12 Proofs in calculus

We now move onto sequences and series. These are important in their own right, but they also help lay the foundation for limits, continuity of functions, derivatives, integrals etc. And most importantly they introduce the idea of convergence.

If we have time then we might look at limits of functions too.

12.1 Limits of sequences

To introduce convergence we look at what you might consider to be a special type of function from the natural numbers. Now these can really have any co-domain, but for the purposes of this course, let us take the codomain to be $\mathbb{R}$. So with that caveat, let us make a definition.

Definition.

- A sequence is a real-valued function defined on the set of natural numbers. That is, a sequence is a function $f : \mathbb{N} \to \mathbb{R}$.
- If $f(n) = a_n$ then we can simply refer to the sequence by a_n. That is we can say "the sequence is $a_1, a_2, a_3, \ldots$." We also often write this as $\{a_n\}$. This is unhelpfully similar to the set notation.
- a_n is the n^{th} term in the sequence.

We usually describe the sequence by giving a formula for the n^{th} term, or by giving the first few terms (hopefully this gives enough info for the reader to understand it).

For example (a few taken from the text)

- $\left\{\frac{1}{n^2}\right\}$ is the sequence $1, 1/4, 1/9, 1/16, \ldots$.
- $\left\{\frac{n}{2n+1}\right\}$ is the sequence $1/3, 2/5, 3/7, \ldots$.
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$ is the sequence $\left\{\frac{1}{2n}\right\}$
- $1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{8}, \ldots$ is the sequence $\{1 + 2^{-n}\}$
- $1, \frac{3}{5}, \frac{1}{2}, \frac{5}{11}, \frac{3}{7}, \frac{7}{17}, \ldots$ is the sequence $\left\{\frac{n+1}{3n-1}\right\}$

Consider now the following example $a_n = 1 + 1/n$. The first few terms of the sequence are

$$2, 3/2, 4/3, 5/4, 6/5, 7/6, \ldots$$

Now if we were to plot this we would see that the values of the sequence are getting closer and closer to 1 as n gets larger and larger. Of course we understand that this must be the case because $1/n$ goes to 0 as n goes to ∞.

But what do we mean by "goes to" or "limit"? And what do we mean by "closer and closer"? How do we make this precise?

Let us start with an idea of distance (or perhaps the simplest idea of distance)

Definition.

- Recall that $|x|$ is a function such that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

so that $|x| < y$ iff $-y < x < y$.

- Let $a, b \in \mathbb{R}$. The distance between a and b is $|a - b|$.

With not too much effort (basically some careful case analysis) we can prove

Lemma. *Let $a, b \in \mathbb{R}$ then*

$$|ab| = |a||b| \qquad\qquad |a + b| \leq |a| + |b|$$

Another useful function is the following

Definition. Let $x \in \mathbb{R}$

- the ceiling of x is the smallest integer greater than or equal to x, denoted $\lceil x \rceil$.
- the floor of x is the biggest integer smaller or equal to x, denoted $\lfloor x \rfloor$.

Now to define the limit of a sequence we have to make this idea of "closer and closer" more precise. So let us think of our example $a_n = 1 + 1/n$ — we think that this goes to 1. So what we mean is that the distance between a_n and 1 will become very small as $n \to \infty$.

So we can (in some sense) have a dialog with the sequence. We can pick an arbitrary little window around 1 — say $(1 - \epsilon, 1 + \epsilon)$ where ϵ is some very small positive number. We can then ask "How big do I have to make n so that all the remaining terms in the sequence lie in this little window?"

So for our sequence $1 + 1/n$ the distance between a_n and 1 is simply

$$|a_n - 1| = |1/n| = 1/n$$

So if we want this distance to be smaller than ϵ (so that $a_n \in (1 - \epsilon, 1 + \epsilon)$, then we need to pick n so that

$$\begin{aligned} |a_n - 1| = 1/n &< \epsilon \\ n &> 1/\epsilon \end{aligned}$$

So if we take n very big, so that $n > 1/\epsilon$ then the distance will be small enough. More precisely, let N be an integer bigger than $1/\epsilon$. Then if $n > N$ we will have

$$|a_n - 1| = 1/n \qquad\qquad 1/n < 1/N < \epsilon$$

Now no matter how small we make ϵ we know we can always make N big enough so all the remaining terms of our sequence lie in $(1 - \epsilon, 1 + \epsilon)$.

This is convergence. More generally...

Definition.

- A sequence $\{a_n\}$ is said to converge to the real number L iff

  for every real $\epsilon > 0$ there exists a positive integer N such that if n is an integer with $n > N$ then $|a_n - L| < \epsilon$.

  So no matter how we choose epsilon, we can find such an N so all the subsequent terms are no further than ϵ from L.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } (n > N) \Rightarrow (|a_n - L| < \epsilon)$$

- If $\{a_n\}$ converges to L then L is called the limit of the sequence, and we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$.
- If a sequence does not converge to any real number, then we say that the limit diverges.

One must be very careful to get the order of the quantifiers correct in this expression. Think of it a bit like a dialogue.

- Suppose we think $s_n \to \sigma$.
- Choose any $\epsilon > 0$ — when is the sequence closer than ϵ?
- Given that choice of ϵ all the terms in the sequence after N are closer to σ than ϵ.

If we go back to our example $(1 + 1/n)$ we are pretty sure it is converging to 1. But let us prove this

Result. *The sequence* $(1 + 1/n)$ *converges to* 1.

So convergence proofs are examples like surjective proofs where we need to somehow know the answer before we write down the proof. "Given ϵ there is an N such that blah blah". So we have some scrap work to do.

So we think $a_n \to 1$ and to prove it we need to find N given *epsilon* so that $|a_n - 1| < \epsilon$. So what does this formula give us?

$$|a_n - 1| = |1 + 1/n - 1| = |1/n| = 1/n$$

So we need $1/n < \epsilon$ or $n > 1/\epsilon$. So it makes sense to choose $N = \lceil 1/\epsilon \rceil$.

Note how when ϵ gets smaller, N gets bigger — this makes sense. You should observe this almost always. A "reality check".

Proof. Given $\epsilon > 0$ let $N = \lceil 1/\epsilon \rceil$. Then for all $n > N$, $n > N > 1/\epsilon$, so $1/n < \epsilon$. Hence $|a_n - 1| = |1/n| < \epsilon$. Thus $a_n \to 1$. □

Exercise.

- Prove that $\frac{1}{n^2} \to 0$.

  Proof. Let $N = \lceil 1/\sqrt{\epsilon} \rceil$. Then if $n > N$, $n^2 > N^2 > 1/\epsilon$. Hence

  $$|\frac{1}{n^2} - 0| = \frac{1}{n^2} < \epsilon$$

  Hence $n^{-2} \to 0$. □

- Prove that $\frac{n}{2n+1} \to \frac{1}{2}$.

  $$|\frac{n}{2n+1} - \frac{1}{2}| = |\frac{2n - (2n-1)}{2(2n+1)}| = |\frac{1}{4n+2}|$$

  Proof. Let $N = \lceil \frac{1}{4\epsilon} \rceil$. Then for all $n > N$, $4n + 2 > 4N > \frac{1}{\epsilon}$. Thus

  $$|\frac{n}{2n+1} - \frac{1}{2}| = \frac{1}{4n+2} < \epsilon$$

  Thus $\frac{n}{2n+1} \to 1/2$. □

Let us now look at a divergent example

Result (12.4)**.** *The sequence* $\{(-1)^n\}$ *is divergent.*

Now this is clearly the case since the sequence is $1, -1, 1, -1, 1, -1, \dots$ and the terms always bounce around. The easiest way to do this is a proof by contradiction. So we assume that the sequence does converge and it converges to a number L. So pick some epsilon (since the distance between terms =2, we should pick something much smaller) and show that we can always make the distance bigger than ϵ.

So pick an $\epsilon = 1/2$. Then since the series converges, there is an N such that if $n > N$ then $|(-1)^n - L| < \epsilon$. Lets think about what happens when things are odd or even

- Let $k > N$ be an even number, then

$$|(-1)^k - L| = |1 - L| = |L - 1| < 1/2$$
$$-1/2 < L - 1 < 1/2 \qquad\qquad 1/2 < L < 3/2$$

- Let $\ell > N$ be an odd number

$$|(-1)^\ell - L| = |1 + L| < 1/2$$
$$-1/2 < L + 1 < 1/2 \qquad\qquad -3/2 < L < -1/2$$

So we have two facts about L. Let us put them together

$$-3/2 < L < -1/2 < 1/2 < L < 3/2$$

But this tells us $L < L$ which does not make sense. This gives the contradiction. Note that we don't have to use $\epsilon = 1/2$, any $0 < \epsilon \leq 1$ will do. Also note that we can use the triangle inequality: $2 = |(1-L)+(L+1)| \leq ||1-L|+|1+L| < 1/2+1/2 = 1$ to get the contradiction.

Proof. Assume to the contrary that the sequence converges, so $(-1)^n \to L$ for some real number L. Now pick $\epsilon = 1/2$. Since the sequence converges, there is some number N, so that if $n > N$ then $|(-1)^n - L < 1/2$.

Let $k > N$ be an even number, then $|(-1)^k - L| = |1 - L| < 1/2$. Hence $1/2 < L < 3/2$. Similarly, let $\ell > N$ be an odd number then $|(-1)^\ell - L| = |1 + L| < 1/2$. Hence $-3/2 < L < -1/2$.

This implies that $-3/2 < L < -1/2 < 1/2 < L < 3/2$, and so $L < L$ which gives a contradiction. Thus the sequence does not converge. □

Exercise. Prove that the sequence $\{n\}$ does not converge.

Proof. Assume that the sequence does converge to a number L. Now let $\epsilon = 1/2$. Then there exists some N such that if $n > N$ then $|n - L| < 1/2$. Hence we have

$$|L - n| < 1/2 \qquad\qquad n - 1/2 < L < n + 1/2$$
$$|L - (n+1)| < 1/2 \qquad\qquad n + 1/2 < L < n + 3/2$$

Hence we have $n - 1/2 < L < n + 1/2 < L < n + 3/2$, so $L < L$ — which is a contradiction. Hence the sequence does not converge. □

Now - sequences don't always diverge in the same way. Some (like the previous example) just bounce around and don't settle down to a limit. Others, just get bigger and bigger. In this case we can say something more than just "it diverges"

Definition. A sequence $\{a_n\}$ diverges to infinity ($a_n \to \infty$) iff for every $M > 0$ there is some positive integer N such that if $n > N$ then $a_n > M$.

Result. *The sequence $\{n^2 - n\}$ diverges to infinity.*

We need $n^2 - n = n(n-1) > (n-1)^2 > M$. And so $n - 1 > \sqrt{M}$. Hence we pick $n > \sqrt{M} + 1$.

Proof. Let $M > 0$ and choose $N = \lceil \sqrt{M} + 1 \rceil$. Then if $n > N$, $n^2 - n = n(n-1) > (n-1)^2 > M$. Hence the sequence diverges to infinity. $\square$

Result. *Let the sequence $\{a_n\}$ be given by $a_{2n} = n, a - 2n - 1 = 1$. Then $\{a_n\}$ does not converge, and does not converge to nifinity.*

A couple of theorems

These are not in the text but they are useful.

First a quick and useful lemma:

Lemma. *If $|x - y| < c$ then $|x| < c + |y|$*

Proof. We will use the triangle inequality

$$|x| = |(x - y) + y| \leq |x - y| + |y| < c + |y|.$$

$\square$

This "trick" (adding and subtracting the same term) turns out to be quite useful and we will use it a few times before we are done. Now a quick word about boundedness.

Definition. A sequence $\{a_n\}$ is bounded when the set $\{a_1, a_2, a_3, \dots\}$ is a bounded subset of the reals. (i.e. $\exists M > 0$ s.t. $|a_n| < M, \forall n \in \mathbf{N}$)

ie - write down the set of all the terms in the sequence — if that set is bounded then we say the sequence is bounded. Now here is a nice result.

Theorem. *Every convergent sequence is bounded.*

Draw a picture of the situation.

Proof. Let $\{a_n\}$ be a convergent sequence and let $a_n \to L$. Hence if we pick $\epsilon = 1$ then there is some N such that if $n > N$ then $|a_n - L| < 1$. Hence (by the previous lemma

$$|a_n| < |L| + 1$$

Now let $M = \max\{|a_1|, |a_2|, \dots, |a_N|, |L| + 1\}$. Then we have $|a_k| \leq M$ for all k. Hence the sequence is bounded. $\square$

Theorem. *If a sequence converges then its limit is unique.*

Draw a picture of the situation.

Proof. Let $\{a_n\}$ be a sequence and assume that it converges to both L and K. Assume that $K \neq L$ and set $\epsilon = |K - L|/3$. Thus

- there is N_1 such that if $n > N_1$ then $|a_n - L| < \epsilon$.
- there is N_2 such that if $n > N_2$ then $|a_n - K| < \epsilon$.

So when $n > \max\{N_1, N_2\}$ we have $|a_n - L| < \epsilon$ and $|a_n - K| < \epsilon$. Thus

$$\begin{aligned}|K - L| &= |(K - a_n) - (L - a_n)| \\ &\leq |K - a_n| + |L - a_n| \\ &< 2\epsilon = \frac{2}{3}|K - L|\end{aligned}$$

This is contradiction, so we must have $K = L$. □

The following is a useful result for sequences that always increase (clearly there is a similar result for sequences that always decrease).

Theorem. *Let $\{a_n\}$ be a sequence that is bounded and $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. Then $\{a_n\}$ converges.*

Proof. Let a_n be a bounded sequence so that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. Now let $S = \{a_1, a_2, a_3 \ldots\}$. By assumption this is a bounded subset of $\mathbb{R}$, and so by **the completeness axiom** it has a supremum. Let $L = \sup S$. We will show that $a_n \to L$.

Let $\epsilon > 0$. Then since $L - \epsilon$ is smaller than L it is not an upper bound for S (since L is the best upper bound). Thus there is some $a_k \in S$ so that $L - \epsilon < a_k \leq L$. Let $n > k$, then since the sequence is increasing, we must have $a_k \leq a_n$, and since L is an upper bound for S we must also have

$$L - \epsilon < a_k \leq a_n \leq L < L + \epsilon.$$

And thus $|a_n - L| < \epsilon$. □

Hence we have shown that no matter which ϵ we choose, there is some k so that if $n > k$, $|a_n - L| < \epsilon$ — a_n converges.

Limit laws

This is not in the text. However, there are very similar results given for limits of functions (like you did in first year calculus, but now with rigor) — see Chapter 12.4 of the text.

So in a couple of the examples we have done, it is pretty fiddly to prove convergence. The purpose of this section is to give us some more general machinery to work out limits piece by piece. It turns out that limits are fairly robust and provided the pieces converge and stay away from bad places (like 0 and ∞) we can really mess around with limits quite a bit and still be rigorous. In particular we can commute "taking the limit" with various operations like addition, multiplication and division. This makes life much easier.

Theorem. *Let (s_n) and (t_n) be convergent sequences with $s_n \to s$ and $t_n \to t$. Then*

(a) $\lim(s_n + t_n) = s + t$.

(b) if $k \in \mathbb{R}$ *then* $\lim(ks_n) = ks$ *and* $\lim(k + s_n) = k + s$.

(c) $\lim(s_n t_n) = st$.

(d) $\lim(s_n/t_n) = s/t$ *provided* $t \neq 0$ *and* $t_n \neq 0$ *for all* n.

The proof is fairly straightfoward and we will do a good chunk of it. But before we prove this, let us apply it to that previous example

Lemma. *The sequence* $s_n = \frac{4n^2-3}{5n^2-2n} \to 4/5$.

We need to break the sequence down into nice convergent pieces. We can't take $4n^2 - 3$ and $5n^2 - 2n$ since they both diverge. On the other hand we can write

$$s_n = \frac{4n^2 - 3}{5n^2 - 2n}\frac{1/n^2}{1/n^2} = \frac{4 - 3/n^2}{5 - 2/n}$$

Now the numerator is $4 - 3/n^2$ which clearly converges to 4. Similarly the denominator converges to 5. The ratio of the limits is then $4/5$ as required.

Proof. Write $s_n = \frac{4-3/n^2}{5-2/n}$. By part (d) of the above theorem, the limit of the ratio is the ratio of the limits (and the denominator is nowhere 0). Again, by part (a) of the theorem, the limit of the sum is the sum of the limits, so $4-3/n^2 \to (\lim 4)+\lim(-3/n^2) = 4+0 = 4$. Similarly the denominator $5 - 2/n$ converges to $5 + 0 = 5$. Hence $s_n \to 4/5$. □

We prove the various parts of the theorem by creative use of inequalities and the triangle inequality.

$$|a + b| \leq |a| + |b| \qquad\qquad |a + b| \geq |a| - |b|$$

For example we need to show that $|s_n + t_n - (s + t)|$ is small. So we can massage this a little using the triangle inequality to get

$$|s_n + t_n - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t|$$

Similarly for the limit of the product we need to show that $|s_n t_n - st|$ is small so we write

$$\begin{aligned} |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| = |s_n(t_n - t) + t(s_n - s)| \\ &\leq |s_n||t_n - t| + |t||s_n - s| \end{aligned}$$

Since s_n converges, it is bounded. Thus we need to pick n big enough to make these terms very small.

Finally to do the ratio, it suffices to show that $\lim(1/t_n) = 1/t$. This requires us to show that

$$|\frac{1}{t_n} - \frac{1}{t}| = |\frac{t - t_n}{tt_n}| < \epsilon$$

For n big, we know that $t_n \approx t$, so we will have something like $|t - t_n| < \epsilon|t|^2$. We have to make this a big more rigorous.

Proof of Theorem. (a) Let $\epsilon > 0$. Since $s_n \to s$ and $t_n \to t$ there exists N_1 and N_2 such that

$$n > N_1 \Rightarrow |s - s_n| < \epsilon/2 \qquad\qquad n > N_2 \Rightarrow |t - t_n| < \epsilon/2$$

So now take $N = \max\{N_1, N_2\}$ then if $n > N$ we have

$$\begin{aligned} |s_n + t_n - (s+t)| &= |(s_n - s) + (t_n - t)| \le |s_n - s| + |t_n - t| \\ &< \epsilon \end{aligned}$$

as required.

(b) Can derive the first from (a) and the second from (c).

(c) Since (s_n) convergent, it is bounded by M_1. Now let $M = \max\{M_1, |t|\}$. Thus we have

$$\begin{aligned} |s_n t_n - st| &= |s_n(t_n - t) + t(s_n - s)| \le |s_n||t_n - t| + |t||s_n - s| \\ &\le M|t_n - t| + M|s_n - s| \end{aligned}$$

Hence we need to choose n big enough that both $|.|$ are really small — smaller than $\epsilon/2M$.

Now since both sequences are convergent there exist N_1, N_2 such that

$$n > N_1 \Rightarrow |s - s_n| < \epsilon/2M \qquad\qquad n > N_2 \Rightarrow |t - t_n| < \epsilon/2M$$

Let $N = \max\{N_1, N_2\}$, then if $n > N$ we have

$$|s_n t_n - st| \le M|t_n - t| + M|s_n - s| < \epsilon.$$

Hence the limit of the product is the product of the limits.

(d) Since $s_n/t_n = s_n(1/t_n)$, we can prove (d) from (c) provided we show that $\lim(1/t_n) = 1/t$. Thus given $\epsilon > 0$ we have to show that

$$\begin{aligned} |\frac{1}{t_n} - \frac{1}{t}| = |\frac{t - t_n}{tt_n}| &< \epsilon \\ |t - t_n| &< \epsilon|tt_n| \end{aligned}$$

We know that this denominator is never zero, and that for big n it will be approximately t^2. There is N_1 such that if $n > N_1$ then $|t - t_n| < |t|/2$. This implies that for $n > N_1$

$$|t_n| = |t_n - t + t| \ge |t_n - t| - |t| < |t|/2$$

So when $n > N_1$ we have $|t_n t| > |t|^2/2$.

Now there exists N_2 such that if $n > N_2$ then $|t - t_2| < \epsilon|t|^2/2$. Let $N = \max\{N_1, N_2\}$ then if $n > N$ we have

$$|\frac{1}{t_n}\frac{1}{t}| < \frac{2}{|t|^2}|t - t_n| < \epsilon$$

as required.

□

Exercise. Find the limits of the following sequences if they exist

$$\begin{aligned} a_n &= \frac{n+1}{3n-1} \\ b_n &= \frac{2n^2}{4n^2+1} \end{aligned}$$

Supremum and infimum

So we are now moving on towards properties of the real numbers, limits etc.

Unfortunately, while I do like the text it is deficient in 1 topic — namely one fundamental property of the reals that we should really explore.

As a motivation for this — remember that $\sqrt{2}$ is not a rational number. We want to see that it really is a real number, and in order to do that we have to talk about upper and lower bounds of sets.

Consider $\{q \in \mathbb{Q} \text{ s.t. } q^2 \leq 2\}$. This set has no maximum — $\sqrt{2} \notin \mathbb{Q}$, but the set $\{x \in \mathbb{R} \text{ s.t. } r^2 \leq 2\}$ does — we all know its $\sqrt{2}$. But we can do all the same things with $\mathbb{Q}$ that we can with $\mathbb{R}$ — it is closed under addition, multiplication etc etc. So how do we know that $\sqrt{2} \in \mathbb{R}$?

Bounds

Let us be a bit more precise about bounds and maximums and minimums.

Definition. Let $S \subseteq \mathbb{R}$.

- A number $m \in \mathbb{R}$ is called an upper bound of S if for every $s \in S$, $s \leq m$.
- Similarly a number $m \in \mathbb{R}$ is called a lower bound of S if for every $s \in S$, $m \leq s$.
- If m is an upper bound of S and $m \in S$ then we call m the maximum of S.
- Similarly if m is a lower bound of S and $m \in S$ then we call m the minimum of S.

Here is a simple lemma

Lemma. *Let $A \subseteq B \subseteq \mathbb{R}$. If B has an upper bound then so does A.*

Proof. Assume B has an upper bound of m. Then every $b \in B$ satisfies $b \leq m$. Since every $a \in A$ is also an element of B we also have $a \leq m$. Hence m is also an upper bound for A. □

Now, just because a set has an upper bound, it does not mean that it has a maximum — consider $(0, 1)$ — upper bound of 2 (for example), but no maximum. Now clearly the number 1 has a special status here — it is going to be the best upper bound. Best in what sense? Best in the sense that it is the tightest upper bound — we cannot find any smaller number that is still going to be a bound. Such "least upper bounds" have a special name.

Definition. Let S be a non-empty subset of $\mathbb{R}$.

- If S is bounded above then its least upper bound is called the supremum of S, and is denoted $\sup S$.
- The number $m = \sup S$ iff
  - $m \geq s$ for all $s \in S$ (it is an upper bound)
  - if $k < m$ then there is some $x \in S$ such that $k < x$ (you cannot find a better upper bound).

- Similarly if S is bounded below then is greatest lower bound is called the infimum of S and is denoted $\inf S$.

So while $(0,1)$ does not have a max or min, it does have a sup and inf.

$$\sup(0,1) = 1 \qquad\qquad \inf(0,1) = 0$$

On the other hand if a set does have a max then it must be equal to the sup (why?) and similarly for the inf.

Some examples here

Example.

- The set $S = (-1,1)$ — sup, max, inf, min = 1,none,-1,none.
- $S = \{r \in \mathbb{Q} \text{ s.t. } r < 5\}$ — sup, max, inf, min = 5, none, none, none.
- $S = \{1/n \text{ s.t. } n \in \mathbb{N}\}$ — sup, max, inf, min = 1,1,0,none.
- $S = \left\{\frac{n}{n+1} \text{ s.t. } n \in \mathbb{N}\right\}$ — sup, max, inf, min = 1,none,1/2,1/2.
- $S = \left\{n + \frac{(-1)^n}{n} \text{ s.t. } n \in \mathbb{N}\right\}$ — sup, max, inf, min = none,none,0,0
- $S = \cap_{n=1}^{\infty} (1-1/n, 1+1/n) = \{1\}$ — sup, max, inf, min = 1,1,1,1

So here is the important thing that separates the reals from the rationals

Axiom (The completeness axiom)**.** *Let S be a nonempty subset of $\mathbb{R}$. If S has an upper bound then it has a supremum. That is $\sup S$ exists and is a real number.*

We will use this to prove that $\sqrt{2} \in \mathbb{R}$ by expressing it as the supremum of the set $\{x \text{ s.t. } x^2 \leq 2\}$.

Let us look at a very important consequence of the completeness axiom — we can show that $\mathbb{N}$ is not bounded in $\mathbb{R}$. Now this seems pretty obvious (it is how we think about naturally think about these sets of numbers), but there are other mathematical contexts in which this is not true.

The first person to codify this is thought to be Archimedes who said that if you take two finite line segments, then you only need a finite number of copies of the first to exceed the last. If you think about this, it is just a way of saying there is no biggest number.

Theorem (Archimedean property)**.** *The set $\mathbb{N}$ is unbounded above in $\mathbb{R}$.*

It is clearly bounded below by 1.

Proof. Assume that $\mathbb{N}$ is bounded above. Then by the completeness axiom, there is a supremum, $\sup \mathbb{N} = m$. We would like to show there is a natural number between m and $m+1$. Since m is the least upper bound, it follows that $m-1$ is not an upper bound — hence there is an $q \in \mathbb{N}$ such that $m - 1 < q$. But now $q + 1 \in \mathbb{N}$ and $m < q + 1$. This contradicts the assumption that $m = \sup \mathbb{N}$. $\square$

We can restate the Archimedean property in different ways

Theorem. *The following are equivalent to the Archimedian property:*

(a) For each $z \in \mathbb{R}$ there is an $n \in \mathbb{N}$ such that $n > z$.

(b) For each $x, y \in \mathbb{R}$ with $x > 0$ there is an $n \in \mathbb{N}$ such that $nx > y$.

(c) For each $x > 0$ there is an $n \in \mathbb{N}$ such that $0 < 1/n < x$.

Proof. We prove (Archmiedian Property) $\Rightarrow$ *(a)* $\Rightarrow$ *(b)* $\Rightarrow$ *(c)* $\Rightarrow$ (Archmiedian Property).

- Assume (a) is false. Then there is some $z \in \mathbb{R}$ such that there is no $n > z$. This z is an upper bound for $\mathbb{N}$. But this contradicts the Archimedian property, so (a) must be true.
- Let $y/x = z$. By (a) there is an $n > z$. Hence $n > x/y$ or $nx > y$.
- Let $y = 1$ in (b) — this gives $nx > 1$, so $x > 1/n$. Since $n > 0$, it also follows that $1/n > 0$.
- Suppose that $\mathbb{N}$ is bounded above by some number z. Then for all $n \in \mathbb{N}$ we have $n < z$. Equivalently $1/n > 1/z$ for all natural numbers n. Set $x = 1/z$ and this contradicts (c). Hence (c) implies the Archimedean property.

□

Before we get to the main result of this section — that $\sqrt{2} \in \mathbb{R}$, let us look at a very imporant property of the reals and rational numbers that we touched on a little when we did cardinality. We use this property all the time when we approximate a real number by the first few terms of its decimal expansion.

Theorem. *The set $\mathbb{Q}$ is dense in the set $\mathbb{R}$. That is, if $x < y$ are real numbers then there is a rational number r such that $x < r < y$.*

We need to find some rational number between x and y. How can we do this? We start by stretching the real line (by multiplying by some integer b) so that the distance between x and y becomes bigger than 1. This means that we know there is an integer point in there somewhere a. Letting everything snap back down will give the fraction we want — a/b.

Proof. Assume $x > 0$. Now $1/(y - x) > 0$, and so by the Archimedian property we know there is some natural number $b > 1/(y-x) > 0$. This means $by - bx > 1$. Hence the interval from bx to by must contain an integer — this will be a homework problem. □

One can extend this (using the fact we will prove in a moment) to show that between any two reals there is also an irrational number.

Theorem. *If $x < y \in \mathbb{R}$ then there exists an irrational number between them.*

We use the fact there is a rational number between any two reals and that multiplying a rational number by an irrational one gives another irrational number (**why**?)

Proof. We rely on the fact (soon to be proved) that $\sqrt{2} \in \mathbb{R}$. Consider the numbers $x/\sqrt{2} < y/\sqrt{2}$. By the previous theorem there is a rational number q such that $x/\sqrt{2} < q < y/\sqrt{2}$. Multiplying everything by $\sqrt{2}$ gives $x < q\sqrt{2} < y$. Thus there is an irrational number between x and y. □

Let us now prove our last result for this section —- that $\sqrt{2}$ is a real number — that is, that the completeness axiom implies its existence.

Theorem. *Let p be a prime number. There exists a real number x such that $x^2 = p$.*

Proof. Let $S = \{r \in \mathbb{R} \text{ s.t. } r > 0, r^2 < p\}$. Clearly we want $(\sup S)^2 = p$, so we first have to show the supremum exists.

- If $S \neq \varnothing$ and S is bounded above then by the completeness axiom the supremum exists.
- The set S is non-empty since $1^2 = 1 < p$, so $1 \in S$.
- If $r \in S$ then $r^2 < p$. But $p < p^2$ so $r^2 < p^2$ and so $r < p$. Hence we have a (real number) upper bound for the set namely p.
- Thus the supremum exists — call it x.

We wish to show that $x^2 = p$. We will do this by assuming $x^2 < p$ and $x^2 > p$ and showing that each of these gives a contradiction. These two cases are nearly identical, so we only do the first one.

Assume $x^2 < p$ — in order to get a contradiction we will show that this implies that there is another (rational) number between x^2 and p — contradicting the assumption that x is the supremum.

- We want to choose some small number ϵ so that $(x+\epsilon)^2 < p$ — this would give us another real number between x^2 and p.
- Expand this — we need ϵ so that $x^2 + \epsilon(2x + \epsilon) < p$.
- Since ϵ is small we can bound $x^2 + \epsilon(2x+\epsilon) < x^2 + \epsilon(2x+1)$. Now if we find ϵ so that $x^2 + \epsilon(2x+1) < p$ then we will have our desired ϵ.
- This implies we need epsilon so that

  $$\epsilon < \frac{p - x^2}{2x+1}$$

  the RHS is a real number > 0, so we know there is some rational number between it and 0 — call it ϵ.
- This shows that $(x+\epsilon)^2 < p$. But then $x + \epsilon \in S$.
- This contradicts the assumption that $x = \sup S$.

□

So — this proof is not especially pleasant, but it does illustrate an important point — $\sqrt{2}$ really is a real number.

12.2 Series

The study of sequences of numbers leads naturally to the study of the sum of a sequence of numbers — series. Actually this is perhaps the other way around — series are very important in mathematics and we study sequences to build some tools and ideas to study series.

So we wish to get some idea of how to treat infinite sums of numbers — we need these for calculus and many other parts of mathematics. But what does an infinite sum mean?

$$1 + 2 + 3 + 4 + \ldots = ?$$
$$1/1 + 1/2 + 1/3 + 1/4 + \ldots = ?$$
$$1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \ldots = \pi^2/6$$
$$1/1 \cdot 2 + 1/2 \cdot 3 + 1/3 \cdot 4 + 1/4 \cdot 5 + \ldots = 1$$
$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \qquad |r| < 1$$

How can we tell if these sums make any sense? The first clearly does not. Why?

Definition.

- Let $\{a_n\}$ be a sequence of real numbers. We denote the sum
$$a_m + a_{m+1} + \cdots a_n = \sum_{k=m}^{n} a_k$$
- We use the sequence $\{a_n\}$ to define the sequence of partial sums $\{s_n\}$ by
$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$
- The sequence of partial sums $\{s_n\}$ is referred to as the infinite series or series $\sum_{n=1}^{\infty} a_n = \sum_{n\geq 1} a_n$.
- If the sequence $\{s_n\}$ converges to a real number s then we say that the series $\sum_{n\geq 1} a_n$ is convergent and write
$$\sum_{n=1}^{\infty} a_n = s$$
- A sequence that is not convergent is divergent and if the sequence $s_n \to +\infty$ then we say that the series diverges to $+\infty$ and write $\sum_{n\geq 1} a_n = +\infty$.

So note how we have use the symbol $\sum a_n$ in two ways — to denote the sequence of partial sums and also to denote the limit of that sequence. Instead of writing things with the $\sum$ symbol we might write

$$a_1 + a_2 + \cdots + a_n + \cdots$$

with the general n^{th} term written in the middle to be sure we know which series we are talking about. This notation really means

$$a_1 + a_2 + \cdots + a_n + \cdots = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n)$$

So let us prove a couple of things now

Lemma (12.7)**.** *The series* $\sum_{n\geq 1} \frac{1}{n(n+1)} = 1$.

This is an example of a "telescoping series". The terms have a very nice (hidden) form $a_n = b_n - b_{n+1}$ — so many pairs of terms cancel nicely.

Proof. We can write $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and hence the sequence of partial sums are given by

$$\begin{aligned} s_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Now $s_n \to 1$ and hence the series converges to 1. □

Similarly one can do

$$\sum_{n=1}^{\infty} \frac{3}{9n^2 + 3n - 2} = \frac{1}{2}$$

since the summand is $\frac{1}{3n-1} - \frac{1}{3n+2}$.

It is (with a few exceptions) pretty hard to work out the actual limit of a series — we can do telescoping ones and geometric ones (shortly), but everything else is pretty hard.

Lemma. *If* $|r| < 1$ *then the series* $\sum_{n\geq 0} r^n = \frac{1}{1-r}$.

Again we write down the partial sums and see what their limit is

Proof. We can write

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^n \\ rs_n &= r + r^2 + r^3 \cdots + r^{n+1} \\ (1-r)s_n &= 1 - r^{n+1} \end{aligned}$$

and so $s_n = \frac{1-r^{n+1}}{1-r}$. Now if $|r| < 1$ then $r^{n+1} \to 0$ and $s_n \to 1/(1-r)$ as required. □

What happens when $|r| = 1$ and $|r| > 1$? The sequence diverges (for different reasons).

Let us look at a couple of divergent examples

Lemma. *The series* $\sum_{n\geq 1} n$ *and* $\sum_{n\geq 1} 1/n$ *diverge.*

The first of these is pretty obvious, but let us do it anyway to get the idea of how to write this down. The second one is less obvious — it diverges extremely slowly, but there is a beautiful proof by Oresme in the 14th century. We have to group together the terms of the series in a clever way.

Proof.
- We write $s_n = 1 + 2 + \cdots + n = n(n+1)/2$. The sequence of partial sums is not bounded and so diverges. Hence the series diverges.

- To show that the second series diverges we note that the sequence of partial sums is increasing and then find a lower bound for the partial sums.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\cdots \\ &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\cdots \\ &\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\cdots+\frac{1}{16}\right)+\cdots\end{aligned}$$

Hence $s_{2^k} \geq 1+k/2$. Thus $s_n \to +\infty$ and so the series diverges.

□

To show things like

$$\begin{aligned}1/1-1/2+1/3-1/4+\ldots &= \log(2) \\ 1/1^2+1/2^2+1/3^2+1/4^2+\ldots &= \pi^2/6\end{aligned}$$

one needs more machinary — namely calculus and Taylor series. You have all seen things like

$$\begin{aligned}\exp(x) &= 1+x/1!+x^2/2!+x^3/3!+\cdots \\ \cos(x) &= 1-x^2/2!+x^4/4!-x^6/6!+\cdots\end{aligned}$$

in order for these expressions to make sense we need to know for what values of x they converge. Unfortunately we don't have time in this course to study this type of series — power series, but they help us to define new mathematical functions so that we are not stuck with $\cos, \exp$ etc.

Adding and multiplying series

Theorem. *Let $k \in \mathbb{R}$, $\sum a_n = s$ and $\sum b_n = t$ then*

$$\sum(a_n+b_n) = s+t \qquad\qquad \sum(ka_n) = ks$$

We can prove this very easily by writing down the partial sums and using our theorem about adding sequences.

The following is a reasonably obvious result — if a series converges then the summands had better become smaller and smaller.

Theorem. *If $\sum a_n$ converges then $a_n \to 0$.*

Contrapositive — if $a_n \not\to 0$ then the series does not converge.

Proof. If $\sum a_n$ converges then the partial sums converge to some limit $s_n \to s$. Now $s_n - s_{n-1} = a_n$. Since the limit of the sum is the sum of the limits, we have $s_n \to s$ and $s_{n-1} \to s$ so $(s_n - s_{n-1}) = a_n \to 0$. □

Note that the converse is false — harmonic series. This is a good way to test if something does not converge, but it is not so useful for telling if something does converge (a bit like "convergent sequence is bounded").

So let us now turn to some tests for convergence.

Convergence tests

There are many different tests of convergence — we will only touch on a couple of important ones. Some of them parallel the test of sequence convergence we did.

Theorem (comparison test)**.** *Let $\sum a_n$ and $\sum b_n$ be infinte series with $0 \leq a_n \leq b_n$ for all n. Then*

(a) If $\sum b_n$ converges then so does $\sum a_n$.

(b) If $\sum a_n = +\infty$ then $\sum b_n = +\infty$.

The proof of this follows easily by considering the sequences of partial sums and using some of our sequence convergence results.

Proof.
- Since the summands are non-negative, the sequence of partial sums form increasing sequences. Further, the sequence of partial $\sum a_n$ is bounded above by $b = \sum b_n$. Since the sequence is bounded and increasing it converges.
- The sequence of partial sums $\sum b_n$ is bounded below by the sequence of partial sums of $\sum a_n$. Since this second sequence diverges to infinty, it follows that the first must too.

□

We can use this result to prove, quite easily, that $\sum_{n\geq 2} 1/(n-\sqrt{2})$ diverges and that $\sum 1/(n+1)^2$ converges. One is bounded below by the harmonic series, and the other is bounded above by $\sum 1/n(n+1)$.

Definition. If the series $\sum |a_n|$ converges then the series $\sum a_n$ is said to converge absolutely. If $\sum |a_n|$ diverges but $\sum a_n$ converges, then the series is said to be conditionally convergent.

Theorem. *If a series converges absolutely then it converges.*

We won't do the proof of this as we need "Cauchy convergence" which is actually equivalent to the definition of convergence we have done but makes the proof much easier.

This is perhaps the most useful series convergence test. We state it slightly differently from the text as we have not done limsups and liminfs.

Theorem (ratio test)**.** *Let $\sum a_n$ be a series of non-zero terms, and let $r_n = a_{n+1}/a_n$.*

1. *If $\lim |r_n| < 1$ then the series converges absolutely.*
2. *If $\lim |r_n| > 1$ then the series diverges.*
3. *If $\lim |r_n| = 1$ then the root test does not give information.*

Note that the last case covers series such as $a_n = n$ and $a_n = 1/n^2$.

Proof.
- Let $\lim |r_n| = L < 1$. Now pick $L < c < 1$. Since $r_n \to L$ it follows that there exists N such that if $n \geq N$ then $|r_n| = |a_{n+1}/a_n| < c$. By induction $|a_{N+k}| < c^k|a_N|$. Hence $\sum_{n=N}^{\infty} |a_n| < \sum_{k=0}^{\infty} c^k|a_N| = \frac{|a_N|}{1-c}$. Adding the finite number of terms shows that the series converges absolutely and hence converges.

- If $r_n \to L > 1$ then $|a_{n+1}| > |a_n|$ for n sufficiently large. Hence $a_n \not\to 0$ and so the series cannot converge.

$\square$

This theorem proves that $\sum \frac{n^{13}+2n}{3^n}$ converges (for example).
A similar theorem is the root test which we won't do. But we do the integral test:

Theorem (Integral test)**.** *Let f be a continuous function defined on $[0, \infty)$ and further suppose f is positive and decreasing. The series $\sum f(n)$ converges iff $\lim_{n\to\infty} \int_1^n f(x)\mathrm{d}x$ exists and is a real number.*

Proof. Let $a_n = f(n)$ and $b_n = \int_n^{n+1} f(x)\mathrm{d}x$. Since f is decreasing we have $f(n+1) \leq f(x) \leq f(n)$ for all $x \in [n, n+1]$. Hence (one can show this)

$$f(n+1) \leq \int_n^{n+1} f(x)\mathrm{d}x \leq f(n).$$

Thus if $\sum b_n$ converges then $\sum a_n$ converges and vice-versa. $\square$

We can use this to show that

Lemma. *The series $\sum n^p$ converges if $p < -1$ and it diverges if $p > -1$.*

Proof. Let $p < -1$ and let $f(x) = x^p$. Since $p \neq -1$ we have $\int f(x)\mathrm{d}x = \frac{1}{1+p}x^{1+p}$ and hence

$$\int_1^n x^p \mathrm{d}x = \frac{n^{1+p} - 1}{1+p}$$

The limit of this expression as $n \to \infty$ is $\frac{1}{1+p}$, provided $p < -1$. If $p > -1$ the above expression diverges as $n^{1+p} \to \infty$ as $n \to \infty$ and so does the sum. $\square$

Let us finish here.