§ 5. Self-Similar Solutions by Dimensional Analysis

Consider the diffusion problem from last section, with pointwise release (Ref: Bluman & Cole, §2.3):

$$\begin{cases} \dfrac{\partial c}{\partial t} = D\dfrac{\partial^2 c}{\partial x^2} + Q_0\delta(x)\delta(t) \\ c(x,0)=0, \quad c(\pm\infty,t)=0 \end{cases}$$

Initial release within infinitely narrow neighborhood of $x = 0$, such that $\Pi(x)/d = \delta(x)$ and $L/d \to \infty$. Note Q_0 has different dimension as the previous Q because of the cross-sectional area S and time contained in $\delta(t)$.

1. *Dimensional analysis:*

   $\{c\}=\mathrm{ML}^{-3}$, $\{D\}=\mathrm{L}^2\mathrm{T}^{-1}$, $\{Q_0\}=\mathrm{ML}^{-2}$ (Mass releases per unit cross-sectional area)

   $\{x\}=\mathrm{L}$, $\{t\}=\mathrm{T}$

   Thus, we expect 2 Pi groups:

   $$\Pi_1 = \frac{\sqrt{Dt}}{Q_0}c, \qquad \Pi_2 = \frac{x}{\sqrt{Dt}}$$

   And the solution to the PDE problem must be of the form $\Pi_1 = f(\Pi_2)$ or

   $$c = \frac{Q_0}{\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right)$$

   Normally we expect dimensional analysis to reduce the number of variables and parameters. But here we reduced the number of independent variables from 2 to 1!

2. *Transformation of PDE to ODE:*

   Now we can plug this form back into the PDE. First, the partial derivatives:

   $$\frac{\partial c}{\partial t} = -\frac{Q_0}{2t\sqrt{Dt}}f - \frac{Q_0 x}{2Dt^2}f', \quad \frac{\partial c}{\partial x} = \frac{Q_0}{Dt}f', \quad \frac{\partial^2 c}{\partial x^2} = \frac{Q_0}{(Dt)^{3/2}}f''$$

   For $t > 0$, there is no more injection: $\delta(t) = 0$. After inserting the above into the PDE:

   $$-\frac{f}{2} - \frac{x}{2\sqrt{Dt}}f' = f'' \quad \text{or} \quad f'' + \frac{\xi}{2}f' + \frac{f}{2} = 0, \tag{1}$$

   where $\xi = \dfrac{x}{\sqrt{Dt}}$ is our new independent variable. We have successfully transformed the PDE into an ODE. How about the initial and boundary conditions? Note that $t = 0$ and $x=\infty$ both correspond to $\xi = \infty$, so that the initial and boundary conditions can be rolled into one:

$$f(\pm\infty) = 0. \tag{2}$$

But we need another condition on f, one that reflects the amount of initial injection. This is obtained by integrating the PDE over the following intervals:

$\int_{0^-}^{t} dt \int_{-\infty}^{+\infty} [PDE]dx$, where $t = 0^-$ means "just before $t = 0$".

Now the left-hand-side is

$$\int_{0^-}^{t} dt \int_{-\infty}^{+\infty} \frac{\partial c}{\partial t} dx = \int_{-\infty}^{+\infty} dx \int_{0^-}^{t} \frac{\partial c}{\partial t} dt = \int_{-\infty}^{+\infty} [c(x,t) - c(x,0)]dx = \int_{-\infty}^{+\infty} c(x,t)dx$$

and the two terms on the right-hand-side are:

$$\int_{0^-}^{t} dt \int_{-\infty}^{+\infty} \frac{\partial^2 c}{\partial x^2} dx = \int_{0^-}^{t} \left. \frac{\partial c}{\partial x} \right|_{-\infty}^{+\infty} dt = 0 \ \text{ because } \frac{\partial c}{\partial t} = 0 \text{ at } \pm\infty;$$

$\int_{0^-}^{t} dt \int_{-\infty}^{+\infty} Q_0 \delta(t)\delta(x)dx = Q_0$ by virtue of the definition of the delta function.

Now we have $\int_{-\infty}^{+\infty} c(x,t)dx = Q_0$, which can be transformed, using the variable ξ, into

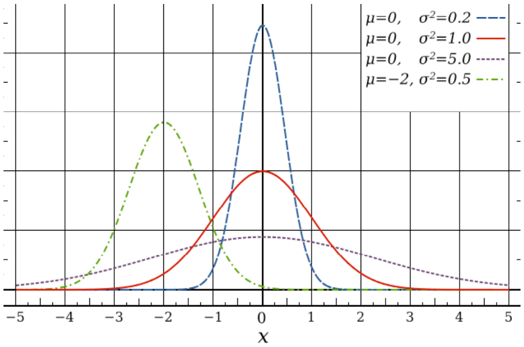
$$\int_{-\infty}^{+\infty} f(\xi)d\xi = 1 \tag{3}$$

ODE (1), along with conditions (2) and (3) will uniquely determine $f(\xi)$, from which we get $c(x,t)$. We are not concerned with the actual solution of the new ODE problem. Rather, the interesting question is: How did we manage to turn a PDE to an ODE?

3. *Discussion*

a. The problem admits a *self-similar solution*: if x is scaled by the diffusion length $(Dt)^{1/2}$, then the $c(x,t)$ profiles at different times can be collapsed onto each other if c is scaled by $Q_0/(Dt)^{1/2}$.

b. This means that x and t are not really 2 independent variables; as far as c is concerned, they can be rolled into one independent variable ξ.



c. Similarity solutions are "happy coincidences" in physical processes. Can we always find them for any PDE's? No. This problem is special in that there is no inherent length scale. Thus, we are not able to form dimensionless groups for each of the variables x, t, and c; instead, we have to combine them and end up with only 2 Pi groups. That's how we ended up with the ODE. If we had the release length dS or the domain length L, the self-similarity will be ruined.

d. Can we always find similarity solutions by dimensional analysis? No. But we will study another example next, and then introduce the general "stretching transformation" idea for detecting similarity solutions.

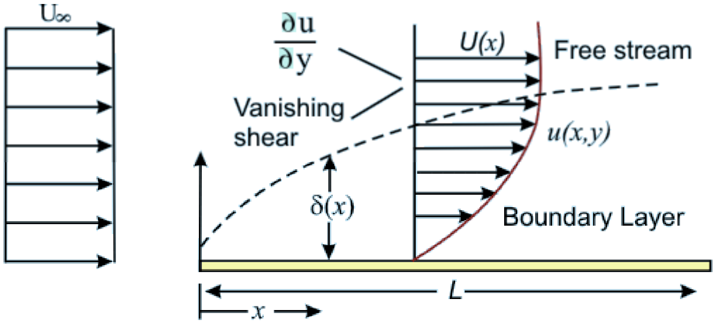
§ 6. Similarity Solutions by Stretching Transformation

It is rare that similarity solutions can be obtained from dimensional analysis. In this section we introduce the idea of stretching transformation which is a more general procedure for seeking out similarity in PDE problems. The materials are based on Barenblatt (§5.2) and Bluman & Cole (§2.5).

As a concrete example, we will take Prandtl's boundary layer equation for flow over a flat semi-plane. After the boundary layer approximation (that viscosity acts only within a thin layer, that the gradient in the flow direction (x) is much smaller than in the transverse direction (y), and that the pressure is constant in the y direction), the governing equations are

$$\begin{cases} u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=\nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\ u(x,0)=0, \quad v(x,0)=0 \\ u(x,\infty)=U_\infty, \;\; u(0,y)=U_\infty \end{cases}$$

where U_∞ is the free-stream velocity, and ν is the kinematic viscosity. If you recall your fluid mechanics, this problem does have a similarity solution (Blasius' solution), and the PDE can be reduced to ODE. (Try to distinguish the velocity v from the viscosity ν. We could use different symbols but these are the conventional ones.)



1. *Would dimensional analysis work?*

Let's write out the dimensions of all the variables and parameters:

$\{u\}=\{v\}=\{U_\infty\}=L/T$, $\{\nu\}=L^2/T$, $\{x\}=\{y\}=L$

There are 2 independent dimensions involved (L and T), and we can construct 4 dimensionless groups out of these. For instance:

$$\Pi_1=\frac{u}{U_\infty},\;\; \Pi_2=\frac{v}{U_\infty},\;\; \Pi_3=\frac{U_\infty x}{\nu},\;\; \Pi_4=\frac{U_\infty y}{\nu}$$

and we expect solutions such as

$$\Pi_1=f(\Pi_3,\Pi_4),\;\; \Pi_2=g(\Pi_3,\Pi_4).$$

Plugging these back into the equations, and we will see that we have **NOT** achieved a reduction of the number of independent variable. Dimensional analysis has failed to give us the similarity solution. Why? Even though the problem has no intrinsic time or length scales, similar to the diffusion problem in the last section, there are only 2 independent dimensions (L and T) instead of 3. Thus, it is possible for x and y to form their own Pi groups; they don't have to be forced into a single one.

It turns out that in this particular example, a trivial manipulation can "cure" the above problem. This is not a general technique, but nevertheless is fun to illustrate here. We will take this little detour before marching into the general technique that is the focus of this section.

Based on the physical insight that things happen at different scales along the x and y directions, which is the fundamental idea behind the boundary layer approximation, we assign two different dimensions to x and y, L and H, and for the moment *pretend* that they are *different* dimensions. Now the list of variables and unknowns are scaled as such:

$\{u\}=\{U_\infty\}=$ L/T, $\{v\}=$ H/T, $\{\nu\}=$H^2/T, $\{x\}=$ L, $\{y\}=$ H.

There are now 3 independent dimensions involved (L, H and T), and we can construct only 3 dimensionless groups out of these:

$$\tilde{\Pi}_1 = \frac{u}{U_\infty},\quad \tilde{\Pi}_2 = \frac{v}{\sqrt{\nu U_\infty / x}},\quad \tilde{\Pi}_3 = \frac{y}{\sqrt{\nu x / U_\infty}} = \zeta .$$

Now we expect a similarity solution in this form:

$$u = U_\infty f(\zeta),\quad v = \sqrt{\frac{\nu U_\infty}{x}}\, g(\zeta) .$$

Plugging this into the original PDE will show that indeed, we have reduced the PDE problem to a couple of ODEs, whose solution is detailed in Fluid Mechanics textbooks. For another example of such "ingenious" dimensional analysis, see the Rayleigh problem analyzed in the next section (see also Bluman & Cole, p. 195). We typically seek to increase the number of independent dimensions (as done above) or decrease the number of dimensional parameters (as done in Bluman & Cole's example).

2. *Stretching transformation*

The "ingenious" dimensional analysis method is specific to the problems. There is, however, a general scheme for seeking out possible similarity solutions. The scheme sometimes goes by the name of "renormalization groups" or "invariant transformation groups", and is based on rather formalistic mathematical manipulations. We will skip the proofs and focus on the technique itself.

Since the essence of similarity is that the solution is *invariant* after certain scaling of the independent and dependent variables, we consider the following *stretching transformation*, and see if such transformations will leave the PDE and the boundary conditions invariant. Consider:

$$\begin{cases} U = \alpha^a u, & V = \alpha^b v \\ X = \alpha^c x, & Y = \alpha^d y \end{cases},$$

where α is a positive number. Under this transformation, we have

$$\frac{\partial u}{\partial x} = \alpha^{c-a}\frac{\partial U}{\partial X},\quad \frac{\partial u}{\partial y} = \alpha^{d-a}\frac{\partial U}{\partial Y},\quad \frac{\partial v}{\partial y} = \alpha^{d-b}\frac{\partial V}{\partial Y},\quad \frac{\partial^2 u}{\partial y^2} = \alpha^{2d-a}\frac{\partial^2 U}{\partial Y^2} .$$

Plugging these into the original PDE's and boundary conditions, we'll see what choices of a, b, c, d may maintain the invariance of the problem. The continuity equation yields:

$$c - a = d - b .$$

The three terms of the momentum equation requires:

$$c-2a=d-a-b=2d-a\,.$$

Note that the first equation above is identical to the preceding equation, and thus the momentum equation adds only 1 additional constraint on the power indices. Finally the boundary conditions require

$$a=0$$

because for the problem in the new variables to be invariant, the non-homogeneous BC should remain as $U\,(\mathrm{X},\infty)=U_\infty$. Now we have 3 equations constrain the 4 indices, and we rewrite the transformation as:

$$\begin{cases} U=u, \quad V=\dfrac{v}{\varepsilon} \\ X=\varepsilon^2 x, \quad Y=\varepsilon y \end{cases}, \text{ where } \varepsilon=\alpha^d.$$

This transformation will leave the problem the same as before, in the new "stretched" and scaled variables. The fact that this *one-parameter family* of transformations will maintain the invariance of the PDE problem reveals the intrinsic self-similarity of the problem. In other words, if we stretch the coordinate y by a factor ε, then we must stretch x by ε^2 and the velocity component v by $1/\varepsilon$ in order to collapse the velocity profiles. From this argument, we recognize that

$$u,\ v\sqrt{x},\ \frac{y}{\sqrt{x}}$$

shall remain the same no matter how we stretch the coordinates. These are known as the invariants of the transformation, and immediately suggest the following similarity solution:

$$\begin{cases} u=f(\zeta) \\ v=\dfrac{1}{\sqrt{x}}g(\zeta) \end{cases}, \text{ with the similarity variable } \zeta=\frac{y}{\sqrt{x}}.$$

This is the same form as obtained from the "ingenious dimensional analysis", aside from a few constant factors. Note that we reached the conclusion here not through dimensional considerations, but through the idea of invariance under general stretching transformations.

Now it's a simple matter to plug these forms into the original PDE problem, and transform it into the following ODE problem:

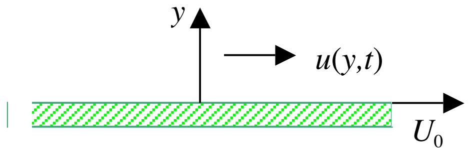
$$\begin{cases} \nu f''+f'\left(\dfrac{\zeta}{2}f-g\right)=0, \\ \zeta f'-2g'=0, \\ f(\infty)=U_\infty,\ f(0)=0,\ \ g(0)=0 \end{cases},$$

the solution of which will not be of immediate interest to us here. Note that the two BC's at x = 0 and $y=\infty$ both project onto $\zeta=\infty$.

3. *Similarity Solution for the Rayleigh Problem*

The Rayleigh problem is another classical example with a self-similar solution. Consider the transient motion in a viscous fluid induced by a flat plate moving in its own plane. Initially both the plate and the fluid are at rest. Starting at $t = 0$, the plate moves with a constant velocity U_0. The Navier-Stokes equations, simplified for this problem, along with the initial and boundary conditions, can be written as:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2} \\ u(y,0) = 0, \quad u(0,t) = U_0, \quad u(\infty,t) = 0 \end{cases}$$



(a) *Dimensional analysis*. From the following dimensions:

$$\{u\}=\{U_0\}= \mathrm{L/T},\ \{\nu\}=\mathrm{L}^2/\mathrm{T},\ \{t\}= \mathrm{T},\ \{y\}= \mathrm{L},$$

we can make 3 Pi groups, say u/U_0, $U_0 y/\nu$, $U_0^2 t/\nu$, and there is no reduction to ODE. Again, we can play tricks here, by either *increasing* the number of "independent dimensions", or *decreasing* the number of parameters, so as to reduce the number of Pi groups.

Using the physical observation that the viscous diffusion happens along the y direction, while the primary flow is in the x direction, we can introduce different length scales:

$$\{u\}=\{U_0\}= \mathrm{L/T},\ \{\nu\}=\mathrm{H}^2/\mathrm{T},\ \{t\}= \mathrm{T},\ \{y\}= \mathrm{H}.$$

Now there are only 2 Pi groups:

$$\Pi_1 = \frac{u}{U_0}, \qquad \Pi_2 = \frac{y}{\sqrt{\nu t}}$$

and we can try a similarity solution of the form

$$u(y,t) = U_0\, f\left(\frac{y}{\sqrt{\nu t}}\right).$$

Alternatively, we can reduce the number of parameters by scaling u by U_0, and calling $\tilde{u}(y,t) = u(y,t)/U_0$ the new dependent variable. Now the problem has one less parameter, and again only admits 2 Pi groups. In the following, however, let us carry out the formal procedure of *stretching transformation* as an exercise.

(b) *Stretching transformation.*

Consider:

$$U = \alpha^a u, \quad Y = \alpha^b y, \quad T = \alpha^c t\,,$$

where α is a positive number. Under this transformation, we have

$$\frac{\partial u}{\partial t} = \alpha^{c-a}\frac{\partial U}{\partial T}, \quad \frac{\partial^2 u}{\partial y^2} = \alpha^{2b-a}\frac{\partial^2 U}{\partial Y^2}.$$

To maintain invariance of the PDE, we require

$$c - a = 2b - a, \quad \text{or} \quad c = 2\,b.$$

The boundary condition $u\,(\,0, t\,) = U_0$ requires $a = 0$. Thus, we have the following transformation that renders the problem invariant:

$$U = u, \quad Y = \varepsilon y, \quad T = \varepsilon^2 t, \text{ which } \varepsilon = \alpha^b.$$

This transformation dictates that y and t be transformed in a coordinated way. Thus u and $\zeta = y/\sqrt{t}$ shall be our new variables that remain unchanged for any stretching α or ε:

$$u = f\left(\frac{y}{\sqrt{t}}\right) = f(\zeta).$$

This reduces the original PDE into the following ODE problem:

$$\begin{cases} 2\nu f'' + \zeta f' = 0, \\ f(0) = U_\infty, \; f(\infty) = 0 \end{cases}$$

which can be integrated analytically to give:

$$f = c_1 \int_0^{\zeta} \exp\left(-\frac{z^2}{4\nu}\right) dz + c_2 .$$

Noting that $\int_0^{\infty} \exp\left(-\frac{z^2}{4\nu}\right) dz = 2\sqrt{\nu}\int_0^{\infty} \exp\left(-\xi^2\right) d\xi = \sqrt{\pi\nu}$, the two constants of integration are determined: $c_1 = -U_0/\sqrt{\pi\nu}$ and $c_2 = U_0$. Finally the solution can be written in terms of the *complementary error function*:

$$f = U_0 \operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\nu}}\right) = U_0 \operatorname{erfc}\left(\frac{y}{\sqrt{4\nu t}}\right),$$

with $\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-z^2\right) dz$.