Finding the source

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Which way did the bicycle go?
Simple random walk on $\mathbb{Z}^2$, run for $5 \cdot 10^6$ steps.
Q: Given a snapshot of a (random) process, what can be determined?
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- Starting/ending point?
- Most/least visited points?
- Step distribution/generator?
- Properties of the underlying graph?
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Warmup: simple random walk on $\mathbb{Z}$.

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Given that the range is an interval of length \( N \), what’s the most likely starting point? Purple, red, or green?
A: They’re all equally likely!

Proof sketch: think of the range as a 'coin switching' markov chain, compute transition probabilities recursively.
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Alternatively: last vertex visited by SRW on the ring is uniform.
Previous work

Same SRW as before, with occupation times
Assume *partial* information about the occupation measure.
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(Warren, Yor ‘98) Brownian burgler: BM conditioned on local times
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Assume \textit{partial} information about the occupation measure.

(Warren, Yor ‘98) Brownian burgler: BM conditioned on local times

\begin{tcolorbox}
\textbf{Theorem (Pemantle, Peres, Pitman, Yor ’00)}

Let $d \geq 3$, and consider Brownian motion in $\mathbb{R}^d$ run for time 1.

Given the \textit{occupation measure} of the path projected onto the sphere, you can recover the \textit{range} and the \textit{endpoint} with probability 1.
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Theorem (Pemantle, Peres, Pitman, Yor ’00)

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Given the **occupation measure** of the path projected onto the sphere, you can recover the **range** and the **endpoint** with probability 1.

Conjecture (PPPY ’00)

In dimension $d = 2$, the range cannot be recovered with probability 1.
SRW in $\mathbb{Z}^d$

Q: where is the starting point?
SRW in $\mathbb{Z}^d$
\( R_t = \) range of SRW up to time \( t \).

**Definition**

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Example: \( \hat{v}_{CM} = \) closest point to the center of mass of \( R_t \).
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**Definition**

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**Example:** \( \hat{v}_{CM} = \text{closest point to the center of mass of } R_t. \)

**Definition**

For \( x \in \mathbb{Z}^d \), let \( R_t^x \) be the range of an independent SRW started from \( x \). For any estimator \( \hat{v} \), the likelihood of \( \hat{v} \) is

\[
L(\hat{v}) = \mathbb{P}(R_t^{\hat{v}} = R_t | R_t).
\]
We want to find an estimator with large likelihood. Canonical best guess is the maximum likelihood estimator:

\[ \hat{v}_{\text{MLE}} = \arg \max_{x \in R_t} L(x) \]
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$$\hat{v}_{MLE} = \arg \max_{x \in R_t} L(x)$$

**Goal:** Find a convenient estimator $\hat{v}$ with

$$L(\hat{v}) \approx L(\hat{v}_{MLE})$$
Theorem (Hoffman, R.)

The following hold for SRW in $\mathbb{Z}^d$ as $t \to \infty$.

i. If $d = 2$,

$$\frac{L(\hat{v}_{MLE})}{\sum_{w \in R_T} L(w)} \xrightarrow{p} 0$$

ii. If $d \in \{3, 4, 5, 6\}$, there exists an estimator $\hat{\nu}$ such that

$$\mathbb{P}(\hat{\nu} = 0) \geq \Theta(t^{-c_d})$$

for some constant $c_d \in (0, 1)$.

iii. If $d \geq 7$, there exists an estimator $\hat{u}$ such that

$$\mathbb{P}(\hat{u} = 0) = \Theta(1).$$
Conjecture

\[ \mathbb{P}(\hat{v}_{MLE} = 0) = \begin{cases} 
  o(1), & d = 2 \\
  \Theta(1), & d \geq 5 
\end{cases} \]

Further Q’s:

- SRW on $d$-regular tree, biased RW
- Performance of ‘longest path’ estimator for transient RW’s?
- Good estimator for $\mathbb{Z}^2$?
Proof ideas:

1. Get rid of the ‘middle’ of the range, using transience.

2. Infer chronological info using ‘cut points.’
Proof sketch:

1. Get rid of the ‘middle’ of the range, using **transience**.

2. Infer chronological info using ‘**cut points**.’
Ingredients:

1. Long cycles: return probabilities / self-intersection exponents (Lawler)

Theorem (James, Peres, '96)

In dimension \(d \geq 3\), there are infinitely many cut times. In dimension \(d \geq 5\), cut times have positive density.
Ingredients:

1. Long cycles: return probabilities / self-intersection exponents (Lawler)

2. A cut time for $X$ is a time $s \in [0, t]$ such that

$$X_{[0,s]} \cap X_{(s,t]} = \emptyset$$

If $s$ is a cut time, $X_s$ is called a cut point.

Theorem (James, Peres, '96)

*In dimension $d \geq 3$, there are infinitely many cut times. In dimension $d \geq 5$, cut times have positive density.*
Cutpoints are totally ordered (by their cut times).

Given all the cut points, find the ‘first’ and ‘last’ ones, pick uniformly from their small components.
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**Problem:** not all ‘divider’ points are cut points!

*Figure:* The three red ‘divider’ points can’t all be cut points.
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![Figure](image)

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Need more information about how cutpoints are distributed.
Consider a rumor spreading through a network.

- The rumor starts from a ‘source’ vertex
- At integer times, nodes can pass the rumor to neighbors
- Observer sees which nodes have the rumor at some time
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- At integer times, nodes can pass the rumor to neighbors
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For today, focus on the $d$-regular tree.
The rumor is spread by a random algorithm known to the observer.

Goals for the rumor spreader:

- **Spreading**: spread the rumor to many nodes
- **Obfuscation**: minimize the probability that the observer guesses the source correctly
- **Multiple observations**: obfuscate the source even when the observer has access to multiple independent rumors
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Goals for the rumor spreader:

- **Spreading**: spread the rumor to many nodes
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- **Multiple observations**: obfuscate the source even when the observer has access to multiple independent rumors
- **Local spreading** (new): ensure that all vertices close to the source learn the rumor quickly
Anonymous messaging platforms, e.g. Secret, Yik Yak, Whisper

Obfuscating the source $\leftrightarrow$ protecting user data
Anonymous messaging platforms, e.g. Secret, Yik Yak, Whisper

Obfuscating the source ↔ protecting user data

Contact tracing / finding patient zero

Previous work: SI/SIR. MLE well understood. Rumor centrality
New algorithm: adaptive diffusion

- $G = d$-regular tree
- $G_t = \text{set of nodes that know the rumor at time } t$
- $vs_t = \text{virtual source at time } t$
- $G_t$ is a ball of radius $t/2$ centered at $vs_t$ at even times $t$
- Defined by transition probabilities $\alpha(t, h)$ for the virtual source
Virtual source evolves according to:

- Start with $v_{s0} = v^*$
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- $v_{s2}$ is a uniform neighbor of $v^*$.
- Let $h = \text{dist}(v_{st}, v^*)$
- Probability $\alpha(t, h)$: $v_{st+2} = \text{uniform neighbor of } v_s t$ excluding previous virtual sources
- Probability $1 - \alpha(t, h)$: $v_{st+2} = v_{st}$
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- Probability $1 - \alpha(t, h)$: $vs_{t+2} = vs_t$

When the virtual source moves, it always moves in a uniform direction away from $v^*$. 
Equivalently, work with \( p(t, h) = \mathbb{P}(\text{dist}(v_{st}, v^*) = h) \).
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MLE for the source vertex:

$$\hat{v}_{MLE} = \arg \max_{v \in G_t} L(v),$$

where $L(v) = \mathbb{P}(G^v_t = G_t | G_t)$. 
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Fact

$\hat{v}_{\text{MLE}}$ is uniform over all vertices at distance $h^*$ from $v_{st}$, where

$$h^* = \arg \max_{h \in \{1,2,\ldots,t/2\}} \frac{p(t, h)}{(d - 1)^h}.$$
For adaptive diffusion,

\[ |G_t| = N_t = \frac{1}{d-2} (d-1)^{t/2}. \]

deterministically at even times \( t \). (Order-optimal spreading)

\[
\mathbb{P}(\hat{v}_{MLE} = v^*) = \begin{cases} 
\Theta(N_t^{-1}) & \text{(perfect obfuscation)} \\
\Theta(N_t^{-\gamma}) & \text{(polynomial obfuscation)} \\
o(1) & \text{(weak obfuscation)} 
\end{cases}
\]
SI/SIR: good spreading, weak obfuscation; not even weak obfuscation under multiple observations. [Shah, Zaman, Dong, Tan, Wang, Zhang]

Adaptive diffusion (Fanti, Kairouz, Oh, Viswanath ’15)

Let $G = d$-regular tree. There exists an adaptive diffusion algorithm that achieves perfect obfuscation:

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Proof sketch: choose $p(t, h) \sim (d - 1)^h$, so the MLE picks a uniform random vertex in $G_t$. Show this is realizable for some values $\alpha(t, h)$. 
Q: Does it have good local spreading?
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**Definition**

The *local spread* $R_t$ is the radius of the largest ball centered at $v^*$ and contained in $G_t$.

For adaptive diffusion, $R_t = \#$ of times the virtual source has *not* moved:

$$R_t = t/2 - \text{dist}(v^*, vs_t).$$
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The *local spread* $R_t$ is the radius of the largest ball centered at $v^*$ and contained in $G_t$.

For adaptive diffusion, $R_t = \# \text{ of times the virtual source has not moved}$:

$$R_t = \frac{t}{2} - \text{dist}(v^*, vs_t).$$

The algorithm from the theorem doesn’t even achieve weak local spread!

$$p(t, h) \sim (d - 1)^h \implies \text{dist}(vs_t, v^*) \approx t/2 - O(1).$$
Consider any adaptive diffusion with polynomial obfuscation of order \( \gamma \in (0, 1) \), i.e.

\[
P(\hat{v}_{MLE} = v^*) = O(N_t^{-\gamma}).
\]

Then the local spreading is bounded from above:

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\mathbb{E}[R_t] \leq (1 - \gamma) \frac{t}{2} + O(\log t).
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Obfuscation and local spreading are inversely linked in this case.
The trade-off is essentially tight:

**Spreading/obfuscation trade-off [Racz, Richey ’18]**

For every $\gamma \in (0, 1)$, there exists an adaptive diffusion with both polynomial obfuscation of order $\gamma$, 

$$
\mathbb{P}(\hat{v}_{MLE} = v^*) = O(N_t^{-\gamma}),
$$

and order optimal local spreading 

$$
\mathbb{E}[R_t] \geq (1 - \gamma)^{t/2}.
$$
Suppose the observer has access to $k > 1$ independent snapshots $\{G_t^i\}_{i=1}^k$ of the diffusion started from the same source $v^*$. 
Two independent observations (Racz, Richey ’18)

Suppose the observer has access to two independent observations $G^1_t$ and $G^2_t$ started from a fixed source $v^*$. There exists a nice estimator $\hat{v}$, not depending on the spreading algorithm, such that for any $t$,

$$\mathbb{P}(\hat{v} = v^*) \geq \frac{d - 1}{d} \cdot \frac{2}{t}.$$  

Moreover, there exists a protocol such that for any $t$,

$$\mathbb{P}(\hat{v}_{ML} = v^*) \leq \frac{d - 1}{d} \cdot \frac{7}{t}.$$  

Only weak obfuscation now!
It gets worse:

Three or more independent observations (Racz, Richey ’18)

Suppose the observer has access to \( k \geq 3 \) independent observations \( G_t^i, \ i \in [k] \). There exists a nice estimator \( \hat{w} \), not depending on the spreading algorithm, such that for any \( t \),

\[
P(\hat{w} = v^*) \geq 1 - d \exp\left(-\frac{(d - 2)^2}{2d^2k}\right).
\]

Not even weak obfuscation!
Proof: Pick any three virtual sources and draw the paths between them.

When the three virtual sources lie in different sub-trees away from the root, there will be a unique intersection point $\hat{w}$. 
Simple estimator: guess a green vertex!

Obfuscation and local spreading are positively linked in this case:

$$\Pr(\hat{v}_{\text{MLE}} = v^*) \geq \mathbb{E}[\left| \bigcap_{i=1}^{k} G_i \right| - 1]$$
Simple estimator: guess a green vertex!

Obfuscation and local spreading are \textbf{positively linked} in this case:

\[
\mathbb{P}(\hat{v}_{MLE} = v^*) \geq \mathbb{E} \left[ \left| \bigcap_{i=1}^{k} G_t^i \right|^{-1} \right]
\]
Question

Does there exist a spreading algorithm that achieves order-optimal spreading and polynomial obfuscation under $\geq 2$ observations?

Should look at algorithms that have order-optimal local spreading.

Also, need more randomness: adaptive diffusion is indexed by a single vertex. Too simple!