Homework 10
Solutions

1. Section 11.1, #12
Prove that the relation "divides" on the set $\mathbb{Z}$ is reflexive and transitive.
Solution:

For any $x \in \mathbb{Z}$, $x = 1x$, so $x|x$, hence the relation is reflexive.

Suppose $x, y, z \in \mathbb{Z}$ are such that $x|y$ and $y|z$. Then there exist integers a and b such that $xa = y$ and $yb = z$. Then $ab \in \mathbb{Z}$ and $x(ab) = z$, so $x|z$. We conclude the relation is transitive.

2. Suppose R is a symmetric and transitive relation on a set A, and for every $x \in A$ there exists some $a \in A$ such that aRx. Prove that R is reflexive.

Solution:

Let $x \in A$. By the information given in the question, there exists $a \in A$ such that aRx. Since R is symmetric, xRa. Since R is transitive, and since xRa and aRx, we conclude xRx. Therefore, R is reflexive.

3. Let $A = \{1, 2, 3\}$. Give an example of a relation on A that is symmetric and transitive, but not reflexive.

Solution:

One such relation is $R = \emptyset$. That is, no element of A relates to any other.

Another example is $R = \{(a, b) : \min\{a, b\} > 1\} = \{2, 3\} \times \{2, 3\}$.

4. Below, we describe four relations on the integers. For each, prove or disprove that it is an equivalence relation. For the equivalence relation(s), describe [26], either by writing out all its terms, or by noticing that it is a familiar set.

   (a) $Q \subseteq \mathbb{Z} \times \mathbb{Z}$, $Q = \{(a, b) : \gcd(a, b) > 1\}$
   Solution:

   Q is not an equivalence relation. In particular, it is not transitive. Note that $2, 5, 10 \in \mathbb{Z}$ and $(2, 10) \in Q$ and $(10, 5) \in Q$, but $(2, 5) \notin Q$.

   (b) $R \subseteq \mathbb{Z} \times \mathbb{Z}$, $R = \{(a, b) : |a - b| < 2\}$
   Solution:

   R is not an equivalence relation, because it is not transitive. Note $1, 2, 3 \in \mathbb{Z}$, and $(1, 2) \in R$ and $(2, 3) \in R$, but $(1, 3) \notin R$.

   (c) $S \subseteq \mathbb{Z} \times \mathbb{Z}$, $S = \{(a, b) : a^2 = b^2\}$
   Solution:

   S is an equivalence relation. For any $a \in \mathbb{Z}$, $a^2 = a^2$, so S is reflexive. If $a, b \in \mathbb{Z}$

and aSb, then $a^2 = b^2$, so $b^2 = a^2$, so bSa. Therefore S is symmetric. Suppose there exist $a, b, c \in \mathbb{Z}$ such that aSb and bSc. Then $a^2 = b^2$ and $b^2 = c^2$, so $a^2 = c^2$, hence aSc. So, S is symmetric. We conclude S is an equivalence relation.

We note that $[26] = \{a \in Z : a^2 = 26^2\}$, so $[26] = \{-26, 26\}$.

(d) $T \subseteq \mathbb{Z} \times \mathbb{Z}$, $T = \{(a, b) : a^2 \equiv b^2 \mod 4\}$

Solution:

T is an equivalence relation. Let $a \in \mathbb{Z}$. Since $4|0 = a^2 - a^2$, we see that $a^2 \equiv a^2 \mod 4$, so aTa, hence T is reflexive. Suppose $a, b \in \mathbb{Z}$ and aTb. That is, $a^2 \cong b^2 \mod 4$. Then $4|(a^2 - b^2)$, which means $4x = a^2 - b^2$ for some integer x. Then $-x$ is also an integer, and $4(-x) = b^2 - a^2$, so $b^2 \cong a^2 \mod 4$, hence bTa, so T is symmetric. Finally, suppose $a, b, c \in \mathbb{Z}$ and aTb, bTc. By our definition of T, that means $a^2 \cong b^2 \mod 4$ and $b^2 \cong c^2 \mod 4$. By the definition of congruence, that means $4|(a^2-b^2)$ and $4|(b^2-c^2)$. So, $4|\left[(a^2 - b^2) + (b^2 - c^2)\right]$. That is, $4|a^2-c^2$, so $a^2 \cong c^2 \mod 4$, hence aTc and T is transitive. We conclude T is an equivalence relation.

If a is even, then $a^2 \equiv 0 \mod 4$. If a is odd, then $a^2 \equiv 1 \mod 4$. So, $T = \{(a, b) : a \equiv b \mod 2\}$. Therefore, $[26]$ is the set of all even integers.

5. Cigol is a student who dislikes truth tables. Cigol is considering statements that are made out of statements P and Q (possibly repeated), together with the symbols $\vee$, $\wedge$, and $\sim$. Cigol will call two statements "related" if they agree in at least three of the four columns of a truth table. For example: the statements $P \vee Q$ and $(P \vee Q) \wedge \sim (P \wedge Q)$ agree in three cases (P and Q both false; P true and Q false; P false and Q true), so Cigol calls them related.

Show that Cigol's relation is not an equivalence relation.

Solution:

Consider these three statements: $S_1 = P \vee \sim P$, $S_2 = P \vee Q$ $S_3 = \sim (P \wedge Q)$. The truth table below shows that all S_2 relates to S_1, and S_1 relates to S_3, but S_2 does not relate to S_3.

P	Q	$P \vee (\sim P)$	$P \vee Q$	$\sim (P \wedge Q)$
T	T	T	T	F
T	F	T	T	T
F	T	T	T	T
F	F	T	F	T

6. Section 11.2, #4

Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A. Suppose also that aRd and bRc, eRa, and cRe. How many equivalence classes does R have?

Solution:

Only one. Using symmetry and transitivity, we see that every element relates to every other.

7. List all the partitions of $A = \{a, b, c\}$.

   Solution:

   - $\{\{a\}, \{b\}, \{c\}\}$
   - $\{\{a, b\}, \{c\}\}$
   - $\{\{a\}, \{b, c\}\}$
   - $\{\{a, c\}, \{b\}\}$
   - $\{\{a, b, c\}\}$

8. Let $A = \{0, 1\}^3$: that is, A is the set of all ordered triples with entries from 0 and 1. Then define a relation $R \subseteq A \times A$ such that xRy if and only if x and y have the same number of 0s. Note that R is an equivalence relation.

   Give the partition of A created by the equivalence classes of R.

   Solution:

$$\{\{(0,0,0)\}, \{(1,0,0), (0,1,0), (0,0,1)\}, \{(0,1,1), (1,0,1), (1,1,0)\}, \{(1,1,1)\}\}$$

9. Section 11.4, #2 (see Page 192 for examples)

   Write the addition and multiplication tables for $\mathbb{Z}_3$.

   Solution:

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

+	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

10. Section 11.4, #6

    Suppose $[a], [b] \in \mathbb{Z}_6$ and $[a] \cdot [b] = [0]$. Is it necessarily true that either $[a] = [0]$ or $[b] = [0]$?

    Solution:

No: it could be (for instance) that $[a] = 2$ and $[b] = 3$.

Remark: when n is a *prime*, if $[a] \cdot [b] = [0]$, then one of the two classes must be zero. This is not the case when n is composite.