I
Embedding of Algebras

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1. Abstract

We look at some general conditions for determining when the unit of an adjunction is monic when the generated monad is defined on a category of algebras. Among other possible applications these conditions will be shown to provide a common approach to some classic embedding theorems in algebra.

2. Introduction

Let $T = (T, \eta, \mu)$ be a monad defined on a category $\mathbf{C}$. We consider the following problem. Is η_C monic for a given object C of $\mathbf{C}$?

There is an easy answer in the case of monads defined on **Set**. This is given here but goes back at least to the thesis of Manes.

In this talk we require that T be defined on a category $\mathbf{C}$ of Eilenberg-Moore algebras given by finitary operations and equations. It is further required that T be generated by a pair of adjoint functors fitting into a certain framework.

Necessary and sufficient conditions are given for η_C to be monic for given object C when T fits into the framework described. A key condition is given in terms of a graph assigned to object C.

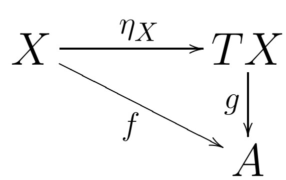
Additional flexibility is attained by relating the key condition to several other graph conditions. This results in another theorem, which we apply in several cases.

3. Monads over Set.

Let $T = (T, \eta, \mu)$ be a monad defined on **Set**. The category $\mathbf{Set}^T$ of Eilenberg-Moore algebras is said to be *nontrivial* if there is at least one algebra (A, α) whose underlying set A has more than one element.

Lemma. *If the category of Eilenberg-Moore algebras for a monad (T, η, μ) on* **Set** *is nontrivial, then η_X is monic for all sets X.*

Proof. Let x and y be distinct elements of a set X with more than one element and suppose that (A, α) is an algebra with more than one element. Then there is a function $f : X \to A$ with $f(x)$ distinct from $f(y)$. By the universal mapping property there is a unique algebra morphism $g : (TX, \mu_X) \to (A, \alpha)$ such that the following diagram commutes



thus $\eta_X(x)$ is not equal to $\eta_X(y)$. □

Thus for example, in the case of semigroups, this shows that any set may be embedded in the free semigroup.

It is clearly rare for the category of Eilenberg-Moore algebras to be trivial since, if trivial, each TX is either empty or consists of one element since it is the underlying object of the algebra (TX, μ_X). Furthermore, TX is the codomain of η_X, hence cannot be empty unless X is.

Now, we remark that the unique function $T1 \to 1$ is a T-algebra, where 1 is the one point set, and the empty set has a T-structure if and only if $T(\emptyset) = \emptyset$. Accordingly, up to isomorphism, there are only two monads (T_1, η_1, μ_1) and (T_2, η_2, μ_2) yielding just trivial algebras. These are given by $T_1 X = 1$ for each set X and $T_2(\emptyset) = \emptyset$ otherwise $T_2(X) = 1$

4. Reduction and Graph Components.

Let $\mathbf{G}$ be a small subcategory of a category $\mathbf{C}$ and let $\mathcal{P}(\mathbf{G})$ be the *power category* of $\mathbf{G}$. The objects of $\mathcal{P}(\mathbf{G})$ are the subclasses of objects of $\mathbf{G}$ and the morphisms are the inclusions.

The *reduction functor* $\mathcal{R}_{\mathbf{G}} : \mathbf{C}^{\mathrm{op}} \to \mathcal{P}(\mathbf{G})$ is defined by

$$\mathcal{R}_{\mathbf{G}}X = \{A | A \text{ is in} \mathbf{G} \text{ and} X \to A \text{ exists in} \mathbf{C}\}$$

with the obvious definition on morphisms.

An object A is *reduced* in $\mathbf{C}$ if the only $\mathbf{C}$-morphism with domain A is the identity. The subcategory $\mathbf{G}$ is *reduced in* $\mathbf{C}$ if each of its objects is reduced in $\mathbf{C}$. An object X of $\mathbf{C}$ is $\mathbf{G}$-*reducible* if $\mathcal{R}_{\mathbf{G}}X$ is nonempty.

The *component class* $[X]$ of an object X of a category $\mathbf{C}$ (or a graph $\mathbf{C}$) is the class of all objects Y which can be connected to X by a finite sequence of morphisms (e.g. $X \to X_1 \longleftarrow X_2 \to Y$). We let $\mathrm{Comp}\mathbf{C}$ denote the collection of component classes.

Strong Embedding Principle for G If $[X] = [Y]$ in $\mathrm{Comp}\mathbf{C}$, then $\mathcal{R}_{\mathbf{G}}X = \mathcal{R}_{\mathbf{G}}Y$. Furthermore there is at most one morphism $X \to A$ for each pair (X, A) consisting of an object X of $\mathbf{C}$ and an object A of $\mathbf{G}$.

5. A Framework for Embeddings.

Let $\mathbf{Alg}(\Omega, E)$ denote the category of algebras defined by a set of operators Ω and identities E. Suppose

$$\mathbf{A} \xrightarrow{U} \mathbf{B} \xrightarrow{V} \mathbf{D} \tag{1}$$

is a diagram such that

(a) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ are the categories of (Ω, E), (Ω', E') and (Ω'', E'') algebras, respectively, with $\Omega'' \subseteq \Omega'$ and $E'' \subseteq E'$, and

(b) V is the forgetful functor on operators $\Omega' - \Omega''$ and identities $E' - E''$ and U is a functor commuting with the underlying set functors on $\mathbf{A}$ and $\mathbf{B}$. Note that U is *not* necessarily a functor forgetting part of Ω and E.

We next describe a functor $C_V : \mathbf{B} \to \mathbf{Grph}$ associated to each pair consisting of a diagram (1) of algebras and an adjunction $(L, VU, \varphi') : \mathbf{D} \to \mathbf{A}$, where $\mathbf{Grph}$ is the category of directed graphs.

Given an object G of $\mathbf{B}$ let the objects of the graph $C_V(G)$ be the elements of the underlying set $|LVG|$ of LVG.

Recursive definition of the arrows of $C_V(G)$:

$$\omega_{ULVG}(|\eta'_{VG}|x_1, \cdots, |\eta'_{VG}|x_n) \to |\eta'_{VG}|\omega_G(x_1, \cdots, x_n)$$

is an arrow if ω is in the set $\Omega' - \Omega''$ of operators forgotten by V and $(x_1, \cdots, x_n)$ is an n-tuple of elements of $|G|$ for which $\omega_G(x_1, \cdots, x_n)$ is defined.

If $d \to e$ is an arrow of $C_V(G)$, then so is

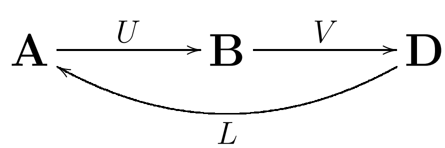
$$\rho_{LVG}(d_1, \cdots, d, \cdots, d_q) \to \rho_{LVG}(d_1, \cdots, e, \cdots, d_q)$$

for ρ an operator of arity q in Ω and $d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_q$ arbitrary elements of $|LVG|$.

If $\beta : G \to G' \in \mathbf{B}$, then $C_V(\beta) : C_V(G) \to C_V(G')$ is the graph morphism which is just the function $|LV\beta| : |LVG| \to |LVG'|$ on objects and defined recursively on arrows in the obvious way.

In the following proposition note that if, in the diagram (1),**D** is the category of sets, then an adjunction (L, VU, φ') is given by letting LX be the free (Ω, E) algebra on the set X.

Proposition. *Suppose*



is a diagram of algebras as in (1)*, with given adjunction* $(L, VU, \varphi') : \mathbf{D} \to \mathbf{A}$*. Then there is an adjunction* $(F, U, \varphi) : \mathbf{B} \to A$ *with the following specific properties:*

(a) The underlying set of FG *is* $\mathrm{Comp}C_V(G)$.

(b) If ρ *is an operator of arity* n *in* Ω, *then* ρ_{FG} *is defined by*

$$\rho_{FG}([c_1], \cdots, [c_n]) = [\rho_{LVG}(c_1, \cdots, c_n)]$$

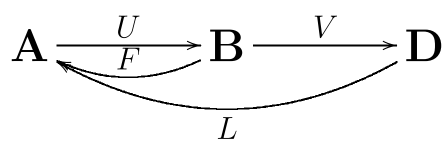
where $c_1, \cdots, c_n$ *are members of the set* $|LVG|$ *of objects of the graph* $C_V(G)$.

(c) The unit morphism $\eta_G : G \to UFG$ *of* (F, U, ϕ) *has an underlying set map which is the composition* $[-]. \mid \eta'_{VG}|$, *where* $|\eta'_{VG}| : |G| \to |LVG| = \mathrm{Obj}C_V(G)$ *is the set map underlying the unit* $\eta'_{VG} : VG \to VULVG$ *of the adjunction* (L, VU, ϕ') *and* $[-] : \mathrm{Obj}C_V(G) \to CompC_V(G)$ *is the component function.*

Suppose the hypotheses of the proposition hold. Let S_V be a subgraph of $C_V(G)$ having the same objects $|LVG|$ and the same components as $C_V(G)$. Then the proposition remains valid under substitution of S_V for $C_V(G)$ throughout. This allows us to "picture" the adjoint using a possibly smaller set of arrows than those present in $C_V(G)$. Accordingly, we define a *V picture of the adjoint F to U at $G \in |\mathbf{B}|$* to be any quotient category $\mathbf{C}(= \mathbf{C}(S_V))$ of the free category generated by such a subgraph S_V of $C_V(G)$. This proposition is then valid upon substitution of the underlying graph of a V picture $\mathbf{C}$ for $C_V(G)$ throughout.

We now present the first embedding theorem.

Theorem. *Let*



be given with adjunctions $(L, VU, \varphi') : \mathbf{D} \to \mathbf{A}$ *and* $(F, U, \varphi) : \mathbf{B} \to A$ *as described in the Proposition.*

Given $G \in |\mathbf{B}|$ *let* $\mathbf{C}(S_V)$ *be any* V *picture of the adjoint* F *to* U *at* G.

Then the unit morphism $\eta_G : G \to UFG$ *of the adjunction* (F, U, φ) *is monic if and only if the following hold*

(a) The discrete subcategory $\mathbf{G} = \eta'_{VG}(|G|)$ *is reduced in* $\mathbf{C}(S_V)$ *for* η'_{VG} *the unit of* (L, VU, ϕ').

(b) If $[A] = [B]$ *in* $\mathrm{Comp}\mathbf{C}(S_V)$ *with* $A, B \in |\mathbf{G}|$, *then* $\mathcal{R}_{\mathbf{G}}A = \mathcal{R}_{\mathbf{G}}B$, *where* $\mathcal{R}_{\mathbf{G}} : \mathbf{C}(S_V)^{\mathrm{op}} \to \mathcal{P}(\mathbf{G})$ *is the reduction functor.*

(c) The unit morphism η'_{VG} *is monic.*

6. Diamonds, Strong Embeddings and Connectedness.

We make use of the following principles.

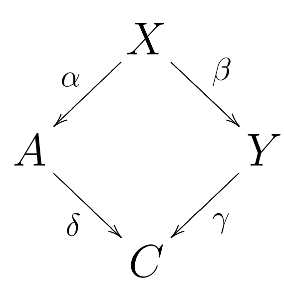
Diamond Principle for G. Let **G** be a subcategory of **C**, then the objects $\alpha : X \to A$ of $X/\mathbf{C}$ with A an object of **G** are terminal in $X/\mathbf{C}$ for each object X of **C**.

Lemma A. *Let* **G** *be a reduced subcategory of* **C**. *Then the following statements are equivalent:*

(a) The Diamond Principle for **G**.

(b) Each pair $Y \leftarrow X \rightarrow Z$ *of* **C***-morphisms with* X *a* **G***-reducible object can be completed to a commutative diamond in* **C**.

Proof. Suppose that (b) holds and let $\alpha : X \to A$ be an object of $X/\mathbf{C}$ with A in **G** and $\beta : X \to Y$ be any other object. Then, by hypothesis there exists a commutative diagram



in $\mathbf{C}$. But A in $\mathbf{G}$ implies that $\delta = 1$. Thus $\gamma : \beta \to \alpha$. If $\gamma' : \beta \to \alpha$ then, by hypothesis, $A \stackrel{\gamma}{\leftarrow} Y \stackrel{\gamma'}{\to} A$ can be completed to a commutative square since Y is G-reducible. Thus $\gamma = \gamma'$ since A is reduced and so α is terminal.

It is trivial to show that (a) implies (b). □

For the next Lemma we recall the definition of the

Strong Embedding Principle for G. If $[X] = [Y]$ in Comp$\mathbf{C}$, then $\mathcal{R}_{\mathbf{G}}X = \mathcal{R}_{\mathbf{G}}Y$. Furthermore there is at most one morphism $X \to A$ for each pair (X, A) consisting of an object X of $\mathbf{C}$ and an object A of $\mathbf{G}$.

Lemma B. *The Strong Embedding Principle for* $\mathbf{G}$ *implies the Diamond Principle for* $\mathbf{G}$.

Proof. Let $A \stackrel{\alpha}{\leftarrow} X \stackrel{\beta}{\to} Y$ be a diagram in $\mathbf{C}$ with A in $\mathbf{G}$. Thus $[A] = [Y]$ in Comp$\mathbf{C}$ and $\mathcal{R}_{\mathbf{G}}X = \mathcal{R}_{\mathbf{G}}Y$, by hypothesis. Hence A is in $\mathcal{R}_{\mathbf{G}}Y$ and there exists $\gamma : Y \to A$. But then $\gamma\beta$ and α are morphisms $X \to A$ and $\alpha = \gamma\beta$ by hypothesis. Thus $\gamma : \beta \to \alpha$ in $X/\mathbf{C}$ and γ is unique since as a $\mathbf{C}$ morphism it is the only morphism $Y \to A$ by hypothesis. □

The implication also goes in the other direction. For a proof see [6].

Given an object X of $\mathbf{C}$ let $(X/\mathbf{C})_{\mathcal{P}}$ be the full subcategory of the slice category $X/\mathbf{C}$ obtained by omitting the object $1_X : X \to X$.

Principle of Connectedness for T. The categories $(X/\mathbf{C})_{\mathcal{P}}$ are connected for each object X of $\mathbf{T}$, where $\mathbf{T}$ is a subcategory of $\mathbf{C}$.

Lemma C. *If $\alpha : X \to A$ is a terminal object of $X/\mathbf{C}$ with A reduced, then $(X/\mathbf{C})_{\mathcal{P}}$ is connected.*

Proof. If X is reduced, then $X/\mathbf{C}$ contains only one object, namely 1_X, and $(X/\mathbf{C})_{\mathcal{P}}$ is empty, hence trivially connected. If X is not reduced and $\alpha : X \to A$ is terminal in $X/\mathbf{C}$ with A reduced, then $\alpha \neq 1_X$ and α is terminal in $(X/\mathbf{C})_{\mathcal{P}}$. Thus $(X/\mathbf{C})_{\mathcal{P}}$ is connected. □

Lemma D. *If the Diamond Principle holds for a reduced subcategory $\mathbf{G}$ of $\mathbf{C}$, then the Principle of Connectedness holds for the full subcategory $\mathbf{T_G}$ of $\mathbf{C}$ consisting of all G-reducible objects of $\mathbf{C}$.*

Proof. Let X be in $\mathbf{T_G}$. Then there is a morphism $X \to A$ in $\mathbf{C}$ with A in $\mathbf{G}$. By the Diamond Principle for $\mathbf{G}$ the morphism $X \to A$ is terminal in $X/\mathbf{C}$. By Lemma C then $(X/\mathbf{C})_{\mathcal{P}}$ is connected. □

Definition. Let $\mathbb{N}$ be the preorder of nonnegative integers with $n \to m$ iff $n \geq m$. A *rank functor* for a category $\mathbf{C}$ is a functor $R : \mathbf{C} \to \mathbb{N}$ with $R\alpha \neq 1$ whenever $\alpha \neq 1$.

Theorem. *Let* $\mathbf{C}$ *be a category with rank functor given and* $\mathbf{G}$ *a subcategory which is reduced in* C*. Then the following are equivalent.*

(a) The Principle of Connectedness for the full subcategory $\mathbf{T_G}$ *of* $\mathbf{C}$ *determined by all* $\mathbf{G}$ *reducible objects of* $\mathbf{C}$*.*

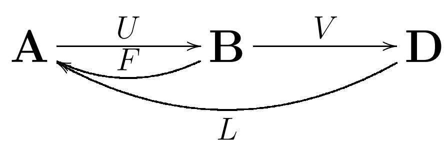
(b) The Diamond Principle for $\mathbf{G}$*.*

(c) The Strong Embedding Principle for $\mathbf{G}$*.*

7. The Second Embedding Theorem.

In the presence of a rank functor we have seen in Theorem 6 that the three principles are equivalent. We apply this Theorem to the hypotheses of Theorem 5 to obtain the following result:

Theorem. *Let*



be given with hypotheses as in Theorem 6.

Then the unit morphism $\eta_G : G \to UFG$ *of* (F, U, φ) *is monic provided there exists a* V *picture* $\mathbf{C}$ *of the adjoint* F *to* U *at* G *for which the following conditions hold:*

(a) $\mathbf{C}$ *has a rank functor.*

(b) The discrete subcategory $\mathbf{G} = \eta'_{VG}(|G|)$ *is reduced in* $\mathbf{C}$ *for* η'_{VG} *the unit of* (L, VU, φ')*.*

(c) The categories $(X/\mathbf{C})_{\mathcal{P}}$ *are connected for each* $X \in |\mathbf{C}|$ *which is* $\mathbf{G}$ *reducible.*

(d) The unit morphism η'_{VG} *of* (L, VU, φ') *is monic.*

8. Associative Embedding of Lie Algebras.

Let $U : \mathbf{A} \to \mathbf{L}$ be the usual algebraic functor from associative algebras over K to Lie algebras over K, for K a commutative ring. That is, for $A \in |\mathbf{A}|$, UA is the same as A except that a new multiplication $[ab] = a.b - b.a$ replaces the associative multiplication $a.b$ of $A \in |\mathbf{A}|$.

It is well known that an adjunction $(F, U, \varphi) : \mathbf{L} \to \mathbf{A}$ exists. The question of embeddability of a Lie algebra in its universal associative algebra (i.e. the question as to whether the unit morphisms η_G of (F, U, φ) are monomorphisms) has been investigated by various authors (cf. Birkhoff[3] and Serre[11]). Not all Lie algebras can be so embedded (cf. Higgins[4]).

We demonstrate how such a question can be put in the context of the previous sections. Let $V : \mathbf{L} \to Mod_K$ be the functor forgetting the Lie multiplication. We then have the diagram

$$\mathbf{A} \xrightarrow{\;U\;} \mathbf{L} \xrightarrow{\;V\;} Mod_K$$
$$L : Mod_K \to \mathbf{A}$$

where the conditions (a) and (b) of diagram (1) hold. The existence of the adjunction $(L, VU, \varphi') : Mod_K \to \mathbf{A}$ is assured on theoretical grounds, but can also be described explicitly as follows.

Given $G \in |\mathbf{L}|$ it is known that LVG is the tensor algebra of VG. Thus

$$LVG = \oplus_{n \geq 0}(\otimes_{i=1}^{n}(VG)).$$

Furthermore $\eta'_{VG} : VG \rightarrow K \oplus VG \oplus (VG \otimes VG) \oplus \cdots$ is monic. In this section let $\mathbf{G}$ be the discrete subgraph $\eta'_{VG}(|G|)$ of $C_V(G)$. Thus $\mathbf{G}$ is a discrete subcategory of any V picture of F at G. Applying Theorem 5 the following Lie algebra embedding result holds.

Theorem A. *Let G be a Lie algebra and $\mathbf{C}$ any V picture of F at G. Then a Lie algebra G can be embedded in its universal associative algebra FG if and only if the following hold:*

(a) $[A] = [B]$ *in* Comp$\mathbf{C}$ *implies that* $\mathcal{R}_{\mathbf{G}}A = \mathcal{R}_{\mathbf{G}}B$ *for all* $A, B \in |\mathbf{G}|$ *where* $\mathcal{R}_{\mathbf{G}} : \mathbf{C}^{\text{op}} \rightarrow \mathcal{P}(\mathbf{G})$ *is the reduction functor.*

(b) $\mathbf{G}$ *is a reduced subcategory of* $\mathbf{C}$*.*

Similarly, by applying Theorem 7 and noting that η'_{VG} is monic we have the following sufficient conditions:

Theorem B. *A Lie algebra G can be embedded in its universal associative algebra FG if there exists any*

V picture $\mathbf{C}$ of the adjoint F to U at G with the following properties.

(a) The categories $(X/\mathbf{C})_{\mathcal{P}}$ are connected for each $X \in |\mathbf{C}|$ which is $\mathbf{G}$ reducible.

(b) $\mathbf{C}$ has a rank functor.

(c) The discrete subcategory $\mathbf{G} = \eta'_{VG}(|G|)$ is reduced in $\mathbf{C}$ for η'_{VG} the unit of (L, VU, ϕ').

Theorem (Birkhoff-Witt). *A Lie algebra G whose underlying module VG is free can be embedded in its universal associative algebra FG.*

Proof. The conditions of the previous theorem are to be verified for the following V picture of the adjoint F to U at G. Let $\mathbf{C}$ be the preorder which is a quotient of the free category on the following subgraph S_V of $C_V(G)$. The objects of S_V are the elements of the free K module LVG on all finite strings $x_{i_1} \cdots x_{i_n}$ of elements from a basis $(x_i)_{i \in I}$ of the free K module VG. Given a well ordering of I we let the arrows of S_V be those of the form

$$k_i x_{i_1} \cdots x_{i_n} + \alpha \to$$

$$\mathrm{k}_i x_{i_1} \cdots x_{i_{j+1}} x_{i_j} \cdots x_{i_n} + k_i x_{i_1} \cdots [x_{i_j}, x_{i_{j+1}}] \cdots x_{i_n} + \alpha$$

for $i_{j+1} < i_j$, $k_i \in K$, and α any element of LVG (not involving $x_{i_1} \cdots x_{i_n}$).

To show that the categories $(X/\mathbf{C})_{\mathcal{P}}$ are connected for each $X\in|\mathbf{C}|$ which is $\mathbf{G}$ reducible, it turns out that the key idea is to show that for $c < b < a$ in I the objects

$$\beta : x_a x_b x_c \rightarrow x_b x_a x_c + [x_a, x_b]x_c$$

and

$$\gamma : x_a x_b x_c \rightarrow x_a x_c x_b + x_a[x_b, x_c]$$

can be connected in $((x_a x_b x_c)/\mathbf{C})_{\mathcal{P}}$. This is done by further reduction of the ranges of β and γ and use of the Jacobi identity and the identity $[x, y] = -[y, x]$.

The rank functor for $\mathbf{C}$ is given as follows. Given $X = kx_{a_1} \cdots x_{a_n}$ let $R(X) = (R_n(X))$ be a sequence of nonnegative integers defined by $R_n(X) = \sum_{i=1}^{n} p_{a_i}$ where p_{a_i} is the number of x_{a_j} to the right of x_{a_i} with $a_j < a_i$ and $R_s(X) = 0$ for $s \neq n$. We extend by linearity to all elements of $LVG = |\mathbf{C}|$. If $X \to Y$ is an arrow, then $R(Y) < R(X)$ where the latter inequality means that $R_n(Y) < R_n(X)$ for n the largest integer with $R_n(Y) \neq R_n(X)$. Thus R extends to a rank functor.

Finally, we verify condition (c) by observing that any element of $|G|$ may be expressed in the form $\sum_{i\in I} k_i x_i$ in terms of the basis $(x_i)_{i\in I}$ of G, which is regarded as a subset $\mathbf{G}$ of LVG via the embedding η'_{VG}. From the preceding description of arrows of S_V there is no arrow with domain an element of $\mathbf{G} = \eta'_{VG}(|G|)$. Thus $\mathbf{G}$ is reduced in $\mathbf{C}$. $\square$

9. Sets with a partially defined binary operation and a result of Schreier.

We show how the following classical theorem follows from sections 5 and 7.

Theorem (Schreier). *If S is a common subgroup of the groups X and Y and if*

$$\begin{array}{ccc} S & \xrightarrow{\subseteq} & X \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow[\beta]{} & P \end{array} \tag{2}$$

is the pushout in the category of groups, then α and β are monomorphisms. The group P is referred to as the free product of X and Y with amalgamated subgroup S.

Let **B** be the category of sets with a single partially defined binary operation. The diagram $X \longleftarrow S \to Y$ of groups can be regarded as a diagram in **B** and can be completed to a diagram

$$\begin{array}{ccc} S & \xrightarrow{\subseteq} & X \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\gamma} \\ Y & \xrightarrow[\delta]{} & G \end{array} \tag{3}$$

commuting in **B** where G is the disjoint union of X and Y with common subset S identified and $a.b$ defined if both

$a, b \in X$ or if both $a, b \in Y$, otherwise $a.b$ is undefined. The morphisms γ and δ are the obvious monomorphisms. The next Lemma and Proposition describe how this approach yields the Schreier Theorem.

Lemma. *The Schreier Theorem holds if in 3 the pushout codomain G in $\mathbf{B}$ is embeddable in a group.*

Proof. Let $\iota : G \to P'$ be a monomorphism in $\mathbf{B}$ with P' a group. Then

$$\begin{array}{ccc} S & \xrightarrow{\subseteq} & X \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\iota\gamma} \\ Y & \xrightarrow[\iota\delta]{} & P' \end{array}$$

commutes in groups. Thus for some group homomorphism ϕ we have $\iota\gamma = \phi\alpha$ and $\iota\delta = \phi\beta$ since (2) is a pushout in groups. Thus α, β are monic since ι, γ and δ are. □

Proposition. *Let G be as in the lemma. Then G is embeddable in a group.*

Proof. We embed G in a particular semigroup which turns out to be a group. Begin with the diagram

$$\mathbf{A} \xrightarrow{U} \mathbf{B} \xrightarrow{V} Sets, \qquad Sets \xrightarrow{L} \mathbf{A}$$

where **A** is the category of semigroups(not necessarily with 1), U forgetful, V forgetful and (L, VU, φ') an adjunction. We then have a preorder $\mathbf{C}_G$ which is a quotient of the free category $\mathbf{F}_G$ generated by $C_V(G)$. Proposition 5 (which also holds for the category **B** of partial algebras, see[6]) shows that an adjunction $< F, U, \varphi >: \mathbf{B} \rightarrow \mathbf{A}$ exists and describes it. It is sufficient to show that the unit $\eta_G : G \rightarrow UFG$ of the adjunction is a monomorphism. By Theorem 7 it is sufficient to verify conditions (a) through (d). These conditions are trivial except for (c), which requires that the categories $(X/\mathbf{C})_{\mathcal{P}}$ be connected for each **G**-reducible object X of **C**. The objects of $\mathbf{C}_G$ are elements of the free semigroup LVG on VG. An object X may be written as a string $(a_1, \cdots, a_n)$ of length $n \geq 1$ where $a_i \in VG$ for $i = 1, \cdots, n$. It is sufficient to show that C_V arrows

$$(\cdots, a_i a_{i+1}, \cdots) \xleftarrow{\alpha} (a_1, \cdots, a_n) \xrightarrow{\beta} (\cdots, a_j a_{j+1}, \cdots)$$

regarded as $(X/\mathbf{C}_G)_{\mathcal{P}}$ objects can be connected by a finite sequence of morphisms in the same category. This requires a detailed argument when $i = j-1$ or $i = j+1$, otherwise it is trivial (cf. Baer[1], MacDonald[7]). □

10. Classical coherence. In considering categories with operations and natural equivalences (replacing the equations of algebras) we immediately discover a relationship between connectedness and the commutativity of diagrams arising from the isomorphisms. This is illustrated by the following example.

Let $\mathbf{V}$ be a category and $\otimes : \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$ a functor associative up to a natural isomorphism

$$a : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$$

Since we have isomorphisms and not (in general) equalities it is then natural to ask whether all diagrams built up from the natural isomorphism a commute. This is an example of a *coherence* question. It turns out that the answer is affirmative in this case if a certain type of *pentagon* diagram commutes in a subcategory $\mathbf{C}$ of the category of shapes $\mathcal{N}()$ that we are going to define next.

Shapes are defined inductively by

S1. 1 is a shape

S2. If T and S are shapes, so is $T \oslash S$

For each shape there is a variable set $\nu(T)$ defined inductively by

V1. $\nu(1)$ is a chosen one element set.

V2. $\nu(T \oslash S)$ is the disjoint union of $\nu(T)$ and $\nu(S)$.

Given , then for each shape T there is a functor $|T|$ given inductively by

F1. $|1|$: $\rightarrow$ is the identity functor.

F2. $|T \oslash S| = \otimes \cdot (|T| \times |S|) : W_T \times W_S \rightarrow \times \rightarrow$ for W_T and W_S products of $\nu(T)$ and $\nu(S)$ copies of , respectively.

Let $\mathcal{N}()$ be the category whose objects are all shapes and whose morphisms $F : T \rightarrow S$ are the natural transformations $F : |T| \rightarrow |S|$.

Given T, S and R we obtain $\alpha_{TSR} : T \oslash (S \oslash R) \rightarrow (T \oslash S) \oslash R)$ in $\mathcal{N}()$ by letting $\alpha_{TSR}(X, Y, Z)$ be the component

$$|T|X \otimes (|S|Y \otimes |R|Z) \rightarrow (|T|X \otimes |S|Y) \otimes |R|Z)$$

of the natural transformation a on for X, Y, Z objects of the domain categories for $|T|$, $|S|$ and $|R|$, respectively.

Let $\mathbf{C}$ ($=\mathbf{C}()$) be the subcategory of $\mathcal{N}()$ whose objects are all shapes and whose morphisms, called the *allowable* morphisms of $\mathcal{N}()$, are given by

AM1. $1 : T \rightarrow T$ and $\alpha : T \oslash (S \oslash R) \rightarrow (T \oslash S) \oslash R)$ are in $\mathbf{C}$ for any shapes T, S, R.

AM2. If $f : T \rightarrow T'$ and $g : S \rightarrow S'$ are in $\mathbf{C}$, then so is $f \oslash g : T \oslash S \rightarrow T' \oslash S'$.

AM3. If $f : T{\to}S$ and $g : S{\to}R$ are in $\mathbf{C}$, then so is $gf : T{\to}R$.

Lemma. *Let* $\mathbf{D}$ *be a category with a rank functor and assume that* $(X/\mathbf{D})_{\mathcal{P}}$ *is connected for each object* X *of* $\mathbf{D}$*. Then* $\mathbf{D}$ *is a preorder if and only if the morphisms of* $\mathbf{D}$ *are all monomorphisms.*

Proof. Let $\mathbf{G}$ be the subcategory of all reduced objects. Then, since $\mathbf{D}$ has a rank functor, every object of $\mathbf{D}$ is $\mathbf{G}$-reducible. Thus, if $f, g : X{\to}Y$ in $\mathbf{D}$, then there is $h : Y{\to}G$ with G in $\mathbf{G}$. By Theorem 3.2 the Diamond Principle for $\mathbf{G}$ holds. Thus objects of $X/\mathbf{D}$ with codomain in $\mathbf{G}$ are terminal. Thus $h \cdot f = h \cdot g$ since G is reduced and $f = g$ since h is monic. $\square$

A rank functor ρ for the subcategory $\mathbf{C}$ of $\mathcal{N}()$ described above is defined recursively by $\rho(1) = 0$ and $\rho(T{\oslash}S) = \rho(T) + \rho(S) + |\nu(S)| - 1$, where $|\nu(S)| = \text{card}\ \nu(S)$.

Theorem. *The category* $(X/\mathbf{C})_{\mathcal{P}}$ *is connected for every shape* $X{\in}|\mathbf{C}|$ *provided that the pentagon diagram*

$$\begin{array}{ccccc}
A{\otimes}(B{\otimes}(C{\otimes}D)) & \xrightarrow{a} & (A{\otimes}B){\otimes}(C{\otimes}D) & \xrightarrow{a} & ((A{\otimes}B){\otimes}C){\otimes}D \\
\downarrow{\scriptstyle 1\otimes a} & & & & \uparrow{\scriptstyle a\otimes 1} \\
A{\otimes}((B{\otimes}C){\otimes}D) & & \xrightarrow{\qquad a \qquad} & & (A{\otimes}(B{\otimes}C)){\otimes}D
\end{array}$$

commutes for all A, B, C, D in .

Corollary. *The category* $\mathbf{C}$ *is a preorder provided all pentagon diagrams commute.*

A functor $T : \mathbf{C} \to \mathbf{D}$ whose domain is a preorder is called a *coherence functor* for $\mathbf{D}$. Intuitively, T describes a class of commutative diagrams in $\mathbf{D}$.

The subcategory of $\mathcal{N}()$ generated by $\mathbf{C}$ and the inverses to the associativity isomorphisms is also a preorder. For further details see ([6]) or ([8]).

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