INFINITELY MANY POSITIVE SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$

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Abstract. We consider the following nonlinear problem in $\mathbb{R}^N$

$$-\Delta u + V(|y|)u = u^p, \quad u > 0 \ \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N) \tag{0.1}$$

where $V(r)$ is a positive function, $1 < p < \frac{N+2}{N-2}$. We show that if $V(r)$ has the following expansion: There are constants $a > 0$, $m > 1$, $\theta > 0$, and $V_0 > 0$, such that

$$V(r) = V_0 + \frac{a}{r^m} + O\big(\frac{1}{r^{m+\theta}}\big), \quad \text{as } r \to +\infty,$$

then (0.1) has **infinitely many non-radial positive** solutions, whose energy can be made arbitrarily large.

1. Introduction

Standing waves for the following nonlinear Schrödinger equation in $\mathbb{R}^N$:

$$-i\frac{\partial \psi}{\partial t} = \Delta \psi - \tilde{V}(y)\psi + |\psi|^{p-1}\psi, \tag{1.1}$$

where $p > 1$, are solutions of the form $\psi(t, y) = \exp(i\lambda t)u(y)$. Assuming that the amplitude $u(y)$ is positive and vanishes at infinity, we see that ψ satisfies (1.1) if and only if u solves the nonlinear elliptic problem

$$-\Delta u + V(y)u = u^p, \quad u > 0, \quad \lim_{|y|\to+\infty} u(y) = 0, \tag{1.2}$$

where $V(y) = \tilde{V}(y) + \lambda$. In the rest of this paper, we will assume that V is bounded, and $V(y) \ge V_0 > 0$.

A problem which is similar to (1.2) is the following scalar field equation:

$$-\Delta u + u = Q(y)u^p, \quad u > 0, \quad \lim_{|y|\to+\infty} u(y) = 0, \tag{1.3}$$

where $Q(y)$ is bounded, and $Q(y) \ge Q_0 > 0$.

If

$$\inf_{y\in\mathbb{R}^N} V(y) < \lim_{|x|\to\infty} V(x), \quad (\text{or } \sup_{y\in\mathbb{R}^N} Q(y) > \lim_{|x|\to\infty} Q(x)), \tag{1.4}$$

then, using the concentration compactness principle [24, 25], one can show that (1.2) and (1.3) have a least energy solution. See for example [17, 24, 25, 28]. But if (1.4) does not hold, (1.2) (or (1.3)) may not have least energy solution. So, one needs to find solution with higher energy level. For results on this aspect, the readers can refer to [4, 5, 6]. Note that the energy of the solutions in [4, 6] is less than twice of the first level at which the Palais-Smale condition fails.

Recently, Cerami, Devillanova and Solimini [9] showed that the following problem

$$-\Delta u + V(y)u = |u|^{p-1}u, \quad \lim_{|y|\to+\infty} u(y) = 0,$$

has **infinitely many sign-changing solutions** if $V(y)$ tends to its limit at infinity from below with a suitably rate. Except [10], where $Q(y)$ is periodic, there is no result on the multiplicity of positive solutions for (1.2) (or (1.3)).

On the other hand, if we consider the following singularly perturbed problem:

$$-\varepsilon^2\Delta u + V(y)u = u^p, \quad u > 0, \quad \lim_{|y|\to+\infty} u(y) = 0, \tag{1.5}$$

or

$$-\varepsilon^2\Delta u + u = Q(y)u^p, \quad u > 0, \quad \lim_{|y|\to+\infty} u(y) = 0, \tag{1.6}$$

where $\varepsilon > 0$ is a small parameter, then the number of the critical points of $V(y)$ (or $Q(y)$) (see for example [1, 7, 8],[13]–[16],[18, 27, 29]), the type of the critical points of $V(y)$ (or $Q(y)$) (see for example [12, 21, 26], and the topology of the level set of $V(y)$ (or $Q(y)$) [2, 3, 11, 19], can affect the number of the solutions for (1.5) (or (1.6)). But for the singularly perturbed problems (1.5) and (1.6), the parameter ε will tend to zero as the number of the solutions tends to infinity. So, all these results do not give any multiplicity result for (1.2) (or (1.3)).

In this paper, we assume that $V(y)$ is radial. That is, $V(y) = V(|y|)$. Thus, we consider the following problem

$$-\Delta u + V(|y|)u = u^p, u > 0 \ \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{1.7}$$

where $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $1 < p < +\infty$ if $N = 2$. We assume

$$\lim_{|y|\to+\infty} V(|y|) = V_0 > 0.$$

Note that if $V(r)$ is non-decreasing, by [20], any solution of (1.7) is radial.

The aim of this paper is to obtain **infinitely many non-radial positive solutions** for (1.7) under an assumption for $V(r)$ near the infinity. We assume that $V(r) > 0$ satisfies the following condition:

(V): There is are constants $a > 0$, $m > 1$, $\theta > 0$, and $V_0 > 0$, such that

$$V(r) = V_0 + \frac{a}{r^m} + O\big(\frac{1}{r^{m+\theta}}\big), \tag{1.8}$$

as $r \to +\infty$. (Without loss of generality, we may assume that $V_0 = 1$.)

Our main result in this paper can be stated as follows:

Theorem 1.1. *If $V(r)$ satisfies (V), then problem (1.7) has infinitely many non-radial positive solutions.*

Remark 1.2. To obtain the result in Theorem 1.1, (1.8) can not be changed to

$$V(r) = V_0 - \frac{a}{r^m} + O\big(\frac{1}{r^{m+\theta}}\big), \tag{1.9}$$

In fact, it is easy to find a function $V(r)$, satisfying $V'(r) \geq 0$ and (1.9). So, for this V, all the solutions must be radial.

Remark 1.3. The radial symmetry can be replaced by the following weaker symmetry assumption: after suitably rotating the coordinate system,

(V1) $V(y) = V(y', y'') = V(|y'|, |y_3|, ..., |y_N|)$, where $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$,

(V2) $V(y) = V_0 + \frac{a}{|y|^m} + O\big(\frac{1}{|y|^{m+\theta}}\big)$ as $|y| \to +\infty$, where $a > 0$, $m > 1$, $\theta > 0$, and $V_0 > 0$ are some constants.

We believe that Theorem 1.1 is still true for non-radial potential $V(y)$. So, we make the following conjecture:

Conjecture: *Problem (1.2) has infinitely many solutions, if*

$$V(y) = V_0 + \frac{a}{|y|^m} + O\big(\frac{1}{|y|^{m+\theta}}\big),$$

as $|y| \to +\infty$, where $V_0 > 0$, $a > 0$, $m > 0$ and $\theta > 0$ are some constants.

Remark 1.4. Using the same argument, we can prove that if

$$Q(r) = Q_0 - \frac{a}{r^m} + O\big(\frac{1}{r^{m+\theta}}\big),$$

where $Q_0 > 0$, $a > 0$, $m > 1$ and $\theta > 0$ are some constants, then

$$-\Delta u + u = Q(|y|)u^p, \quad u > 0, \quad u \in H^1(\mathbb{R}^N),$$

has infinitely many positive non-radial solutions.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1. We will construct solutions with large number of bumps near the infinity. Since we assume

$$\lim_{|y|\to+\infty} V(|y|) = 1,$$

we will use the solution of

$$\begin{cases} -\Delta u + u = u^p, \ \ u > 0, & \text{in } \mathbb{R}^N, \\ u(y) \to 0, & \text{as } |y| \to \infty, \end{cases} \tag{1.10}$$

to build up the approximate solutions for (1.7). It is well-known that (1.10) has a unique solution U, satisfying $U(y) = U(|y|)$, $U' < 0$.

Let

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, $r \in [r_0 k\ln k, r_1 k \ln k]$ for some $r_1 > r_0 > 0$.

Set $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. Define

$$\begin{aligned} H_s = \big\{u : & u \in H^1(\mathbb{R}^N), u \text{ is even in } y_h, h = 2, \cdots, N, \\ & u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta + \frac{2\pi j}{k}), r\sin(\theta + \frac{2\pi j}{k}), y'')\big\}. \end{aligned}$$

Let

$$W_r(y) = \sum_{j=1}^{k} U_{x_j}(y),$$

where $U_{x_j}(y) = U(y - x_j)$.

Theorem 1.1 is a direct consequence of the following result:

Theorem 1.5. *Suppose that $V(r)$ satisfies (V). Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (1.7) has a solution u_k of the form*

$$u_k = W_{r_k}(y) + \omega_k,$$

where $\omega_k \in H_s$, $r_k \in [r_0 k \ln k, r_1 k \ln k]$ and as $k \to +\infty$,

$$\int_{\mathbb{R}^N} \big(|D\omega_k|^2 + \omega_k^2\big) \to 0.$$

We will use the techniques in the singularly perturbed elliptic problems to prove Theorem 1.5. We know that there is always a small parameter in a singularly perturbed elliptic problem. Although there is no parameter in (1.7), we use k, **the number of the bumps** of the solutions, as the parameter in the construction of spike solutions for (1.7). This is the **new idea** of this paper. This is partly motivated by recent paper of Lin-Ni-Wei [23] where they constructed multiple spikes to a singularly perturbed problem. There they allowed the number of spikes to depend on the small parameter.

This paper is organized as follows. In section 2, we will carry out the reduction. Then, we will study the reduced finite dimensional problem and prove Theorem 1.5. We will leave all the technical calculations in the appendix.

2. Proof the the main result

Let

$$Z_j = \frac{\partial U_{x_j}}{\partial r}, \quad j = 1, \cdots, k,$$

where $x_j = \big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\big)$. In this paper, we always assume

$$r \in S_k =: \big[(\frac{m}{2\pi} - \beta)k\ln k, (\frac{m}{2\pi} + \beta)k\ln k\big], \tag{2.1}$$

where m is the constant in the expansion for V, and $\beta > 0$ is a small constant.

Define

$$E=\big\{v: v\in H_s,\ \int_{\mathbb{R}^N}U_{x_j}^{p-1}Z_jv=0,\ j=1,\cdots,k\big\}$$

The norm of $H^1(\mathbb{R}^N)$ is defined as follows:

$$\|v\|=\sqrt{\langle v,v\rangle},\quad v\in H^1(\mathbb{R}^N),$$

where

$$\langle v_1,v_2\rangle=\int_{\mathbb{R}^N}\big(Dv_1Dv_2+V(|y|)v_1v_2\big).$$

It is easy to check that

$$\int_{\mathbb{R}^N}\big(Dv_1Dv_2+V(|y|)v_1v_2-pW_r^{p-1}v_1v_2\big),\quad v_1,v_2\in E,$$

is a bounded bilinear functional in E. Thus, there is a bounded linear operator L from E to E, such that

$$\langle Lv_1,v_2\rangle=\int_{\mathbb{R}^N}\big(Dv_1Dv_2+V(|y|)v_1v_2-pW_r^{p-1}v_1v_2\big),\quad v_1,v_2\in E.$$

The next lemma shows that L is invertible in E.

Lemma 2.1. *There is a constant $\rho>0$, independent of k, such that for any $r\in S_k$,*

$$\|Lv\|\ge\rho\|v\|,\quad v\in E.$$

Proof. We argue by contradiction. Suppose that there are $n\to+\infty$, $r_k\in S_k$, and $v_k\in E$, with

$$\|Lv_k\|=o(1)\|v_k\|.$$

Then

$$\langle Lv_k,\varphi\rangle=o(1)\|v_k\|\|\varphi\|,\quad\forall\,\varphi\in E. \tag{2.2}$$

We may assume that $\|v_k\|^2=k$.

Let

$$\Omega_j=\big\{y=(y',y'')\in\mathbb{R}^2\times\mathbb{R}^{N-2}:\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\rangle\ge\cos\frac{\pi}{k}\big\}.$$

By symmetry, we see from (2.2),

$$\int_{\Omega_1}\big(Dv_kD\varphi+V(|y|)v_k\varphi-pW_r^{p-1}v_k\varphi\big)=\frac{1}{k}\langle Lv_k,\varphi\rangle=o\big(\frac{1}{\sqrt{k}}\big)\|\varphi\|,\quad\forall\,\varphi\in E. \tag{2.3}$$

In particular,

$$\int_{\Omega_1}\big(|Dv_k|^2+V(|y|)v_k^2-pW_r^{p-1}v_k^2\big)=o(1),$$

and

$$\int_{\Omega_1}\big(|Dv_k|^2+V(|y|)v_k^2\big)=1. \tag{2.4}$$

Let $\bar v_k(y)=v_k(y-x_1)$. Then for any $R>0$, since $|x_2-x_1|=r\sin\frac{\pi}{k}\ge\frac{m}{4}\ln k$, we see that $B_R(x_1)\subset\Omega_1$. As a result, from (2.4), we find that for any $R>0$,

$$\int_{B_R(0)}\big(|D\bar v_k|^2+V(|y|)\bar v_k^2\big)\le 1.$$

So, we may assume that there is a $v \in H^1(\mathbb{R}^N)$, such that as $k \to +\infty$,

$$\bar{v}_k \to v, \quad \text{weakly in } H^1_{loc}(\mathbb{R}^N),$$

and

$$\bar{v}_k \to v, \quad \text{strongly in } L^2_{loc}(\mathbb{R}^N).$$

Since $\bar{v}_k$ is even in y_h, $h = 2, \cdots, N$, it is easy to see that v is even in y_h, $h = 2, \cdots, N$. On the other hand, from

$$\int_{\mathbb{R}^N} U_{x_1}^{p-1} Z_1 v_k = 0,$$

we obtain

$$\int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} \bar{v}_k = 0.$$

So, v satisfies

$$\int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} v = 0. \tag{2.5}$$

Now, we claim that v satisfies

$$-\Delta v + v - pU^{p-1} v = 0, \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

Define

$$\tilde{E} = \Big\{\varphi : \varphi \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} \varphi = 0\Big\}.$$

For any $R > 0$, let $\varphi \in C_0^\infty(B_R(0)) \cap \tilde{E}$ be any function, satisfying that φ is even in y_h, $h = 2, \cdots, N$. Then $\varphi_k(y) =: \varphi(y - x_1) \in C_0^\infty(B_R(x_1))$. Inserting φ_k into (2.3), using Lemma A.1, we find

$$\int_{\mathbb{R}^N} \big(DvD\varphi + v\varphi - pU^{p-1} v\varphi\big) = 0. \tag{2.7}$$

On the other hand, since v is even in y_h, $h = 2, \cdots, N$, (2.7) holds for any function $\varphi \in C_0^\infty(\mathbb{R}^N)$, which is odd in y_h, $h = 2, \cdots, N$. Therefore, (2.7) holds for any $\varphi \in C_0^\infty(B_R(0)) \cap \tilde{E}$. By the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$, it is easy to show that

$$\int_{\mathbb{R}^N} \big(DvD\varphi + v\varphi - pU^{p-1} v\varphi\big) = 0, \quad \forall\, \varphi \in \tilde{E}. \tag{2.8}$$

But (2.8) holds for $\varphi = \frac{\partial U}{\partial x_1}$. Thus (2.8) is true for any $\varphi \in H^1(\mathbb{R}^N)$. So, we have proved (2.6).

Since U is non-degenerate, we see that $v = c\frac{\partial U}{\partial x_1}$ because v is even in y_h, $h = 2, \cdots, N$. From (2.5), we find

$$v = 0.$$

As a result,

$$\int_{B_R(x_1)} v_k^2 = o(1), \quad \forall\, R > 0.$$

On the other hand, it follows from Lemma A.1 that for any small $\eta > 0$, there is a constant $C > 0$, such that

$$W_{r_k}(y) \le Ce^{-(1-\eta)|y - x_1|}, \quad y \in \Omega_1. \tag{2.9}$$

Thus,

$$\begin{aligned}o(1)=&\int_{\Omega_1}\big(|Dv_k|^2+V(|y|)v_k^2-pW_{r_k}^{p-1}v_k^2\big)\\=&\int_{\Omega_1}\big(|Dv_k|^2+V(|y|)v_k^2\big)+o(1)+O(e^{-(1-\eta)(p-1)R})\int_{\Omega_1}v_k^2\\\ge&\frac{1}{2}\int_{\Omega_1}\big(|Dv_k|^2+V(|y|)v_k^2\big)+o(1).\end{aligned}$$

This is a contradiction to (2.4).

□

Define

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}\big(|Du|^2+V(|y|)u^2-\frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}.$$

Let

$$J(\phi)=I(W_r+\phi),\quad \phi\in E.$$

We have

Proposition 2.2. *There is an integer $k_0>0$, such that for each $k\ge k_0$, there is a C^1 map from S_k to H_s: $\phi=\phi(r)$, $r=|x_1|$, satisfying $\phi\in E$, and*

$$\big\langle\frac{\partial J(\phi)}{\partial\phi},\varphi\big\rangle=0,\quad \forall\,\varphi\in E.$$

Moreover, there is a small $\sigma>0$, such that

$$\|\phi\|\le\frac{C}{k^{\frac{m-1}{2}+\sigma}}. \tag{2.10}$$

Proof. Expand $J(\phi)$ as follows:

$$J(\phi)=J(0)+l(\phi)+\frac{1}{2}\big\langle L\phi,\phi\big\rangle+R(\phi),\quad \phi\in E,$$

where

$$l(\phi)=\sum_{i=1}^k\int_{\mathbb{R}^N}(V(|y|)-1)U_{x_i}\phi+\int_{\mathbb{R}^N}\big(W^p-\sum_{i=1}^kU_{x_i}^p\big)\phi,$$

L is the bounded linear map from E to E in Lemma 2.1, and

$$R(\phi)=\frac{1}{p+1}\int_{\mathbb{R}^N}\big(|W+\phi|^{p+1}-W^{p+1}-(p+1)W^p\phi-\frac{1}{2}(p+1)pW^{p-1}\phi^2\big).$$

Since $l(\phi)$ is a bounded linear functional in E, we know that there is an $l_k\in E$, such that

$$l(\phi)=\big\langle l_k,\phi\big\rangle.$$

Thus, finding a critical point for $J(\phi)$ is equivalent to solving

$$l_k+L\phi+R'(\phi)=0. \tag{2.11}$$

By Lemma 2.1, L is invertible. Thus, (2.11) can be rewritten as

$$\phi=A(\phi)=:-L^{-1}l_k-L^{-1}R'(\phi).$$

Let

$$S=\big\{\phi:\phi\in E,\|\phi\|\le\frac{1}{k^{\frac{m-1}{2}}}\big\}.$$

If $p\le 2$, then it is easy to check that

$$\|R'(\phi)\| \le C\|\phi\|^p.$$

So, from Lemma 2.3 below,

$$\|A(\phi)\| \le C\|l_k\| + C\|\phi\|^p \le \frac{C}{k^{\frac{m-1}{2}+\sigma}} + \frac{C}{k^{\frac{p(m-1)}{2}}} \le \frac{1}{k^{\frac{m-1}{2}}}. \tag{2.12}$$

Thus, A maps S into S if $p \le 2$.

On the other hand, if $p \le 2$, then

$$\|R''(\phi)\| \le C\|\phi\|^{p-1}.$$

Thus,

$$\begin{aligned}\|A(\phi_1) - A(\phi_2)\| &= \|L^{-1}R'(\phi_1) - L^{-1}R'(\phi_2)\| \\ &\le C\big(\|\phi_1\|^{p-1} + \|\phi_2\|^{p-1}\big)\|\phi_1 - \phi_2\| \le \frac{1}{2}\|\phi_1 - \phi_2\|.\end{aligned}$$

So, we have proved that if $p \le 2$, A is a contraction map. Therefore, we have proved that if $p \le 2$, A is a contraction map from S to S. So, the result follows from the contraction mapping theorem.

It remains to deal with the case $p > 2$.

Suppose that $p > 2$. Since

$$\big|\langle R'(\phi), \xi\rangle\big| \le C\int_{\mathbb{R}^N} W_r^{p-2}|\phi|^2|\xi| \le C\Big(\int_{\mathbb{R}^N}\big(W_r^{p-2}|\phi|^2\big)^{\frac{p+1}{p}}\Big)^{\frac{p}{p+1}}\|\xi\|.$$

we find

$$\|R'(\phi)\| \le C\Big(\int_{\mathbb{R}^N}\big(W_r^{p-2}|\phi|^2\big)^{\frac{p+1}{p}}\Big)^{\frac{p}{p+1}}.$$

On the other hand, it follows from Lemma A.1 that W_r is bounded. Since $2 < \frac{2(p+1)}{p} < p+1$, we obtain

$$\|R'(\phi)\| \le C\Big(\int_{\mathbb{R}^N}|\phi|^{\frac{2(p+1)}{p}}\Big)^{\frac{p}{p+1}} \le \|\phi\|^2.$$

For the estimate of $\|R''(\phi)\|$, we have

$$\begin{aligned}\big|R''(\phi)(\xi,\eta)\big| &\le C\int_{\mathbb{R}^N} W_r^{p-2}|\phi||\xi||\eta| \le C\int_{\mathbb{R}^N}|\phi||\xi||\eta| \\ &\le C\Big(\int_{\mathbb{R}^N}|\phi|^3\Big)^{\frac{1}{3}}\Big(\int_{\mathbb{R}^N}|\xi|^3\Big)^{\frac{1}{3}}\Big(\int_{\mathbb{R}^N}|\eta|^3\Big)^{\frac{1}{3}} \le C\|\phi\|\|\xi\|\|\eta\|,\end{aligned}$$

since $2 < 3 < p+1$. So

$$\|R''(\phi)\| \le C\|\phi\|.$$

Thus,

$$\|R'(\phi)\| \le \frac{C}{k^{m-1}} \le \frac{C}{k^{\frac{m-1}{2}+\sigma}}.$$

As a result,

$$\|A(\phi)\| \le C\|l_k\| + C\|R'(\phi)\| \le \frac{C}{k^{\frac{m-1}{2}+\sigma}} \le \frac{1}{k^{\frac{m-1}{2}}}. \tag{2.13}$$

Thus, A maps S to S.

On the other hand,

$$\|R''(\phi)\| \le C\|\phi\| \le \frac{C}{k^{\frac{m-1}{2}}},$$

which implies that A is a contraction map. So, we have proved that if $p > 2$, then A is a contraction map from S to S. And the result follows from the contraction mapping theorem.

Finally, (2.10) follows from (2.12) and (2.13).

$\square$

Lemma 2.3. *There is a small $\sigma > 0$, such that*

$$\|l_k\| \le \frac{C}{k^{\frac{m-1}{2}+\sigma}}.$$

Proof. By the symmetry of the problem,

$$\begin{aligned}
&\sum_{i=1}^{k}\int_{\mathbb{R}^N}(V(|y|)-1)U_{x_i}\phi = k\int_{\mathbb{R}^N}(V(|y|)-1)U_{x_1}\phi\\
=&k\int_{\mathbb{R}^N}(V(|y-x_1|)-1)U\phi(y-x_1) \le kO\big(\frac{1}{r^m}\big)\|\phi\|\\
\le&\frac{C}{k^{\frac{m-1}{2}+\sigma}}\|\phi\|,
\end{aligned}\tag{2.14}$$

because $m > 1$.

On the other hand, for any $y \in \Omega_1$,

$$U_{x_i}^p \le U_{x_1}^{p-1}U_{x_i}.$$

Thus,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N}(W^p-\sum_{i=1}^{k}U_{x_i}^p)\phi\Big| = k\Big|\int_{\Omega_1}(W^p-\sum_{i=1}^{k}U_{x_i}^p)\phi\Big|\\
\le&Ck\int_{\Omega_1}U_{x_1}^{p-1}\sum_{j=2}^{k}U_{x_j}|\phi| \le Ck\sum_{j=2}^{k}e^{-\frac{p-\tau}{2}|x_j-x_1|}\Big(\int_{\Omega_1}|\phi|^{p+1}\Big)^{\frac{1}{p+1}}\\
\le&k^{\frac{p}{p+1}}\sum_{j=2}^{k}e^{-\frac{p-\tau}{2}|x_j-x_1|}\|\phi\|,
\end{aligned}\tag{2.15}$$

where $\tau > 0$ is any small fixed constant.

From the definition of S_k in (2.1), we see that for any $r \in S_k$

$$\sum_{j=2}^{k}e^{-\frac{p-\tau}{2}|x_j-x_1|} \le Ce^{-\frac{p-\tau}{2}\frac{2\pi}{k}r} \le \frac{C}{k^{\frac{p-\tau}{2}(m-\beta)}}.$$

Since

$$\frac{pm}{2}-\frac{p}{p+1} > \frac{m-1}{2}, \quad \forall\, m > 1,$$

we obtain from (2.15) that

$$\Big|\int_{\mathbb{R}^N}(W^p-\sum_{i=1}^{k}U_{x_i}^p)\phi\Big| \le \frac{C}{k^{\frac{m-1}{2}+\sigma}}\|\phi\|.\tag{2.16}$$

The result follows from (2.14) and (2.16).

$\square$

We are ready to prove Theorem 1.5. Let $\phi = \phi(r)$ be the map obtained in Proposition 2.2. Define

$$F(r) = I(W_r+\phi), \quad \forall\, r \in S_k.$$

It is well known that if r is a critical point of $F(r)$, then $W_r + \phi$ is a solution of (1.7). (See [21] or [23].)

Proof of Theorem 1.5. It follows from Propositions 2.2 and A.3 that

$$\begin{aligned} F(r) &= I(W) + l(\phi) + \frac{1}{2}\langle L\phi, \phi\rangle + R(\phi) \\ &= I(W) + O\big(\|l_k\|\|\phi\| + \|\phi\|^2\big) \\ &= k\Big(A + \frac{B_1}{r^m} - B_2 e^{-\frac{2\pi r}{k}} + O\big(\frac{1}{k^{m+\sigma}}\big)\Big) \end{aligned}$$

Consider

$$\max\{F(r) : r \in S_k\}. \tag{2.17}$$

For the definition of S_k, see (2.1). Since the function

$$\frac{B_1}{r^m} - B_2 e^{-\frac{2\pi r}{k}}$$

has a maximum point

$$\bar{r}_k = \big(\frac{m}{2\pi} + o(1)\big) k \ln k,$$

which is an interior point of S_k, it is easy to check that (2.17) is achieved by some r_k, which is in the interior of S_k. Thus, r_k is a critical point of $F(r)$. As a result

$$W_{r_k} + \phi(r_k)$$

is a solution of (1.7).

□

Appendix A. Energy Expansion

In this section, we will give the energy expansion for the approximate solutions. Recall

$$x_j = \big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\big), \quad j = 1, \cdots, k,$$

$$\Omega_j = \{y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\rangle \geq \cos\frac{\pi}{k}\}, \quad j = 1, \cdots, k,$$

and

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|Du|^2 + u^2\big) - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}.$$

Firstly, we have the following basic estimate:

Lemma A.1. *For any $y \in \Omega_1$, and $\eta \in (0,1]$, there is a constant $C > 0$, such that*

$$\sum_{j=2}^{k} U_{x_j}(y) \leq C e^{-\eta|x_1|\frac{\pi}{k}} e^{-(1-\eta)|y-x_1|}.$$

Proof. For any $y \in \Omega_1$, we have $|y - x_j| \geq |y - x_1|$.

If $|y - x_1| \geq 2|x_j - x_1|$, then for any $y \in \Omega_1$,

$$\begin{aligned} U_{x_j}(y) &\leq C e^{-|y-x_j|} \leq C e^{-|y-x_1|} \\ &= C e^{-\eta|y-x_1|} e^{-(1-\eta)|y-x_1|} \leq C e^{-2\eta|x_j-x_1|} e^{-(1-\eta)|y-x_1|}. \end{aligned}$$

If $|y - x_1| \leq 2|x_j - x_1|$, then

$$|y-x_j| \ge |x_j - x_1| - |y-x_1| \ge \frac{1}{2}|x_j - x_1|.$$

So for any $y \in \Omega_1$,

$$U_{x_j}(y) \le Ce^{-\eta|y-x_j|}e^{-(1-\eta)|y-x_1|} \le Ce^{-\frac{1}{2}\eta|x_j-x_1|}e^{-(1-\eta)|y-x_1|}.$$

Thus,

$$\begin{aligned}\sum_{j=2}^{k} U_{x_j}(y) \le Ce^{-(1-\eta)|y-x_1|}\sum_{j=2}^{k} e^{-\frac{1}{2}\eta|x_j-x_1|}\\ \le Ce^{-(1-\eta)|y-x_1|}\sum_{j=2}^{k} e^{-\eta|x_1|\sin\frac{j\pi}{k}} \le C_1 e^{-\eta|x_1|\frac{\pi}{k}}e^{-(1-\eta)|y-x_1|}.\end{aligned}$$

□

In this appendix, we denote $r = |x_1|$, and we always assume that

$$r \in S_k,$$

where S_k is defined in (2.1).

Proposition A.2. *We have*

$$I(U_{x_1}) = A + \frac{B_1}{r^m} + O\big(\frac{1}{k^{m+\theta}}\big),$$

where

$$A = \big(\frac{1}{2} - \frac{1}{p+1}\big)\int_{\mathbb{R}^N} U^{p+1}, \quad B_1 = \frac{a}{2}\int_{\mathbb{R}^N} U^2.$$

Proof. We have

$$\begin{aligned}I(U_{x_1}) =& \frac{1}{2}\int_{\mathbb{R}^N}\big(|DU|^2 + U^2\big) - \frac{1}{p+1}\int_{\mathbb{R}^N} U^{p+1} + \frac{1}{2}\int_{\mathbb{R}^N}\big(V(|y|)-1\big)U_{x_1}^2\\ =& A + \frac{1}{2}\int_{\mathbb{R}^N}\big(V(|y-x_1|)-1\big)U^2.\end{aligned} \tag{A.1}$$

On the other hand, for any small $\tau > 0$, using (V),

$$\begin{aligned}\int_{\mathbb{R}^N}\big(V(|y-x_1|)-1\big)U^2 =& \int_{B_{\frac{1}{2}r}(0)}\big(V(|y-x_1|)-1\big)U^2 + O\big(e^{-(1-\tau)r}\big)\\ =& \int_{B_{\frac{1}{2}r}(0)}\Big(\frac{a}{|y-x_1|^m} + O\big(\frac{1}{|y-x_1|^{m+\theta}}\big)\Big)U^2 + O\big(e^{-(1-\tau)r}\big).\end{aligned} \tag{A.2}$$

But for any $\alpha > 0$,

$$\frac{1}{|y-x_1|^\alpha} = \frac{1}{|x_1|^\alpha}\Big(1 + O\big(\frac{|y|}{|x_1|}\big)\Big), \quad y \in B_{\frac{1}{2}|x_1|}(0).$$

Thus,

$$\int_{B_{\frac{1}{2}r}(0)}\frac{1}{|y-x_1|^\alpha}U^2 = \frac{1}{|x_1|^\alpha}\int_{\mathbb{R}^N}U^2 + O\big(\frac{1}{|x_1|^{\alpha+1}} + e^{-(1-\tau)|x_1|}\big). \tag{A.3}$$

Inserting (A.3) into (A.2), we obtain

$$\int_{\mathbb{R}^N}\big(V(|y-x_1|)-1\big)U^2 = \frac{B_1}{r^m} + O\big(\frac{1}{k^{m+\theta}\ln^{m+\theta}k}\big). \tag{A.4}$$

Thus, the result follows from (A.1) and (A.4).

□

Proposition A.3. *There is a small constant $\sigma > 0$, such that*

$$I(W_r) = k\Big(A + \frac{B_1}{r^m} - B_2 e^{-\frac{2\pi r}{k}} + O\big(\frac{1}{k^{m+\sigma}}\big)\Big),$$

where A and B_1 are the constants in Proposition A.2, and $B_2 > 0$ is a positive constant.

Proof. Using the symmetry,

$$\begin{aligned}\int_{\mathbb{R}^N} (|DW_r|^2 + W_r^2) =& \sum_{j=1}^k \sum_{i=1}^k \int_{\mathbb{R}^N} U_{x_j}^p U_{x_i} \\ =& k\int_{\mathbb{R}^N} U^{p+1} + k\sum_{i=2}^k \int_{\mathbb{R}^N} U_{x_1}^p U_{x_i}.\end{aligned} \tag{A.5}$$

Recall

$$\Omega_j = \{y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\rangle \geq \cos\frac{\pi}{k}\}, \quad j = 1, \cdots, k.$$

It follows from Lemma A.1 that

$$\begin{aligned}&\int_{\mathbb{R}^N} (V(|y|) - 1)W_r^2 = k\int_{\Omega_1} (V(|y|) - 1)W_r^2 \\ =& k\int_{\Omega_1} (V(|y|) - 1)\Big(U_{x_1} + O\big(e^{-\frac{1}{2}|x_1|\frac{\pi}{k}} e^{-\frac{1}{2}|y-x_1|}\big)\Big)^2 \\ =& k\int_{\Omega_1} (V(|y|) - 1)U_{x_1}^2 + kO\Big(\int_{\Omega_1} |V(|y|) - 1| e^{-\frac{1}{2}|x_1|\frac{\pi}{k}} e^{-|y-x_1|}\Big) \\ =& k\Big(\frac{B_1}{|x_1|^m} + O\big(\frac{1}{k^{m+\theta}}\big)\Big).\end{aligned} \tag{A.6}$$

Suppose that $p \leq 3$. Then, for any $y \in \Omega_1$,

$$W_r^{p+1} = U_{x_1}^{p+1} + (p+1)U_{x_1}^p \sum_{j=2}^k U_{x_j} + O\Big(U_{x_1}^{\frac{p+1}{2}} \big(\sum_{j=2}^k U_{x_j}\big)^{\frac{p+1}{2}}\Big).$$

Using Lemma A.1, we have

$$\begin{aligned}U_{x_1}^{\frac{p+1}{2}} \big(\sum_{j=2}^k U_{x_j}\big)^{\frac{p+1}{2}} =& U_{x_1}^{\frac{p+1}{2}} \sum_{j=2}^k U_{x_j} \big(\sum_{j=2}^k U_{x_j}\big)^{\frac{p-1}{2}} \\ \leq& Ce^{-\eta\frac{p-1}{2}\frac{|x_1|\pi}{k}} U_{x_1}^{(1-\eta)p} \sum_{j=2}^k U_{x_j}\end{aligned}$$

But for any $r \in S_k$,

$$\sum_{j=2}^k e^{-|x_j - x_1|} \leq Ce^{-\frac{2\pi}{k}r} \leq \frac{C}{k^{m-\beta}}.$$

So, we obtain that for $p \in (1, 3]$, if $\beta > 0$ is small enough,

$$\begin{aligned}&\int_{\mathbb{R}^N} W_r^{p+1}=k\int_{\Omega_1} W_r^{p+1}\\ =&k\int_{\Omega_1}\Big(U_{x_1}^{p+1}+(p+1)\sum_{i=2}^k U_{x_1}^p U_{x_i}\Big)+O\big(\sum_{i=2}^k e^{-\eta\frac{p-1}{2}\frac{|x_1|\pi}{k}}e^{-|x_i-x_1|}\big)\Big)\\ =&k\Big(\int_{\mathbb{R}^N} U_{x_1}^{p+1}+(p+1)\sum_{i=2}^k\int_{\mathbb{R}^N} U_{x_1}^p U_{x_i}+O\big(\frac{1}{k^{m+\sigma}}\big)\Big).\end{aligned} \tag{A.7}$$

Suppose that $p>3$. Then for any $y\in\Omega_1$,

$$W_r^{p+1}=U_{x_1}^{p+1}+(p+1)U_{x_1}^p\sum_{j=2}^k U_{x_j}+O\Big(U_{x_1}^{p-1}\big(\sum_{j=2}^k U_{x_j}\big)^2\Big).$$

Since $p-1>2$, similar to the proof of (A.7), we can obtain the following estimate for $p>3$:

$$\begin{aligned}&\int_{\mathbb{R}^N} W_r^{p+1}=k\int_{\Omega_1} W_r^{p+1}\\ =&k\Big(\int_{\mathbb{R}^N} U_{x_1}^{p+1}+(p+1)\sum_{i=2}^k\int_{\mathbb{R}^N} U_{x_1}^p U_{x_i}+O\big(\frac{1}{k^{m+\sigma}}\big)\Big).\end{aligned} \tag{A.8}$$

Combining (A.5), (A.6), (A.7) and (A.8), we are led to

$$I(W_r)=k\Big(A+\frac{B_1}{r^m}-\frac{1}{2}\sum_{i=2}^k\int_{\mathbb{R}^N} U_{x_1}^p U_{x_i}+O\big(\frac{1}{k^{m+\sigma}}\big)\Big).$$

But there are constants $\sigma>0$, $\tilde B_2>0$ and $B_2'>0$, such that

$$\begin{aligned}&\sum_{i=2}^k\int_{\mathbb{R}^N} U_{x_1}^p U_{x_i}=\tilde B_2\sum_{i=2}^k e^{-|x_i-x_1|}+O\big(\sum_{i=2}^k e^{-(1+\sigma)|x_i-x_1|}\big)\\ =&B_2' e^{-\frac{2|x_1|}{k}}+O\big(e^{-\frac{2(1+\sigma)|x_1|}{k}}\big).\end{aligned}$$

So the result follows.

□

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