-6N+4)}], \\ d_i \in [(\bar{b} - Ls^{-\theta}) s^{-2(N-4)/(N^2-6N+4)}, (\bar{b} + Ls^{-\theta}) s^{-2(N-4)/(N^2-6N+4)}]\Big\}, \end{aligned} \tag{1.9}$$

where $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_k)$, $\mu = (\mu_1, \cdots, \mu_k)$, $d = (d_1, \cdots, d_k)$, $L > 0$ is a large constant, and $\theta > 0$ is a fixed small constant.

Let

$$M_k = V_k \times W_k. \tag{1.10}$$

We will prove that there exists a C^1 map $\omega_{s,x,\mu}$ from M_k to $H^1_0(\Omega)$, such that $\omega_{s,x,\mu} \in E_{x,\mu,k}$, and

$$\frac{\partial J_s(x,\mu,\omega_{s,x,\mu})}{\partial\omega}=\sum_{j=1}^k A_j\frac{\partial(PU_{x_j,\mu_j})}{\partial\mu_j}+\sum_{j=1}^k\sum_{h=1}^N B_{jh}\frac{\partial(PU_{x_j,\mu_j})}{\partial x_{j,h}},$$

for some constants A_j and B_{jh}, where

$$J_s(x,\mu,\omega)=I_s\Big(\sum_{j=1}^k PU_{x_j,\mu_j}+\omega\Big).$$

Here we regard $x_i=\bar{x}_i-d_in(\bar{x}_i)$.

To show that $\sum_{j=1}^k PU_{x_j,\mu_j}+\omega_{s,x,\mu}$ is actually a solution of (1.3), we need to find a $(x_s,\mu_s)\in M_k$, such that the corresponding constants A_j and B_{jh} are all equal to zero. It is well known that if $(x_s,\mu_s)\in M_k$ is a critical point of the function

$$K(\bar{x},d,\mu)=:J_s(x,\mu,\omega_{s,x,\mu}),\quad(\bar{x},d,\mu)\in M_k,$$

where $x_i=\bar{x}_i-d_in(\bar{x}_i)$, $\bar{x}=(\bar{x}_1,\cdots,\bar{x}_k)$, $d=(d_1,\cdots,d_k)$, then the corresponding constants A_j and B_{jh} are all equal to zero. See for example [10] and [26].

In [8, 9], the nonlinearities are subcritical and the critical points for the reduced finite dimensional problems are obtained by using a maximization procedure. These techniques do not work here, because by (3.1), we can expect that $K(\bar{x},d,\mu)$ has a *saddle point* in M_k such that $K(\bar{x},d,\mu)$ attains the minimum in both the d_i directions and the μ_i directions; but $K(\bar{x},d,\mu)$ attains the maximum in the $\bar{x}_i$ directions. In view of the above observation, we will use a *min-max procedure* to find a critical point for $K(\bar{x},d,\mu)$. To achieve this goal, we need to carefully analyze the gradient flow of $K(\bar{x},d,\mu)$.

In section 2, we will reduce the problem of finding peak solutions for (1.3) to a finite dimensional problem. We will prove the main theorem in section 3. We put the lengthy calculations needed in the expansion of the energy and its derivatives in the appendices.

2. The reduction

In this section, we will reduce the problem of finding a k-peak solution for (1.3) to a finite dimension problem. We always assume $N\ge 6$, and $(x,\mu)\in M_k$.

Proposition 2.1. *There is an $s_k>0$, such that for each $s\ge s_k$, there exists a C^1-map $\omega_{s,x,\mu}$: $M_k\to H_0^1(\Omega)$, such that $\omega_{s,x,\mu}\in E_{x,\mu,k}$, and*

$$\frac{\partial J_s(x,\mu,\omega_{s,x,\mu})}{\partial\omega}=\sum_{i=1}^k A_i\frac{\partial(PU_{x,\mu_i})}{\partial\mu_i}+\sum_{i=1}^k\sum_{h=1}^N B_{ih}\frac{\partial(PU_{x_i,\mu_i})}{\partial x_{i,h}},\tag{2.1}$$

for some constants A_i and B_{ih}. Moreover, we have

$$\|\omega_{s,x,\mu}\|=O\Big(\frac{1}{s^{2(1+\sigma)(N-2)/(N^2-6N+4)}}\Big),$$

where $\sigma>0$ is a constant.

Proof. For each $(x,\mu)\in M_k$, we expand $J_s(x,\mu,\omega)$ at $\omega=0$ as follows:

$$J_s(x,\mu,\omega)=J_s(x,\mu,0)+\big\langle l_{s,x,\mu},\omega\big\rangle+\frac{1}{2}\big\langle Q_{s,x,\mu}\omega,\omega\big\rangle+R_{s,x,\mu}(\omega),$$

where $l_{s,x,\mu}\in E_{x,\mu,k}$ satisfying

$$\begin{aligned}\big\langle l_{s,x,\mu},\omega\big\rangle=&\int_\Omega D\Big(\sum_{j=1}^k PU_{x_j,\mu_j}\Big)D\omega\,dy-\lambda\int_\Omega\sum_{j=1}^k PU_{x_j,\mu_j}\omega\,dy\\&-\int_\Omega\Big(\sum_{j=1}^k PU_{x_j,\mu_j}-s\varphi_1\Big)_+^{2^*-1}\omega\,dy,\quad\forall\omega\in E_{x,\mu,k},\end{aligned}\tag{2.2}$$

and $Q_{s,x,\mu}$ is a bounded linear map from $E_{x,\mu,k}$ to $E_{x,\mu,k}$, satisfying

$$\begin{aligned}\big\langle Q_{s,x,\mu}\omega,\eta\big\rangle=&\int_\Omega D\omega D\eta\,dy-\lambda\int_\Omega\omega\eta\,dy\\&-(2^*-1)\int_\Omega\Big(\sum_{j=1}^k PU_{x_j,\mu_j}-s\varphi_1\Big)^{2^*-2}\omega\eta\,dy,\quad\omega,\ \eta\in E_{x,\mu,k},\end{aligned}\tag{2.3}$$

and $R_{s,x,\mu}(\omega)$ collects all the other terms, satisfying

$$R^{(j)}_{s,x,\mu}(\omega)=O(\|\omega\|^{2^*-j}),\quad j=0,1,2.$$

Thus, to find a critical point for $J_s(x,\mu,\omega)$ in $E_{x,\mu,k}$ is equivalent to solving

$$l_{s,x,\mu}+Q_{s,x,\mu}\omega+R'_{s,x,\mu}(\omega)=0.\tag{2.4}$$

By Lemma 2.3 below, we see that $Q_{s,x,\mu}$ is invertible in $E_{x,\mu,k}$, and there is a constant $C>0$, such that $\|Q^{-1}_{s,x,\mu}\|\le C$. It follows from the implicit function theory that there is a $\omega_{s,x,\mu}\in E_{x,\mu,k}$, such that (2.4) holds. Moreover,

$$\|\omega_{s,x,\mu}\|\le C\|l_{s,x,\mu}\|.$$

Finally, by Lemma 2.2, we have

$$\|l_{s,x,\mu}\|\le\frac{C}{s^{2(1+\sigma)(N-2)/(N^2-6N+4)}}.$$

Thus the estimate follows.

□

To finish the proof of Proposition 2.1, it remains to prove the following two lemmas.

Lemma 2.2. *Assume that $N\ge 6$ and $(x,\mu)\in M_k$. We have*

$$
\begin{aligned}
&\langle l_{s,x,\mu}, \omega\rangle \\
=&O\Big(\sum_{i=1}^{k}\Big(\frac{1}{\mu_i^{1+\sigma}}+\big(\frac{sd_i}{\mu_i^{(N-2)/2}}\big)^{\frac{1+\sigma}{2}}\Big)+\sum_{i=1}^{k}\frac{1}{(d_i\mu_i)^{(N+2)/2}}+\sum_{i\neq j}\bar{\varepsilon}_{ij}^{\frac{1+\sigma}{2}}\Big)\|\omega\| \\
=&O\Big(\frac{1}{s^{2(1+\sigma)(N-2)/(N^2-6N+4)}}\Big)\|\omega\|,
\end{aligned}
$$

where $\sigma > 0$ is a constant, and $\bar{\varepsilon}_{ij}$ is defined in (A.2).

Proof. For any $\omega \in E_{x,\mu,k}$, we have

$$
\begin{aligned}
\langle l_{s,x,\mu}, \omega\rangle =& \int_\Omega\Big(\sum_{j=1}^{k}(PU_{x_j,\mu_j}-s\varphi_1)^{2^*-1}-\big(\sum_{j=1}^{k}PU_{x_j,\mu_j}-s\varphi_1\big)_+^{2^*-1}\Big)\omega \\
&+\sum_{j=1}^{k}\int_\Omega\Big(U_{x_j,\mu_j}^{2^*-1}-(PU_{x_j,\mu_j}-s\varphi_1)^{2^*-1}\Big)\omega-\lambda\sum_{j=1}^{k}\int_\Omega PU_{x_j,\mu_j}\omega.
\end{aligned}
$$

But

$$
\Big|\int_\Omega PU_{x_j,\mu_j}\omega\Big|\le\int_\Omega U_{x_j,\mu_j}|\omega|\le C\Big(\int_\Omega U_{x_j,\mu_j}^{2N/(N+2)}\Big)^{(N+2)/2N}\|\omega\|\le\frac{C\ln\mu_j}{\mu_j^2}\|\omega\|,
$$

$$
\begin{aligned}
&\Big|\int_\Omega\Big(U_{x_j,\mu_j}^{2^*-1}-\big(PU_{x_j,\mu_j}-s\varphi_1\big)_+^{2^*-1}\Big)\omega\Big| \\
\le&\Big|\int_\Omega\Big(U_{x_j,\mu_j}^{2^*-1}-\big(U_{x_j,\mu_j}-s\varphi_1\big)_+^{2^*-1}\Big)\omega\Big|+O\Big(\int_\Omega U_{x_j,\mu_j}^{2^*-2}\psi_{x_j,\mu_j}|\omega|\Big) \\
\le&\int_\Omega\Big(U_{x_j,\mu_j}^{2^*-1-\frac{1}{2}-\sigma}(s\varphi_1)^{\frac{1}{2}+\sigma}|\omega|\Big)+O\Big(\frac{1}{(d_i\mu_i)^{(N+2)/2}}\Big)\|\omega\| \\
\le&C\Big(\big(\frac{sd_j}{\mu_j^{(N-2)/2}}\big)^{(1+\sigma)/2}+\frac{1}{(d_i\mu_i)^{(N+2)/2}}\Big)\|\omega\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_\Omega\Big(\sum_{j=1}^{k}(PU_{x_j,\mu_j}-s\varphi_1)_+^{2^*-1}-\big(\sum_{j=1}^{k}PU_{x_j,\mu_j}-s\varphi_1\big)_+^{2^*-1}\Big)\omega\Big| \\
=&\Big|\int_\Omega\Big(\sum_{j=1}^{k}(PU_{x_j,\mu_j})^{2^*-1}-\big(\sum_{j=1}^{k}PU_{x_j,\mu_j}\big)^{2^*-1}\Big)\omega\Big|+O\Big(\big(\frac{sd_j}{\mu_j^{(N-2)/2}}\big)^{(1+\sigma)/2}\Big)\|\omega\| \\
\le&C\Big(\sum_{i\neq j}\bar{\varepsilon}_{ij}^{(1+\sigma)/2}+\big(\frac{sd_j}{\mu_j^{(N-2)/2}}\big)^{(1+\sigma)/2}\Big)\|\omega\|.
\end{aligned}
$$

In view of (1.8)–(1.10), Lemmas A.2 and A.3, the result follows.

□

Lemma 2.3. *There is a constant $\rho > 0$, independent of s, $(x, \mu) \in M_k$, such that*

$$\|Q_{s,x,\mu}\omega\| \geq \rho\|\omega\|, \quad \omega \in E_{x,\mu,k}. \tag{2.5}$$

Proof. We just sketch the proof of this lemma, since it is similar to the proof of Lemma 2.3 in [24].

We argue by contradiction. Suppose that there are $s_n \to +\infty$, $(x_n, \mu_n) \in M_k$ and $\omega_n \in E_{x_n,\mu_n,k}$, such that

$$\|Q_{s_n,x_n\mu_n}\omega_n\| = o(1)\|\omega_n\|, \tag{2.6}$$

where $o(1) \to 0$ as $n \to +\infty$. In (2.6), we may assume $\|\omega_n\| = 1$.

By using the standard blow-up argument, we can prove that for any $R > 0$,

$$\int_{B_{\mu_{n,j}^{-1}R}(x_{n,j})} \omega_n^2 = o(1), \quad j = 1, \cdots, k,$$

which implies that

$$\int_\Omega (\sum_{j=1}^k PU_{x_{n,j},\mu_{n,j}} - s\varphi_1)_+^{2^*-2}|\omega_n|^2 = o(1). \tag{2.7}$$

Combining (2.6) and (2.7), we are led to

$$\int_\Omega (D\omega_n D\eta - \lambda\omega_n\eta) = o(1)\|\eta\|, \quad \forall\, \eta \in E_{x_n,\mu_n,k}. \tag{2.8}$$

Since ω_n is bounded in $H_0^1(\Omega)$, we may assume that there is an $\omega^* \in H_0^1(\Omega)$, such that

$$\omega_n \rightharpoonup \omega^* \quad \text{weakly in } H_0^1(\Omega).$$

From (2.8), we can deduce

$$\int_\Omega (D\omega^* D\eta - \lambda\omega^*\eta) = 0, \quad \forall\, \eta \in H_0^1(\Omega). \tag{2.9}$$

From (Λ_1), or (Λ_2), λ is not an eigenvalue. So we obtain from (2.9) $\omega^* = 0$. Thus $\int_\Omega \omega_n^2 = o(1)$, which, together with (2.8), gives

$$\int_\Omega |D\omega_n|^2 = o(1).$$

This is a contradiction.

□

3. Analysis of Gradient Flow and Proof of Theorem 1.2

Let

$$K(\bar{x}, d, \mu) = J_s(x, \mu, \omega_{s,x,\mu}), \quad (\bar{x}, d, \mu) \in M_k,$$

where $\omega_{s,x,\mu}$ is the map obtained in Proposition 2.1, $x_j = \bar{x}_j - n(\bar{x}_j)d_j$. Then, noting that $\varphi_1(x_j) = -\frac{\partial\varphi_1(\bar{x}_j)}{\partial n}d_j(1 + O(d_j))$ and $H(x_j, x_j) = \frac{\tilde{c}}{d_j^{N-2}}\big(1 + O(d_j)\big)$, we obtain from Propositions 2.1 and B.3 that

$$\begin{aligned} K(\bar{x}, d, \mu) &= J_s(x, \mu, 0) + O(\|\omega_{s,x,\mu}\|^2) \\ &= kA - \sum_{j=1}^{k} \frac{\lambda B_1}{\mu_j^2} + \sum_{j=1}^{k} \frac{B_2(-\frac{\partial\varphi_1(\bar{x}_j)}{\partial n})d_j s}{\mu_j^{(N-2)/2}} + \sum_{j=1}^{k} \frac{c_0\tilde{c}B_2}{2\mu_j^{N-2}d_j^{N-2}} \\ &\quad - B_3\sum_{i\neq j}\varepsilon_{ij} + O\Big(\frac{1}{s^{4(1+\sigma)(N-2)/(N^2-6N+4)}}\Big). \end{aligned} \tag{3.1}$$

Next, we need the following expansions of the derivatives of $K(\bar{x}, d, \mu)$.

Lemma 3.1. *Assume* $(x, \mu) \in M_k$. *Then*

$$\begin{aligned} \frac{\partial K(\bar{x}, d, \mu)}{\partial \mu_j} &= \frac{2\lambda B_1}{\mu_j^3} - \frac{N-2}{2}\frac{B_2(-\frac{\partial\varphi_1(\bar{x}_j)}{\partial n})d_j s}{\mu_j^{N/2}} - \frac{(N-2)c_0\tilde{c}B_2}{2\mu_j^{N-1}d_j^{N-2}} \\ &\quad + O\Big(\frac{1}{s^{6(N-2)/(N^2-6N+4)+\theta}}\Big), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \frac{\partial K(\bar{x}, d, \mu)}{\partial d_j} &= \frac{B_2(-\frac{\partial\varphi_1(\bar{x}_j)}{\partial n})s}{\mu_j^{(N-2)/2}} - \frac{(N-2)c_0\tilde{c}B_2}{2\mu_j^{N-2}d_j^{N-1}} \\ &\quad + O\Big(\frac{s^{2(N-4)/(N^2-6N+4)}}{s^{4(N-2)/(N^2-6N+4)+\theta}}\Big). \end{aligned} \tag{3.3}$$

Intuitively, the estimates in Lemma 3.1 can be obtained by differentiating (3.1) with respect to d_j and μ_i. We will postpone the proof of Lemma 3.1 in Appendix C. Now we are ready to prove Theorem 1.2.

Define

$$f(b, t) = -\frac{\lambda B_1}{t^2} + \frac{B_2\bar{a}b}{t^{(N-2)/2}} + \frac{c_0\tilde{c}B_2}{2t^{N-2}b^{N-2}}.$$

Here $\bar{a}$ is defined in (1.5). Then, we have

$$\frac{\partial f(b, t)}{\partial b} = \frac{B_2\bar{a}}{t^{(N-2)/2}} - (N-2)\frac{c_0\tilde{c}B_2}{2t^{N-2}b^{N-1}}, \tag{3.4}$$

and

$$\frac{\partial f(a,b,t)}{\partial t} = \frac{2\lambda B_1}{t^3} - \frac{N-2}{2}\frac{B_2 \bar{a} b}{t^{N/2}} - (N-2)\frac{c_0 \tilde{c} B_2}{2t^{N-1}b^{N-2}}. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\frac{\partial f(b,t)}{\partial b} = 0, \quad \frac{\partial f(b,t)}{\partial t} = 0,$$

has a unique solution $t = \bar{t} > 0$, $b = \bar{b} > 0$. For the definition of $\bar{t}$ and $\bar{b}$, see (1.6) and (1.7).

On the other hand, using (3.4) and (3.5), we find that if $N \geq 6$, at $t = \bar{t}$ and $b = \bar{b}$,

$$\begin{aligned}\frac{\partial^2 f(\bar{b},\bar{t})}{\partial t^2} &= -\frac{6\lambda B_1}{\bar{t}^4} + \frac{N(N-2)}{4}\frac{B_2 \bar{a}\bar{b}}{\bar{t}^{(N+2)/2}} + (N-2)(N-1)\frac{c_0 \tilde{c} B_2}{2\bar{t}^N \bar{b}^{N-2}} \\ &= \frac{(N-6)(N-2)}{4}\frac{B_2 \bar{a}\bar{b}}{\bar{t}^{(N+2)/2}} + (N-2)(N-4)\frac{c_0 \tilde{c} B_2}{2\bar{t}^N \bar{b}^{N-2}} > 0,\end{aligned}$$

and

$$\frac{\partial^2 f(\bar{b},\bar{t})}{\partial b^2} = (N-2)(N-1)\frac{c_0 \tilde{c} B_2}{2\bar{t}^{N-2}\bar{b}^N} > 0.$$

Let

$$c_{2,s} = kA + \eta,$$

where $\eta > 0$ is a small constant, and

$$c_{1,s} = kA + kf(\bar{b},\bar{t})s^{-4(N-2)/(N^2-6N+4)} - s^{-3\theta/2-4(N-2)/(N^2-6N+4)}.$$

For any c, let $K^c = \{(x,\mu) : \ (x,\mu) \in M_k, \ K(x,\mu) < c\}$. Here, we denote $K(x,\mu) = K(\bar{x}, d, \mu)$.

Consider the following flow:

$$\begin{cases} \frac{dx(t)}{dt} = -D_x K(x(t),\mu(t)), & t > 0; \\ \frac{d\mu(t)}{dt} = -D_\mu K(x(t),\mu(t)), & t > 0; \\ (x(0),\mu(0)) = (x_0,\mu_0) \in K^{c_{2,s}}. \end{cases} \tag{3.6}$$

Proposition 3.2. *Suppose that $N \geq 6$. Then $(x(t),\mu(t))$ will not leave M_k before it reaches $K^{c_{1,s}}$.*

Before we prove Proposition 3.2, we prove the following lemma.

Lemma 3.3. *For any $\bar{x} \in \partial V_k$, we have $(x,\mu) \in K^{\tilde{c}_{1,s}}$, where*

$$\tilde{c}_{1,s} = c_{1,s} - 2s^{-3\theta/2-4(N-2)/(N^2-6N+4)}.$$

Proof. Denote
$$\mu_i = t_i s^{2(N-2)/(N^2-6N+4)}, \quad t_i \in [\bar t - Ls^{-\theta}, \bar t + Ls^{-\theta}],$$
and
$$d_i = b_i s^{-2(N-4)/(N^2-6N+4)}, \quad b_i \in [\bar b - Ls^{-\theta}, \bar b + Ls^{-\theta}].$$
Since $(\bar b, \bar t)$ is the unique minimum point of $f(b,t)$, we have
$$f(b_i, t_i) = f(\bar b, \bar t) + O(s^{-2\theta}). \tag{3.7}$$
Suppose that $|\bar x_i - \bar x_j| = s^{-2(N-4)/(N^2-6N+4)+\theta/N}$. Then, it follows from Lemma A.2 that
$$\varepsilon_{ij} \ge c' s^{-4(N-2)/(N^2-6N+4)-\theta}.$$
By (3.1), noting that $-\frac{\partial \varphi_1(\bar x_i)}{\partial n} \le \bar a$, we have
$$\begin{aligned} &K(x,\mu)\\ \le & kA + \sum_{j=1}^{k} f(b_j, t_j) s^{-4(N-2)/(N^2-6N+4)} - B_3 \varepsilon_{ij} + O\Big(s^{-4(1+\sigma)(N-2)/(N^2-6N+4)}\Big)\\ \le & kA + kf(\bar b, \bar t) s^{-4(N-2)/(N^2-6N+4)} - B_3\varepsilon_{ij} + O\Big(s^{-2\theta-4(N-2)/(N^2-6N+4)}\Big)\\ \le & kA + kf(\bar b, \bar t) s^{-4(N-2)/(N^2-6N+4)} - c' B_3 s^{-4(N-2)/(N^2-6N+4)-\theta}\\ &+ O\big(s^{-4(N-2)/(N^2-6N+4)-2\theta}\big) < \tilde c_{1,s}. \end{aligned} \tag{3.8}$$
Thus, $(x,\mu) \in K^{\tilde c_{1,s}}$.

Suppose that $-\frac{\partial \varphi_1(\bar x_i)}{\partial n} = \bar a - s^{-\theta}$ for some $i = 1, \cdots, k$.

It follows from (3.1) and (3.7) that
$$\begin{aligned} &K\big(x,\mu\big)\\ \le & kA + \sum_{j=1}^{k} f(b_j, t_j) s^{-4(N-2)/(N^2-6N+4)} - \frac{B_2 b_i}{t_i^{N-2}} s^{-4(N-2)/(N^2-6N+4)-\theta}\\ &+ O\big(s^{-4(1+\sigma)(N-2)/(N^2-6N+4)}\big)\\ \le & kA + kf(\bar b, \bar t) s^{-4(N-2)/(N^2-6N+4)} - \frac{B_2 \bar b}{\bar t^{N-2}} s^{-4(N-2)/(N^2-6N+4)-\theta}\\ &+ O\big(s^{-4(N-2)/(N^2-6N+4)-2\theta}\big) < \tilde c_{1,s}. \end{aligned} \tag{3.9}$$
Thus, $(x,\mu) \in K^{\tilde c_{1,s}}$.

□

Proof of Proposition 3.2. Suppose that there is a $t_0 > 0$, such that $(x(t_0), \mu(t_0)) \in \partial M_k$. We will prove that either $(x(t_0), \mu(t_0)) \in K^{c_{1,s}}$, or $\frac{\partial K(x,\mu)}{\partial \nu} > 0$ at $(x(t_0), \mu(t_0))$, where ν is the outward unit normal of ∂M_k at $(x(t_0), \mu(t_0))$.

Note that $\partial M_k = \partial V_k \times W_k \cup V_k \times \partial W_k$.

If $\bar{x}(t_0) \in \partial V_k$, then it follows from Lemma 3.3 that

$$(x(t_0), \mu(t_0)) \in K^{c_{1,s}}.$$

Now we consider the case $(d(t_0), \mu(t_0)) \in \partial W_k$.

Let

$$\mu_i(t_0) = t_i s^{2(N-2)/(N^2-6N+4)}, \quad t_i \in [\bar{t} - Ls^{-\theta}, \bar{t} + Ls^{-\theta}],$$

and

$$d_i(t_0) = b_i s^{-2(N-4)/(N^2-6N+4)}, \quad b_i \in [\bar{b} - Ls^{-\theta}, \bar{b} + Ls^{-\theta}].$$

Suppose that $\mu_j(t_0) = (\bar{t} + Ls^{-\theta})s^{2(N-2)/(N^2-6N+4)}$ for some j. Then, $t_j = \bar{t} + Ls^{-\theta}$. Thus, noting that $-\frac{\partial \varphi_1(\bar{x}_j)}{\partial n} = \bar{a} + O(s^{-\theta})$, from (3.2), we obtain

$$\begin{aligned}
&\frac{\partial K(x,\mu)}{\partial \nu} = \frac{\partial K(x,\mu)}{\partial \mu_j} \\
=&s^{-6(N-2)/(N^2-6N+4)}\Big(\frac{\partial f(t_j, b_j)}{\partial t} + O\big(s^{-\theta}\big)\Big) \\
=&s^{-6(N-2)/(N^2-6N+4)}\Big(\frac{\partial^2 f(\bar{t}, b_j)}{\partial t^2} Ls^{-\theta} + O\big(L^2 s^{-2\theta} + s^{-\theta}\big)\Big) \\
=&s^{-6(N-2)/(N^2-6N+4)}\Big(\frac{\partial^2 f(\bar{t}, \bar{b})}{\partial t^2} Ls^{-\theta} + O\big(L^2 s^{-2\theta} + Ls^{-\theta}|b_j - \bar{b}| + s^{-\theta}\big)\Big) > 0,
\end{aligned}$$

if $L > 0$ is large, since $|b_j - \bar{b}| \le Ls^{-\theta}$. Thus, $(x(t), \mu(t))$ will not leave M_k at $(x(t_0), \mu(t_0))$.

Similarly, if $\mu_j(t_0) = (\bar{t} - Ls^{-\theta})s^{2(N-2)/(N^2-6N+4)}$ for some j, then at $(x(t_0), \mu(t_0))$,

$$\frac{\partial K(x,\mu)}{\partial \nu} = -\frac{\partial K(x,\mu)}{\partial \mu_j} > 0.$$

So, $(x(t), \mu(t))$ will not leave M_k at $(x(t_0), \mu(t_0))$.

Suppose that $d_j(t_0) = (\bar{t} + Ls^{-\theta})s^{-2(N-4)/(N^2-6N+4)}$ for some j. Then, $b_j = \bar{b} + Ls^{-\theta}$. Thus, from (3.3), we have

$$\begin{aligned}
&\frac{\partial K(x,\mu)}{\partial \nu} = \frac{\partial K(x,\mu)}{\partial d_j} \\
=&s^{-2N/(N^2-6N+4)}\Big(\frac{\partial f(t_j, b_j)}{\partial b} + O\big(s^{-\theta}\big)\Big) \\
=&s^{-2N/(N^2-6N+4)}\Big(\frac{\partial^2 f(t_j, \bar{b})}{\partial b^2} Ls^{-\theta} + O\big(L^2 s^{-2\theta} + s^{-\theta}\big)\Big) \\
=&s^{-2N/(N^2-6N+4)}\Big(\frac{\partial^2 f(\bar{t}, \bar{b})}{\partial b^2} Ls^{-\theta} + O\big(L^2 s^{-2\theta} + Ls^{-\theta}|t_j - \bar{t}| + s^{-\theta}\big)\Big) > 0,
\end{aligned}$$

if $L>0$ is large, since $|t_j-\bar{t}|\le Ls^{-\theta}$. Thus, $(x(t),\mu(t))$ will not leave M_k at $(x(t_0),\mu(t_0))$.

Similarly, if $d_j(t_0)=(\bar{t}-Ls^{-\theta})s^{2(N-2)/(N^2-6N+4)}$ for some j, then at $(x(t_0),\mu(t_0))$,

$$\frac{\partial K(x,\mu)}{\partial \nu}=-\frac{\partial K(x,\mu)}{\partial d_j}>0.$$

So, $(x(t),\mu(t))$ will not leave M_s at $(x(t_0),\mu(t_0))$.

□

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We will prove that $K(x,\mu)$ has a critical point in $K^{c_{2,s}}\setminus K^{c_{1,s}}$.

Let Λ be the set of maps $h(\bar{x},d,\mu)$ from M_k to M_k, satisfying

$$h_1(\bar{x},d,\mu)=\bar{x},\quad \text{if } \bar{x}\in\partial V_k,$$

where $h(x,\mu)=(h_1(x,\mu),h_2(x,\mu))$, $h_1(x,\mu)\in V_k$, $h_2(x,\mu)\in W_k$.

Define

$$c_s=\inf_{h\in\Lambda}\sup_{(x,\mu)\in M_k}K(h(x,\mu)).$$

We will show that c_s is a critical value of $K(x,\mu)$. To prove this claim, we need to prove

(i) $c_{1,s}<c_s<c_{2,s}$;

(ii) $\sup_{(x,\mu)\in\partial V_k\times W_k}K(h(x,\mu))<c_{1,s}$, $\forall\, h\in\Lambda$.

To prove (ii), let $h\in\Lambda$. Then, for any $(\bar{x},d,\mu)\in\partial V_k\times W_k$, we have $h(\bar{x},d,\mu)=(\bar{x},\tilde{d},\tilde{\mu})$ for some $(\tilde{d},\tilde{\mu})\in W_k$. By Lemma 3.3, we obtain

$$K(\bar{x},\tilde{d},\tilde{\mu})<\tilde{c}_{1,s},$$

which implies

$$\sup_{(x,\mu)\in\partial V_k\times W_k}K(h(x,\mu))\le\tilde{c}_{1,s}<c_{1,s}.$$

Now, we prove (i). It is easy to see $c_s\le\sup_{(x,\mu)\in M_k}K(x,\mu)<c_{2,s}$.

On the other hand, let

$$d_j^*=\bar{b}s^{-2(N-4)/(N^2-6N+4)},\quad \mu_j^*=\bar{t}s^{2(N-2)/(N^2-6N+4)},\ j=1,\cdots,k,$$

where $\bar{t}$ and $\bar{b}$ are defined in (1.6) and (1.7) respectively.

For any $h\in\Lambda$,. Define $\bar{h}(\bar{x})=h_1(\bar{x},d^*,\mu^*)$. Then, $\bar{h}(\bar{x})$ is a map from V_k to V_k, satisfying

$$\bar{h}(\bar{x})=\bar{x},\quad\forall\,\bar{x}\in\partial V_k.$$

Thus, for any $\bar{z}\in V_k$,

$$deg(\bar{h},V_k,\bar{z})=1.$$

Therefore, for any $z \in V_k$, there is a $\bar{x} \in V_k$, such that $\bar{h}(\bar{x}) = \bar{z}$. Let $(\tilde{d}, \tilde{\mu}) = h_2(\bar{x}, d^*, \mu^*) \in W_k$. We have

$$\sup_{(x,\mu)\in M_k} K(h(x,\mu)) \geq K(\bar{z}, \tilde{d}, \tilde{\mu}).$$

So, we see that to prove $c_s > c_{1,s}$, we only need to choose $\bar{z} \in V_k$, such that for all $(d, \mu) \in W_k$,

$$K(\bar{z}, d, \mu) > c_{1,s} + \frac{1}{2} s^{-3\theta/2-4(N-2)/(N^2-6N+4)}.$$

Let $\bar{x}_0$ be a maximum point of $-\frac{\partial \varphi_1}{\partial n}$ on Γ. Choose $\bar{z}_{s,j} \in B_{s^{-\theta}}(\bar{x}_0) \cap \Gamma$, $j = 1, \cdots, k$, satisfying $|\bar{z}_{s,i} - \bar{z}_{s,j}| \geq c' s^{-\theta}$, $\forall\, i \neq j$, where $c' > 0$ is a small constant. Then

$$-\frac{\partial \varphi_1(\bar{z}_{s,j})}{\partial n} = \bar{a} + O(|\bar{z}_{s,j} - \bar{x}_0|^2) = \bar{a} + O(s^{-2\theta}).$$

As a result, $\bar{z}_s = (\bar{z}_{s,1}, \cdots, \bar{z}_{s,k}) \in V_k$. Now, we estimate $K(\bar{z}_s, d, \mu)$ for any $(d, \mu) \in W_k$.

Denote

$$z_i = \bar{z}_i - d_i n(\bar{z}_i), \; i = 1, \cdots, k,$$

where $n(\bar{z})$ is the outward unit normal of Γ at $\bar{z} \in \Gamma$. We have

$$\begin{aligned}\bar{\varepsilon}_{ij} =& \frac{1}{\mu_i^{(N-2)/2} \mu_j^{(N-2)/2} |z_{s,i} - z_{s,j}|^{N-2}} \\ =& O\big(s^{\theta(N-2)-2(N-2)^2/(N^2-6N+4)}\big), \quad i \neq j\end{aligned}$$

which implies

$$\frac{1}{\mu_i^{(N-2)/2} \mu_j^{(N-2)/2}} G(z_i, z_j) \leq C \bar{\varepsilon}_{ij} = O\big(s^{\theta(N-2)-2(N-2)^2/(N^2-6N+4)}\big).$$

So, from (3.1), we obtain

$$\begin{aligned}K(\bar{z}, d, \mu) =& kA + kf(\bar{b}, \bar{t}) s^{-4(N-2)/(N^2-6N+4)} + O\big(s^{-4(N-2)/(N^2-6N+4)-2\theta}\big) \\ \geq& c_{1,s} + \frac{1}{2} s^{-4(N-2)/(N^2-6N+4)-3\theta/2}.\end{aligned} \tag{3.10}$$

So, we have proved (i). Therefore, $K(x, \mu)$ has a critical point in M_k with critical value c_s. □

Appendix A

Let $G(y,x_j)$ be the Green's function of $-\Delta$ in Ω with the Dirichlet boundary condition, and let $H(y,x_j)$ be the regular part of $G(y,x_j)$. Then, there is a constant $\tilde{c}>0$, such that

$$G(y,x_j)=\frac{\tilde{c}}{|y-x_j|^{N-2}}-H(y,x_j).$$

For any $i\neq j$, define

$$\varepsilon_{ij}=\frac{1}{\mu_i^{(N-2)/2}\mu_j^{(N-2)/2}}G(x_i,x_j), \tag{A.1}$$

and

$$\bar{\varepsilon}_{ij}=\frac{1}{\mu_i^{(N-2)/2}\mu_j^{(N-2)/2}|x_i-x_j|^{N-2}}. \tag{A.2}$$

In this section, we will derive some estimates for the quantities ε_{ij} and $\bar{\varepsilon}_{ij}$ under the assumption that $(x,\mu)\in M_k$.

Firstly, we will give an estimate for the Green function $G(x_i,x_j)$ for x_i and x_j close to $\partial\Omega$. For any $x_j\in\Omega$, which is close to $\partial\Omega$, it follows from [26] that

$$D^iH(y,x_j)=D^i\Big(\frac{\tilde{c}}{|y-\tilde{x}_j|^{N-2}}\Big)\big(1+O(d(x_j,\partial\Omega))\big),\quad i=1,2, \tag{A.3}$$

where $\tilde{x}_j$ is the reflection point of x_j with respect to $\partial\Omega$.

For any $d_i,\ d_j\in[(\bar{b}-Ls^{-\theta})s^{-2(N-4)/(N^2-6N+4)},(\bar{b}+Ls^{-\theta})s^{-2(N-4)/(N^2-6N+4)}]$, $\bar{x}_i\in\partial\Omega$, $\bar{x}_j\in\partial\Omega$, let

$$x_i=\bar{x}_i-d_in(\bar{x}_i),\quad x_j=\bar{x}_j-d_jn(\bar{x}_j).$$

Then

Lemma A.1. *Suppose that $|\bar{x}_i-\bar{x}_i|\geq d_j$. We have*

$$G(x_i,x_j)=\frac{1}{|\bar{x}_i-\bar{x}_j|^{N-2}}\Big(\frac{c^*d_j^2}{|\bar{x}_i-\bar{x}_j|^2}+O\big(\frac{d_j^4}{|\bar{x}_i-\bar{x}_j|^4}+s^{-\theta}\big)\Big),$$

where $c^*>0$ is a constant.

Proof. We have

$$\begin{aligned}G(x_i,x_j)=&\frac{\tilde{c}}{|x_i-x_j|^{N-2}}-H(x_i,x_j)\\=&\frac{\tilde{c}}{|x_i-x_j|^{N-2}}-\frac{\tilde{c}}{|x_i-\tilde{x}_j|^{N-2}}+O\big(\frac{d_j}{|x_i-\tilde{x}_j|^{N-2}}\big).\end{aligned}$$

On the other hand,

$$|x_i - \tilde{x}_j|^2 = |x_i - x_j|^2 + |x_j - \tilde{x}_j|^2 + 2\big\langle x_i - x_j, x_j - \tilde{x}_j\big\rangle.$$

But

$$\begin{aligned} &\big\langle x_i - x_j, x_j - \tilde{x}_j\big\rangle = \big\langle x_i - x_j, -2d_j n(\bar{x}_j)\big\rangle \\ =&\big\langle \bar{x}_i - \bar{x}_j, -2d_j n(\bar{x}_j)\big\rangle + \big\langle d_i n(\bar{x}_i) - d_j n(\bar{x}_j), -2d_j n(\bar{x}_j)\big\rangle \\ =&O\big(d_j|\bar{x}_i - \bar{x}_j|^2 + |d_i - d_j|d_j + d_i d_j|\bar{x}_i - \bar{x}_j|\big), \end{aligned}$$

and

$$\begin{aligned} |x_i - x_j| =&|\bar{x}_i - \bar{x}_j| + O\big(|d_i n(\bar{x}_i) - d_j n(\bar{x}_j)|\big) \\ =&|\bar{x}_i - \bar{x}_j| + O\big(|d_i - d_j| + d_j|\bar{x}_i - \bar{x}_j|\big). \end{aligned}$$

So, we obtain

$$\begin{aligned} |x_i - \tilde{x}_j|^2 =&|\bar{x}_i - \bar{x}_j|^2 + 4d_j^2 \\ &+ O\big(d_j|\bar{x}_i - \bar{x}_j|^2 + |d_i - d_j|d_j + d_i d_j|\bar{x}_i - \bar{x}_j| + |d_i - d_j|^2 + d_j^2|\bar{x}_i - \bar{x}_j|^2\big) \end{aligned}$$

As a result,

$$\begin{aligned} &\frac{1}{|x_i - \tilde{x}_j|^{N-2}} \\ =&\frac{1}{|\bar{x}_i - \bar{x}_j|^{N-2}}\Big(1 - \frac{N-2}{2}\frac{4d_j^2}{|\bar{x}_i - \bar{x}_j|^2} + O\big(\frac{d_j^4}{|\bar{x}_i - \bar{x}_j|^4} + s^{-\theta}\big)\Big). \end{aligned}$$

So, the result follows. □

A direct consequence of Lemma A.1 is

Lemma A.2. *For any* $(x, \mu) \in M_k$,

$$\varepsilon_{ij} \le Cs^{-4(N-2)/(N^2-6N+4)-\theta}.$$

Moreover, if $|\bar{x}_i - \bar{x}_j| = s^{-2(N-4)/(N^2-6N+4)+\theta/N}$, *then*

$$\varepsilon_{ij} \ge C's^{-\theta-4(N-2)/(N^2-6N+4)}.$$

Proof. We just need to use

$$\frac{d_j^2}{|\bar{x}_i - \bar{x}_j|^2} \le Cs^{-2\theta/N}, \quad \forall\ (\bar{x}, d, \mu) \in M_k.$$

□

Next, we need to estimate $\bar{\varepsilon}_{ij}$.

Lemma A.3. *For any $(x,\mu)\in M_k$, we have*

$$\bar\varepsilon_{ij} = O\big(\varepsilon_{ij} + s^{-4(N-2)/(N^2-6N+4)}\big).$$

Proof. By definition, we have

$$\bar\varepsilon_{ij} \le C\varepsilon_{ij} + \frac{C}{\mu_i^{(N-2)/2}\mu_j^{(N-2)/2}} H(x_i,x_j).$$

We may assume that $d_j = \min(d_i,d_j)$.
For any $(x,\mu)\in M_k$, we have

$$\begin{aligned}|x_i - x_j| =&|\bar x_i - \bar x_j| + O\big(d_i s^{-\theta} + d_j|\bar x_i - \bar x_j|\big)\\ \ge& c' d_j s^{\theta/N} + O\big(d_i s^{-\theta} + d_j^2 s^{\theta/N}\big) > d_j.\end{aligned}$$

Then from $G(x_i,x_j)\ge 0$, we find

$$H(x_i,x_j) \le \frac{C}{|x_i-x_j|^{N-2}} \le \frac{C}{d_j^{N-2}}.$$

As a result,

$$\frac{1}{\mu_i^{(N-2)/2}\mu_j^{(N-2)/2}} H(x_i,x_j) \le \frac{C}{\mu_i^{(N-2)/2}\mu_j^{(N-2)/2} d_j^{N-2}} \le C s^{-4(N-2)/(N^2-6N+4)}.$$

□

Lemma A.4. *For any $(x,\mu)\in M_k$, we have*

$$|x_i - x_j|\bar\varepsilon_{ij} = O\big(s^{-2\theta - 4(N-2)/(N^2-6N+4)}\big).$$

Proof. Suppose that $|x_i - x_j| \ge s^{-2\theta}$. Then

$$\begin{aligned}|x_i - x_j|\bar\varepsilon_{ij} \le& \frac{1}{\mu_i^{(N-2)/2}\mu_j^{(N-2)/2}|x_i-x_j|^{N-3}}\\ \le& C s^{2(N-3)\theta - 2(N-2)^2/(N^2-6N+4)} = O\big(s^{-2\theta-4(N-2)/(N^2-6N+4)}\big).\end{aligned}$$

If $|x_i - x_j| \le s^{-2\theta}$, then by Lemmas A.2 and A.3,

$$|x_i - x_j|\bar\varepsilon_{ij} \le s^{-2\theta} s^{-4(N-2)/(N^2-6N+4)} = O\big(s^{-2\theta-4(N-2)/(N^2-6N+4)}\big).$$

□

Appendix B

Denote

$$\psi_{x_j,\mu_j} = U_{x_j,\mu_j} - PU_{x_j,\mu_j}.$$

The following estimates can be found in [26].

Lemma B.1. *We have the following expansion for* ψ_{x_j,μ_j}*:*

$$\psi_{x_j,\mu_j}(y) = \frac{c_0 H(y,x_j)}{\mu_j^{(N-2)/2}}\Big(1+O\big(\frac{1}{(d_j\mu_j)^2}\big)\Big),$$

$$\frac{\partial \psi_{x_j,\mu_j}(y)}{\partial \mu_j} = \frac{-(N-2)c_0 H(y,x_j)}{2\mu_j^{N/2}}\Big(1+O\big(\frac{1}{(d_j\mu_j)^2}\big)\Big),$$

and

$$\frac{\partial \psi_{x_j,\mu_j}(y)}{\partial x_{j,h}} = \frac{c_0}{\mu_j^{(N-2)/2}}\frac{\partial H(y,x_j)}{\partial x_{j,h}}\Big(1+O\big(\frac{1}{(d_j\mu_j)^2}\big)\Big), \quad h=1,\cdots,N,$$

In this section, we will expand $J_s(x,\mu,0)$ and its derivatives. These expansions provide a very good guide on the type of critical points we can expect to obtain for the reduced problem.

Proposition B.2. *Assume* $N \geq 5$*. Then*

$$\begin{aligned} I_s\big(PU_{x_j,\mu_j}\big) =& A - \frac{B_1\lambda}{\mu_j^2} + \frac{B_2\varphi_1(x_j)s}{\mu_j^{(N-2)/2}} + \frac{c_0 B_2 H(x_j,x_j)}{2\mu_j^{N-2}} \\ &+ O\Big(\frac{1}{(d_j\mu_j)^{N-2+\sigma}} + \frac{1}{\mu_j^2(d_j\mu_j)^{(N-4)/2}} + \big(\frac{sd_j}{\mu_j^{(N-2)/2}}\big)^{1+\sigma} + \frac{s}{\mu_j^{N/2}}\Big), \end{aligned}$$

where

$$A = \frac{1}{2}\int_{R^N}|DU|^2\,dy - \frac{1}{2^*}\int_{R^N}U^{2^*}\,dy.$$

$$B_1 = \frac{1}{2}\int_{R^N}U^2\,dy, \quad B_2 = \int_{R^N}U^{2^*-1}\,dy,$$

and σ *is some positive constant.*

Proof. The proof of this proposition can be found in Proposition A.2 of [25]. The the sake of completeness, we sketch it.

Write

$$I_s(u) = \tilde{I}(u) - \frac{1}{2}\lambda\int_\Omega u^2 - \frac{1}{2^*}\int_\Omega\big((u-s\varphi_1)_+^{2^*} - |u|^{2^*}\big),$$

where

$$\tilde{I}(u) = \frac{1}{2}\int_{\Omega} |Du|^2 - \frac{1}{2^*}\int_{\Omega} |u|^{2^*}.$$

We have [3, 26]

$$\tilde{I}\big(PU_{x_j,\mu_j}\big) = A + \frac{c_0 B_2 H(x_j, x_j)}{2\mu^{N-2}} + O\Big(\frac{1}{(d_j\mu_j)^{N-2+\sigma}}\Big), \tag{B.1}$$

and

$$\int_{\Omega} (PU_{x_j,\mu_j})^2 = \frac{1}{\mu_j^2}\int_{R^N} U^2 + O\Big(\frac{1}{\mu_j^2 (d_j\mu_j)^{(N-4)/2}}\Big). \tag{B.2}$$

On the other hand,

$$\begin{aligned}
&\int_{\Omega} \big(PU_{x_j,\mu_j} - s\varphi_1\big)_+^{2^*} - \int_{\Omega} (PU_{x_j,\mu_j})^{2^*} \\
=&\int_{\Omega} \Big(\big(U_{x_j,\mu_j} - s\varphi_1\big)_+^{2^*} - U_{x_j,\mu_j}^{2^*}\Big) \\
&- 2^* \int_{\Omega} \Big(\big(U_{x_j,\mu_j} - s\varphi_1\big)_+^{2^*-1} - U_{x_j,\mu_j}^{2^*-1}\Big)\psi_{x_j,\mu_j} + O\Big(\frac{1}{(d_j\mu_j)^{N-2+\sigma}}\Big) \\
=&- 2^* \int_{R^N} U^{2^*-1} s\mu_j^{-(N-2)/2}\varphi_1(x_j) + O\Big(\frac{1}{(d_j\mu_j)^N} + \frac{s}{\mu_j^{N/2}} + \frac{(sd_j)^{1+\sigma}}{\mu_j^{(1+\sigma)(N-2)/2}}\Big).
\end{aligned} \tag{B.3}$$

So, the results follows from (B.1)–(B.3).

∎

Proposition B.3. *Assume $N \geq 6$ and $(x, \mu) \in M_k$. We have*

$$\begin{aligned}
I_s\big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\big) =&kA + \sum_{j=1}^{k}\Big(-\frac{B_1\lambda}{\mu_j^2} + \frac{B_2(-\frac{\partial\varphi_1(x_j)}{\partial n})d_j s}{\mu_j^{(N-2)/2}} + \frac{c_0\tilde{c}B_2}{2\mu_j^{N-2}d_j^{N-2}}\Big) \\
&- B_3\sum_{i\neq j}\varepsilon_{ij} + O\Big(\frac{1}{s^{2\theta+4(N-2)/(N^2-6N+4)}}\Big),
\end{aligned} \tag{B.4}$$

where $B_3 > 0$ and $\sigma > 0$ are some constants.

Proof. Write

$$
\begin{aligned}
&I_s\big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\big) - \sum_{j=1}^{k} I_s\big(PU_{x_j,\mu_j}\big)\\
=&\tilde I\big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\big) - \sum_{j=1}^{k} \tilde I\big(PU_{x_j,\mu_j}\big) - \frac{1}{2}\lambda \sum_{i\neq j}\int_\Omega PU_{x_i,\mu_i}PU_{x_j,\mu_j}\\
&-\frac{1}{2^*}\int_\Omega\Big(\big(\sum_{j=1}^{k} PU_{x_j,\mu_j} - s\varphi\big)_+^{2^*} - \big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\big)^{2^*}\Big)\\
&-\frac{1}{2^*}\sum_{j=1}^{k}\int_\Omega\Big(\big(PU_{x_j,\mu_j} - s\varphi\big)_+^{2^*} - \big(PU_{x_j,\mu_j}\big)^{2^*}\Big).
\end{aligned}
$$

By [3], we have

$$
\begin{aligned}
&\tilde I\big(\sum_{j=1}^{k} PU_{x_j,\mu_j}\big) - \sum_{j=1}^{k} \tilde I\big(PU_{x_j,\mu_j}\big)\\
=&B_3\varepsilon_{ij} + O(\bar\varepsilon_{ij}^{1+\sigma}) = B_3\varepsilon_{ij} + O\Big(\frac{1}{s^{4(1+\sigma)(N-2)/(N^2-6N+4)}}\Big).
\end{aligned}
$$

Here, in the last relation, we have used Lemmas A.3 and A.2.

On the other hand, for $i \neq j$, from

$$
\begin{aligned}
&\bar\varepsilon_{ij}^{-1}\int_\Omega PU_{x_i,\mu_i}PU_{x_j,\mu_j} \le \bar\varepsilon_{ij}^{-1}\int_\Omega U_{x_i,\mu_i}U_{x_j,\mu_j}\\
\le& C|x_i-x_j|^{N-2}\int_\Omega \frac{1}{|y-x_i|^{N-2}|y-x_j|^{N-2}}\,dy\\
\le& C|x_i-x_j|\Big(\int_\Omega \frac{1}{|y-x_j||y-x_i|^{N-2}}\,dy + \int_\Omega \frac{1}{|y-x_i||y-x_j|^{N-2}}\,dy\Big)\\
\le& C'|x_i-x_j|.
\end{aligned}
$$

and using Lemma A.4, we obtain

$$
\sum_{i\neq j}\int_\Omega PU_{x_i,\mu_i}PU_{x_j,\mu_j} \le C|x_i-x_j|\bar\varepsilon_{ij} = O\Big(\frac{1}{s^{2\theta+4(N-2)/(N^2-6N+4)}}\Big).
$$

It is easy to check

$$\begin{aligned}&\int_\Omega\Big(\big(\sum_{j=1}^k PU_{x_j,\mu_j}-s\varphi\big)_+^{2^*}-\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big)^{2^*}-2^*\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big)^{2^*-1}s\varphi_1\Big)\\ =&O\Big(s^{1+\sigma}\int_\Omega\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big)^{2^*-1-\sigma}\varphi_1^{1+\sigma}\Big)\\ =&O\Big(\sum_{j=1}^k\frac{(sd_j)^{1+\sigma}}{\mu_j^{(1+\sigma)(N-2)/2}}\Big)=O\Big(\frac{1}{s^{4(1+\sigma)(N-2)/(N^2-6N+4)}}\Big),\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\Big(\big(PU_{x_j,\mu_j}-s\varphi\big)_+^{2^*}-\big(PU_{x_j,\mu_j}\big)^{2^*}-2^*\big(PU_{x_j,\mu_j}\big)^{2^*-1}s\varphi_1\Big)\\ =&O\Big(\frac{1}{s^{4(1+\sigma)(N-2)/(N^2-6N+4)}}\Big).\end{aligned}$$

Finally,

$$\begin{aligned}&\Big|\int_\Omega\Big(\big(\sum_{j=1}^k PU_{x_j,\mu_j}\big)^{2^*-1}-\sum_{j=1}^k\big(PU_{x_j,\mu_j}\big)^{2^*-1}\Big)s\varphi_1\Big|\\ \le&Cs\sum_{i\neq j}\int_\Omega U_{x_i,\mu_i}^{(2^*-1)/2}U_{x_j,\mu_j}^{(2^*-1)/2}\varphi_1\\ \le&\sum_{i\neq j}\frac{Cs}{\mu_i^{(N-2)/2}}\int_{R^N}U^{(N+2)/2(N-2)}(y)\frac{\varphi_1(\mu_i^{-1}y+x_i)}{(1+|\mu_i^{-1}\mu_jy+\mu_j(x_i-x_j)|^2)^{(N+2)/4}}\,dy\\ \le&\frac{Csd_i}{\mu_i^{(N-2)/2}}\sum_{i\neq j}\frac{1}{(\mu_j|x_i-x_j|)^{(N+2)/2}}=O\Big(\frac{1}{s^{4(1+\sigma)(N-2)/(N^2-6N+4)}}\Big).\end{aligned}$$

□

Next, we will expand the derivatives of $J_s(x,\mu,0)$ with respect to x and μ. Intuitively, we can differentiate (B.4) and obtain the desired results.

Proposition B.4. *Assume that $N\ge 6$ and $(x,\mu)\in M_k$. For any $i=1,\cdots,k$, we have*

$$\begin{aligned}&\frac{\partial J_s(x,\mu,0)}{\partial\mu_i}\\ =&\frac{2B_1\lambda}{\mu_i^3}-\frac{N-2}{2}\frac{B_2(-\frac{\partial\varphi_1(x_i)}{\partial n})d_is}{\mu_i^{N/2}}-\frac{(N-2)c_0\tilde{c}B_2}{2\mu_i^{N-1}d_i^{N-2}}\\ &+O\Big(\frac{1}{s^{6(N-2)/(N^2-6N+4)+\theta}}\Big),\end{aligned}\tag{B.5}$$

and

$$\begin{aligned}&\frac{\partial J_s(x,\mu,0)}{\partial x_{ih}}\\=&\frac{B_2 s}{\mu_i^{(N-2)/2}}\frac{\partial \varphi_1(x_i)}{\partial x_h}+\frac{c_0\tilde{c}B_2}{2\mu_i^{N-2}}\frac{\partial H(y,x_i)}{\partial y_h}\Big|_{y=x_i}+O\Big(\frac{s^{2(N-4)/(N^2-6N+4)}}{s^{4(N-2)/(N^2-6N+4)+\theta}}\Big).\end{aligned}\tag{B.6}$$

In particular,

$$\begin{aligned}&\frac{\partial J_s(x,\mu,0)}{\partial d_i}\\=&\frac{B_2(-\frac{\partial\varphi_1(x_i)}{\partial n})s}{\mu_i^{(N-2)/2}}-\frac{c_0\tilde{c}B_2(N-2)}{2\mu_i^{N-2}d_i^{N-1}}+O\Big(\frac{s^{2(N-4)/(N^2-6N+4)}}{s^{4(N-2)/(N^2-6N+4)+\theta}}\Big).\end{aligned}$$

Proof. We will prove (B.5). The proof of (B.6) is similar.

Note that for any $(x,\mu)\in M_k$, we have

$$\varepsilon_{ij},\quad |x_i-x_j|\bar{\varepsilon}_{ij},\quad \bar{\varepsilon}_{ij}^{1+\sigma}\le Cs^{-4(N-2)/(N^2-6N+4)-\theta}.$$

It is easy to see that

$$\begin{aligned}\frac{\partial J_s(x,\mu,0)}{\partial \mu_i}&=\Big\langle I_s'(\sum_{j=1}^k PU_{x_j,\mu_j}),\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\Big\rangle\\=&\Big\langle \tilde{I}'(\sum_{j=1}^k PU_{x_j,\mu_j}),\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\Big\rangle\\&-\int_\Omega\Big((\sum_{j=1}^k PU_{x_j,\mu_j}-s\varphi_1)_+^{2^*-1}-(\sum_{j=1}^k PU_{x_j,\mu_j})^{2^*-1}\Big)\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\,dy\end{aligned}\tag{B.7}$$

Similar to [3, 26], we can prove

$$\begin{aligned}&\Big\langle \tilde{I}'(\sum_{j=1}^k PU_{x_j,\mu_j}),\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\Big\rangle\\=&\frac{2B_1\lambda}{\mu_i^3}-\frac{(N-2)c_0\tilde{c}B_2}{2\mu_i^{N-1}d_i^{N-2}}+\sum_{i\ne j}\frac{1}{\mu_j}O\Big(\varepsilon_{ij}+\bar{\varepsilon}_{ij}^{1+\sigma}+|x_i-x_j|\bar{\varepsilon}_{ij}\Big)\\=&\frac{2B_1\lambda}{\mu_i^3}-\frac{(N-2)c_0\tilde{c}B_2}{2\mu_i^{N-1}d_i^{N-2}}+O\Big(\frac{1}{s^{6(N-2)/(N^2-6N+4)+\theta}}\Big).\end{aligned}\tag{B.8}$$

On the other hand,

$$
\begin{aligned}
&\int_\Omega \Big((\sum_{j=1}^k PU_{x_j,\mu_j} - s\varphi_1)_+^{2^*-1} - (\sum_{j=1}^k PU_{x_j,\mu_j})^{2^*-1} \Big) \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i}\, dy \\
=&(2^*-1) \int_\Omega (\sum_{j=1}^k PU_{x_j,\mu_j})^{2^*-2} s\varphi_1 \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i}\, dy \qquad \text{(B.9)} \\
&+ O\Big(\int_\Omega (\sum_{j=1}^k PU_{x_j,\mu_j})^{2^*-2-\sigma} (s\varphi_1)^{1+\sigma} \frac{1}{\mu_i} PU_{x_i,\mu_i} \Big).
\end{aligned}
$$

But

$$
\begin{aligned}
&\int_\Omega (\sum_{j=1}^k PU_{x_j,\mu_j})^{2^*-2-\sigma} (s\varphi_1)^{1+\sigma} \frac{1}{\mu_i} PU_{x_i,\mu_i}\, dy \\
\le& C \sum_{j=1}^k \frac{1}{\mu_i} \int_\Omega U_{x_j,\mu_j}^{2^*-1-\sigma} (s\varphi_1)^{1+\sigma}\, dy \le \frac{C}{\mu_i} \sum_{j=1}^k \frac{(s\varphi_1(x_j))^{1+\sigma}}{\mu_j^{(1+\sigma)(N-2)/2}} \qquad \text{(B.10)} \\
=& O\Big(\frac{1}{s^{(1+\sigma)6(N-2)/(N^2-6N+4)}} \Big).
\end{aligned}
$$

For any $a > 0$ and $b > 0$, we have the following estimate

$$
a\big((a+b)^p - a^p \big) \le C a^{(p+1)/2} b^{(p+1)/2}, \qquad \text{(B.11)}
$$

where $p \in (0,1)$ is a fixed constant. Using the above inequality, we obtain

$$
\begin{aligned}
&\int_\Omega (\sum_{j=1}^k PU_{x_j,\mu_j})^{2^*-2} s\varphi_1 \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i}\, dy \\
=& \int_\Omega (PU_{x_i,\mu_i})^{2^*-2} s\varphi_1 \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i}\, dy + \frac{1}{\mu_i} \sum_{j\ne i}^k O\Big(\int_\Omega U_{x_i,\mu_i}^{(2^*-1)/2} U_{x_j,\mu_j}^{(2^*-1)/2} s\varphi_1 \Big) \\
=& \int_\Omega (PU_{x_i,\mu_i})^{2^*-2} s\varphi_1 \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i}\, dy + \frac{1}{\mu_i} \sum_{j\ne i}^k O\Big(\frac{s\varphi(x_i)}{\mu_i^{N-2)/2}} \frac{1}{(\mu_j |x_i - x_j|)^{(N+2)/2}} \Big) \qquad \text{(B.12)} \\
=& \int_\Omega (PU_{x_i,\mu_i})^{2^*-2} s\varphi_1 \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i}\, dy + O\Big(\frac{1}{s^{(1+\sigma)6(N-2)/(N^2-6N+4)}} \Big).
\end{aligned}
$$

It is easy to see

$$
\begin{aligned}
&(2^*-1)\int_\Omega (PU_{x_i,\mu_i})^{2^*-2}s\varphi_1\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\,dy\\
=&(2^*-1)\int_\Omega U^{2^*-2}_{x_i,\mu_i}s\varphi_1\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}\,dy+\frac{1}{\mu_i}O\Big(\frac{s\varphi_1(x_i)}{\mu_i^{(N-2)/2}}\frac{1}{(d_i\mu_i)^2}\Big)\\
=&-\frac{N-2}{2}\frac{B_2(-\frac{\partial\varphi_1(x_i)}{\partial n})d_is}{\mu_i^{N/2}}+O\Big(\frac{1}{s^{(1+\sigma)6(N-2)/(N^2-6N+4)}}\Big).
\end{aligned}
\tag{B.13}
$$

Combining (B.9)–(B.13), we find

$$
\begin{aligned}
&\int_\Omega\Big((\sum_{j=1}^k PU_{x_j,\mu_j}-s\varphi_1)_+^{2^*-1}-(\sum_{j=1}^k PU_{x_j,\mu_j})^{2^*-1}\Big)\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\,dy\\
=&-\frac{N-2}{2}\frac{B_2(-\frac{\partial\varphi_1(x_i)}{\partial n})d_is}{\mu_i^{N/2}}+O\Big(\frac{1}{s^{(1+\sigma)6(N-2)/(N^2-6N+4)}}\Big).
\end{aligned}
\tag{B.14}
$$

So, the result follows from (B.7), (B.8) and (B.14).

□

Appendix C

In this section, we will estimate the derivatives of the function $K(\bar{x},d,\mu)$. Basically, what we need to prove is that the perturbation term $\omega_{s,x,\mu}$ is negligible in the expansion of these derivatives.

In the following, we will use ∂_i to denote either $\frac{\partial}{\partial\mu_i}$ or $\frac{\partial}{\partial x_{ih}}$.

Using Proposition 2.1, we find

$$
\begin{aligned}
&\partial_iK(\bar{x},d,\mu)=\partial_iJ_s(x,\mu,\omega_{s,x,\mu})+\Big\langle\frac{\partial J_s(x,\mu,\omega_{s,x,\mu})}{\partial\omega},\partial_i\omega_{s,x,\mu}\Big\rangle\\
=&\partial_iJ_s(x,\mu,\omega_{s,x,\mu})+\sum_{j=1}^k A_j\Big\langle\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j},\partial_i\omega_{s,x,\mu}\Big\rangle+\sum_{j=1}^k\sum_{h=1}^N B_{jh}\Big\langle\frac{\partial PU_{x_j,\mu_j}}{\partial x_{jh}},\partial_i\omega_{s,x,\mu}\Big\rangle\\
=&\partial_iJ_s(x,\mu,\omega_{s,x,\mu})-A_i\Big\langle\partial_i\frac{\partial PU_{x_i,\mu_i}}{\partial\mu_i},\omega_{s,x,\mu}\Big\rangle-\sum_{h=1}^N B_{ih}\Big\langle\partial_i\frac{\partial PU_{x_i,\mu_i}}{\partial x_{ih}},\omega_{s,x,\mu}\Big\rangle.
\end{aligned}
\tag{C.1}
$$

Thus, to estimate $\partial_iK(x,\mu)$, we need to estimate $\partial_iJ_s(x,\mu,\omega_{s,x,\mu})$, A_j and B_{jh}.

First, we estimate $\partial_iJ_s(x,\mu,\omega_{s,x,\mu})$.

Lemma C.1. *Let $\omega_{s,x,\mu}$ be the function obtained in Proposition 2.1. Then for any fixed $i=1,\cdots,k$, and $h=1,\cdots,N$,*

$$
\frac{\partial J_s(x,\mu,\omega_{s,x,\mu})}{\partial\mu_i}=\frac{\partial J_s(x,\mu,0)}{\partial\mu_i}+O\Big(\frac{1}{s^{\sigma+6(N-2)/(N^2-6N+4)}}\Big),
\tag{C.2}
$$

and

$$\frac{\partial J_s(x,\mu,\omega_{s,x,\mu})}{\partial x_{ih}} = \frac{\partial J_s(x,\mu,0)}{\partial x_{ih}} + O\Big(\frac{s^{2(N-4)/(N^2-6N+4)}}{s^{\sigma+4(N-2)/(N^2-6N+4)}}\Big), \tag{C.3}$$

where $\sigma > 0$ is some constant.

Proof. We just prove (C.2), since (C.3) can be proved in a similar way.
We have

$$\begin{aligned}
&\frac{\partial J_s(x,\mu,\omega_{s,x,\mu})}{\partial \mu_i} = \Big\langle I_s'\Big(\sum_{j=1}^k PU_{x_j,\mu_j} + \omega_{s,x,\mu}\Big), \frac{\partial(PU_{x_i,\mu_i})}{\partial \mu_i}\Big\rangle \\
=&\frac{\partial J_s(x,\mu,0)}{\partial \mu_i} - \lambda \int_\Omega \omega_{s,x,\mu} \frac{\partial(PU_{x_i,\mu_i})}{\partial \mu_i} \\
&- \int_\Omega \Big[\Big(\sum_{j=1}^k PU_{x_j,\mu_j} + \omega_{s,x,\mu} - s\varphi_1\Big)_+^{2^*-1} - \Big(\sum_{j=1}^k PU_{x_j,\mu_j} - s\varphi_1\Big)_+^{2^*-1}\Big] \frac{\partial(PU_{x_i,\mu_i})}{\partial \mu_i}.
\end{aligned} \tag{C.4}$$

But

$$\Big|\int_\Omega \omega_{s,x,\mu} \frac{\partial(PU_{x_i,\mu_i})}{\partial \mu_i}\Big| \le \frac{C}{\mu_i}\int_\Omega |\omega_{s,x,\mu}| U_{x_i,\mu_i} \le \frac{C\ln \mu_i}{\mu_i^3}\|\omega_{s,x,\mu}\|. \tag{C.5}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega \Big[\Big(\sum_{j=1}^k PU_{x_j,\mu_j} + \omega_{s,x,\mu} - s\varphi_1\Big)_+^{2^*-1} - \Big(\sum_{j=1}^k PU_{x_j,\mu_j} - s\varphi_1\Big)_+^{2^*-1}\Big] \frac{\partial(PU_{x_i,\mu_i})}{\partial \mu_i} \\
=&(2^*-1)\int_\Omega \Big(\sum_{j=1}^k PU_{x_j,\mu_j} - s\varphi_1\Big)_+^{2^*-2} \omega_{s,x,\mu} \frac{\partial(PU_{x_i,\mu_i})}{\partial \mu_i} + O\Big(\frac{1}{\mu_i}\|\omega_{s,x,\mu}\|^2\Big).
\end{aligned} \tag{C.6}$$

For fixed small constant $\sigma > 0$, we have

$$\big|(a-b)_+^p - a^p\big| \le C a^{p-(\frac12+\sigma)} b^{\frac12+\sigma}, \quad \forall\ a>0,\ b>0,$$

where $p \in (0,1)$. So,

$$
\begin{aligned}
&\int_\Omega\Big(\sum_{j=1}^k PU_{x_j,\mu_j}-s\varphi_1\Big)_+^{2^*-2}\omega_{s,x,\mu}\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\\
=&\int_\Omega\Big(\sum_{j=1}^k PU_{x_j,\mu_j}\Big)^{2^*-2}\omega_{s,x,\mu}\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\\
&+O\Big(\frac{1}{\mu_i}\int_\Omega\Big(\sum_{j=1}^k U_{x_j,\mu_j}\Big)^{2^*-2-\frac12-\sigma}(s\varphi_1)^{\frac12+\sigma}|\omega_{s,x,\mu}|U_{x_i,\mu_i}\Big).
\end{aligned}
\tag{C.7}
$$

Since

$$
\begin{aligned}
&\frac{1}{\mu_i}\int_\Omega\Big(\sum_{j=1}^k U_{x_j,\mu_j}\Big)^{2^*-2-\frac12-\sigma}(s\varphi_1)^{\frac12+\sigma}|\omega_{s,x,\mu}|U_{x_i,\mu_i}\\
\le& C\frac{1}{\mu_i}\int_\Omega\Big(\sum_{j=1}^k U_{x_j,\mu_j}^{2^*-1-\frac12-\sigma}(s\varphi_1)^{\frac12+\sigma}|\omega_{s,x,\mu}|\Big)\\
\le& C\sum_{j=1}^k\Big(\frac{s\varphi_1(x_j)}{\mu_j^{(N-2)/2}}\Big)^{\frac12+\sigma}\|\omega_{s,x,\mu}\|,
\end{aligned}
$$

we obtain from (C.7) that

$$
\begin{aligned}
&\int_\Omega\Big(\sum_{j=1}^k PU_{x_j,\mu_j}-s\varphi_1\Big)_+^{2^*-2}\omega_{s,x,\mu}\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\\
=&\int_\Omega\Big(\sum_{j=1}^k PU_{x_j,\mu_j}\Big)^{2^*-2}\omega_{s,x,\mu}\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}+O\Big(\frac{1}{\mu_i}\sum_{j=1}^k\Big(\frac{s\varphi_1(x_j)}{\mu_j^{(N-2)/2}}\Big)^{\frac12+\sigma}\Big)\|\omega_{s,x,\mu}\|.
\end{aligned}
\tag{C.8}
$$

Finally,

$$
\begin{aligned}
&\int_\Omega\Big(\sum_{j=1}^k PU_{x_j,\mu_j}\Big)^{2^*-2}\omega_{s,x,\mu}\frac{\partial(PU_{x_i,\mu_i})}{\partial\mu_i}\\
=&\int_\Omega\Big(\sum_{j=1}^k U_{x_j,\mu_j}\Big)^{2^*-2}\omega_{s,x,\mu}\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}+\frac{1}{\mu_i}\sum_{j=1}^k O\Big(\frac{1}{(d_j\mu_j)^{(N+2)/2}}\Big)\|\omega_{s,x,\mu}\|.
\end{aligned}
\tag{C.9}
$$

Using (B.11), we find

$$\int_\Omega\Big(\sum_{j=1}^k U_{x_j,\mu_j}\Big)^{2^*-2}\omega_{s,x,\mu}\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}=\int_\Omega\Big(\big(\sum_{j=1}^k U_{x_j,\mu_j}\big)^{2^*-2}-U_{x_i,\mu_i}^{2^*-2}\Big)\omega_{s,x,\mu}\frac{\partial U_{x_i,\mu_i}}{\partial\mu_i}$$
$$=O\Big(\frac{1}{\mu_i}\sum_{j\neq i}\int_\Omega U_{x_i,\mu_i}^{(2^*-1)/2}U_{x_j,\mu_j}^{(2^*-1)/2}|\omega_{s,x,\mu}|\Big)=\Big(\frac{1}{\mu_i}\sum_{l\neq j}\bar\varepsilon_{lj}^{(1+\sigma)/2}\Big)\|\omega_{s,x,\mu}\|. \tag{C.10}$$

So, (C.2) follows from (C.4)–(C.10) and Proposition 2.1.

□

Next, we estimate A_j and B_{jh}.

Lemma C.2. *Let A_i and B_{ih} be the constants obtained in Proposition 2.1. Then, we have*

$$A_i=O\big(s^{-2(N-2)/(N^2-6N+4)}\big),$$

and

$$B_{ih}=O\Big(\frac{s^{2(N-4)/(N^2-6N+4}}{s^{8(N-2)/(N^2-6N+4)}}\Big).$$

Proof. From Proposition B.4 and Lemma C.1, we know that A_i and B_{ih} satisfy

$$\sum_{j=1}^k\Big\langle\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j},\frac{\partial PU_{x_i,\mu_i}}{\partial\mu_i}\Big\rangle A_j+\sum_{j=1}^k\sum_{h=1}^N\Big\langle\frac{\partial PU_{x_j,\mu_j}}{\partial x_{jh}},\frac{\partial PU_{x_i,\mu_i}}{\partial\mu_i}\Big\rangle B_{jh}$$
$$=\Big\langle\frac{\partial J_s}{\partial\omega},\frac{\partial PU_{x_i,\mu_i}}{\partial\mu_i}\Big\rangle=O\big(s^{-6(N-2)/(N^2-6N+4)}\big); \tag{C.11}$$

$$\sum_{j=1}^k\Big\langle\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j},\frac{\partial PU_{x_i,\mu_i}}{\partial x_{im}}\Big\rangle A_j+\sum_{j=1}^k\sum_{h=1}^N\Big\langle\frac{\partial PU_{x_j,\mu_j}}{\partial x_{jh}},\frac{\partial PU_{x_i,\mu_i}}{\partial x_{im}}\Big\rangle B_{jh}$$
$$=\Big\langle\frac{\partial J_s}{\partial\omega},\frac{\partial PU_{x_i,\mu_i}}{\partial x_{im}}\Big\rangle=O\Big(\frac{s^{2(N-4)/(N^2-6N+4)}}{s^{4(N-2)/(N^2-6N+4)}}\Big). \tag{C.12}$$

Thus, we can solve (C.11) and (C.12) to obtain the result. □

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. It follows directly from Proposition B.4, Lemmas C.1 and C.2.

□

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