Math 100 Section 102 Test 1 Solutions

1.) Evaluate the following limits

a) [10 marks]

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9}\frac{3-\sqrt{x}}{(3-\sqrt{x})\,(3+\sqrt{x})}$$ **4 marks**

$$= \lim_{x\to 9}\frac{1}{3+\sqrt{x}}$$ **2 marks**

$$= \frac{1}{3+\sqrt{9}}$$ **2 marks**

$$= \frac{1}{6}$$ **2 marks**

b) [10 marks]

$$\lim_{x\to 4}\frac{x^2-16}{2-\sqrt{x}} = \lim_{x\to 4}\frac{(x+4)(x-4)}{2-\sqrt{x}}$$ **2 marks**

$$= \lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)(x+4)}{2-\sqrt{x}}$$

$$= \lim_{x\to 4}\frac{-(2-\sqrt{x})(\sqrt{x}+2)(x+4)}{2-\sqrt{x}}$$ **2 marks**

$$= \lim_{x\to 4} -(\sqrt{x}+2)(4+x)$$ **2 marks**

$$= -(\sqrt{4}+2)(4+4)$$ **2 marks**

$$= -32$$ **2 marks**

2.) Use the definition of the derivative (involving limits) to find the derivative of

$$\frac{x}{x-1}$$

[20 marks]

By the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$ **10 marks**

$$= \lim_{h \to 0} \frac{x+h}{h(x+h-1)} - \frac{x}{h(x-1)}$$

$$= \lim_{h \to 0} \frac{(x+h)(x-1) - x(x+h-1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{h^2 - h - x + hx - x^2 - hx + x}{h(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)}$$ **5 marks**

$$= \frac{-1}{(x+0-1)(x-1)}$$ **2 marks**

$$= \frac{-1}{(x-1)^2}$$ **3 marks**

3.) Write an equation of the line tangent to the curve

$$y = \frac{3x^2}{x^2 + x + 1}$$

[20 marks]

Using the quotient rule to differentiate the function,

$$y' = \frac{6x(x^2 + x + 1) - 3x^2(2x + 1)}{(x^2 + x + 1)^2}$$

$$= \frac{6x^3 + 6x^2 + 6x - 6x^3 - 3x^2}{(x^2 + x + 1)^2}$$

$$= \frac{3x^2 + 6x}{(x^2 + x + 1)^2}$$ **10 marks**

Substitute in x = -1

$$y'(-1) = \frac{3(-1)^2 + 6(-1)}{((-1)^2 + (-1) + 1)^2}$$

= -3 **5 marks**

y = m x + b = -3x +b

Substitute in x = -1 and y = 3 to solve for b

3 = -3(-1) + b

b = 0

y = -3x **5 marks**

4. The period of oscillation P (in seconds) of a simple pendulum of length L (in feet) is given by

[20 marks]

$$P = 2\pi\sqrt{\frac{L}{g}}$$

where g = 32 ft/s^2. Find the rate of change of P with respect to L when P = 2.

First we need to solve for the value of L when P = 2.

$$2 = 2\pi\sqrt{\frac{L}{g}}$$

$$\frac{1}{\pi} = \sqrt{\frac{L}{g}}$$

$$\left(\frac{1}{\pi}\right)^2 = \left(\sqrt{\frac{L}{g}}\right)^2$$

$$\frac{1}{\pi^2} = \frac{L}{g}$$

$$\frac{g}{\pi^2} = L$$ **5 marks**

Differentiating P with respect to L we get:

$$\frac{dP}{dL} = 2\pi \, \frac{1}{2} \left(\frac{L}{g}\right)^{-1/2} \frac{1}{g}$$

$$= \frac{\pi}{g}\left(\frac{L}{g}\right)^{-1/2}$$ **10 marks**

Substituting in L = g/π^2

$$= \frac{\pi}{g}\left(\frac{g}{\pi^2}\,\frac{1}{g}\right)^{-1/2}$$

$$= \frac{\pi^2}{g}$$ second/feet **3 marks for answer**
2 marks for units

5. Find the maximum and minimum values attained by the function

$$y = x\sqrt{1 - x^2}$$

on the closed interval [-1,1].

[20 marks]

Using the product rule to differentiate the function,

$$y' = \sqrt{1 - x^2} + \frac{x}{2}(1 - x^2)^{-1/2}(-2x)$$

$$y' = \sqrt{1 - x^2} - x^2(1 - x^2)^{-1/2}$$ **5 marks**

solve for y’ = 0 to find critical points in the interval [-1, 1]

$$0 = \sqrt{1 - x^2} - x^2(1 - x^2)^{-1/2}$$

$$\sqrt{1 - x^2} = \frac{x^2}{\sqrt{1 - x^2}}$$

by cross multiplying we get:

$$1 - x^2 = x^2$$

$$1 = 2x^2$$

$$\pm\sqrt{\frac{1}{2}} = x$$ **5 marks**

Now check critical points and end points of the interval to find the max and min values of the function.

$$y = x\sqrt{1 - x^2}$$

$$y(1) = 1\sqrt{1 - 1} = 0$$

$$y(-1) = -1\sqrt{1 - 1} = 0$$

$$y\left(-\sqrt{\frac{1}{2}}\right) = -\sqrt{\frac{1}{2}}\sqrt{1 - \frac{1}{2}} = \frac{-1}{2}$$ Minimum value **5 marks**

$$y\left(\sqrt{\frac{1}{2}}\right) = \sqrt{\frac{1}{2}}\sqrt{1 - \frac{1}{2}} = \frac{1}{2}$$ Maximum value **5 marks**