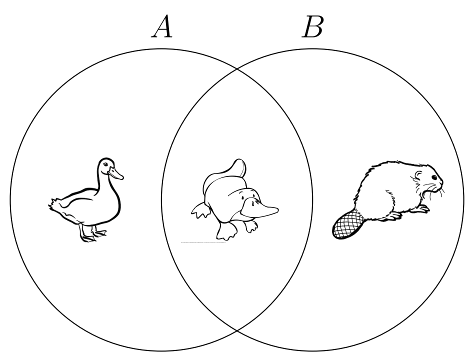
Homework 7 Solutions

- **Chapter 9, Question 10:** In this question, we need to prove or disprove that if A and B are sets and $A \cap B = \emptyset$, then $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

  For this question, we need to realize that if $A \cap B = \emptyset$, then $A - B = \{a \in A : a \notin B\} = A$. Then, if we rewrite the question, we see that we want to prove or disprove the statement, if A and B are sets and $A \cap B = \emptyset$, then $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A)$. But this statement is true. So we need to prove it.

  Proof: Let $X \in \mathcal{P}(A) - \mathcal{P}(B)$. Then we see $X \in \mathcal{P}(A)$ but $X \notin \mathcal{P}(B)$. We also see that if $A \cap B = \emptyset$, then $A - B = \{a \in A : a \notin B\} = A$. Thus, if $X \in \mathcal{P}(A)$, we have $X \in \mathcal{P}(A - B)$. Therefore, if $A \cap B = \emptyset$, then $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$. □

- **Chapter 9, Question 16:** In this question, we are asked to prove or disprove the statement $|A \cup B| = |A| + |B|$ for any finite sets A, B. In these type of questions, it would be useful for us to draw the Venn diagrams for sets A and B and see under what conditions we might have a counterexample. If we don't see any such conditions, we might try to prove the statement. Of course, this is not a perfect way to determine whether a statement is true or false, but it might give us a good intuition nonetheless.



  We see that this statement is false if $A \cap B \neq \emptyset$. So we need to disprove the statement.

  Disproof: We see that this statement is false since for $A = \{1\}$, $A = B$, we see

  $|A \cup B| = |\{1\}| = 1 \neq 2 = |\{1\}| + |\{1\}| = |A| + |B|$.

- **Chapter 9, Question 18:** In this question, we need to prove or disprove that if $a, b, c \in \mathbb{N}$, then at least one of $a - b$, $a + c$, and $b - c$ is even.

  In this question, we need to pay attention to the statement itself. If we look at the statement we see that it says "at least one of $a - b$, $a + c$, and $b - c$ is even" where the expressions inside are all possible pairs of a, b, c under addition and subtraction. We also know that if two numbers have the same parity, then their difference or sum must be even. This observation suggests that the statement must be true since if we have three numbers two of them have to have the same parity. So we need to prove the statement.

  Proof: If we have three numbers $a, b, c \in \mathbb{N}$, we see that at least two of them have to have the same parity, since if they all had different parities, there had to be three different parities, but we know there are only two, even and odd. Thus at least two of a, b, c have to have the same parity. Now, we can use

  Claim: If two integers x, y have the same parity, then both $x - y$ and $x + y$ are even.

  Proof of the claim: Since x, y have the same parity, we have two cases.

  Case 1: They are both odd. Then we see that $x = 2k + 1$ and $y = 2l + 1$ for some $k, l \in \mathbb{Z}$. Then, we see that $x + y = 2k + 1 + 2l + 1 = 2(k + l + 1)$ and $x - y = 2(k - l)$. Thus, since $(k + l + 1) \in \mathbb{Z}$ and $(k - l) \in \mathbb{Z}$, $x + y$ and $x - y$ are both even.

Case 2: They are both even. Then $x = 2k$ and $y = 2l$ for some $k, l \in \mathbb{Z}$. Then, we see that $x + y = 2k + 2l = 2(k + l)$ and $x - y = 2(k - l)$. Thus, since $(k + l) \in \mathbb{Z}$ and $(k - l) \in \mathbb{Z}$, $x + y$ and $x - y$ are both even.

Therefore the claim is true.

Now since we know that at least two of a, b, c have the same parity we can know we have 3 cases.

Case 1: a and b have the same parity. In this case, we can use the claim above to conclude that $a - b$ must be even.

Case 2: a and c have the same parity. In this case, we can use the claim above to conclude that $a + c$ must bve even.

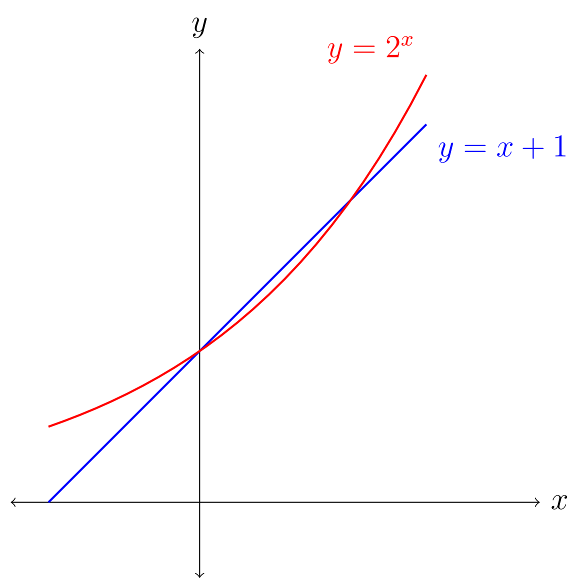
Case 3: b and c have the same parity. In this case, we can, again, use the claim above to conclude that $b - c$ must bve even.

Therefore we see, if $a, b, c \in \mathbb{N}$, then at least one of $a - b$, $a + c$, and $b - c$ is even.

(We could also do a case analysis on the parities of a, b and c, which also proves the statement.)

(We could also prove it using contradiction where we assume for a contradiction that $a-b$, $a+c$, and $b-c$ are all odd. Then we can show that sum of three odd numbers is odd, which gives $(a-b)+(a+c)+(b-c) = 2a$ must be odd, which is a contradiction.)

- **Chapter 9, Question 24:** In this question, we need to prove or disprove the statement: "$2^x \geq x + 1$ for all $x \in \mathbb{R}$, $x > 0$." For these type of problems, it could be useful for us to graph the functions $f(x) = 2^x$ and $g(x) = x + 1$ and see where $f(x) \geq g(x)$.



The graphs suggest that the statement is false for $x \in (0, 1)$. This means that we should disprove the statement.

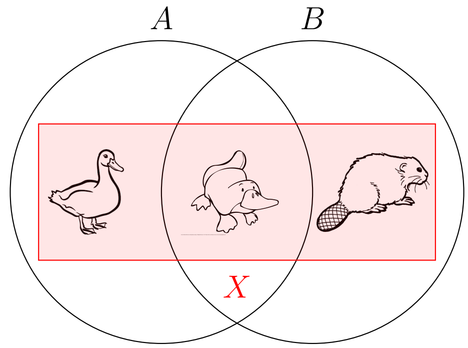
Disproof: This statement is false since for $x = \dfrac{1}{2}$, we see $2^x = \sqrt{2} \approx 1.4 < 1.5 = \dfrac{1}{2} + 1 = x + 1$.

- **Chapter 9, Question 28:** In this question, we need to prove or disprove that if $a, b \in \mathbb{Z}$, and $a \mid b$ and $b \mid a$, then $a = b$.

This statement sounds like a reasonable statement, but we have to be careful, that is how the statements get us. For this question, all we have to keep in mind is that if a number a is divisible by a number b, then it is divisible by $-b$ too. Using this we can show that this statement is false.

Disproof: This statement is false since for $a = 1$ and $b = -1$, we have $a \mid b$ and $b \mid a$, but $a \neq b$.

- **Chapter 9, Question 34:** In this question, we need to prove or disprove that if $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$. In this question, again, it is useful for us to draw the Venn diagram of the sets X, A and B.



The Venn diagram suggests that this statement is false, so we need to disprove it.

Disproof: This statement is false since for $A = \{1\}$, $B = \{2\}$ and $X = A \cup B = \{1, 2\}$, we see $X \subseteq A \cup B$ but $X \nsubseteq A$ and $X \nsubseteq B$.

- Use the well ordering principle to show that if $S \neq \emptyset$ such that $S \subseteq \{n \in \mathbb{Z} : n \leq 0\}$, then $\exists s \in S$ such that $\forall x \in S$, $x \leq s$.

Proof: For this proof, we consider the set

$$T = \{-n : n \in S\}.$$

Then we see that T just flips the elements of S from negative to positive. We also see that if $t \in T$, then $-t \in S$ and thus $-t \leq 0$. Thus, $0 \leq t$, which implies $t \in \{n \in \mathbb{Z} : n \geq 0\}$. This means, $T \subseteq \{n \in \mathbb{Z} : n \geq 0\}$. Moreover, we know that $\{n \in \mathbb{Z} : n \geq 0\} = \mathbb{N} \cup \{0\}$, which is well-ordered. Thus, we know that T has a minimal element, call it t. Then, by definition, we know $t \in T$ and $\forall y \in T$, $t \leq y$. Then, using the definition of the set T and S, we can see that for $-t \in S$ we have$\forall a \in S$, $a \leq -t$, since $\forall a \in S$ we have $-a \in T$ and $-a \geq t$. This means that $-t \in S$ is the maximal element in S.

Therefore, if $S \neq \emptyset$ such that $S \subseteq \{n \in \mathbb{Z} : n \leq 0\}$, then $\exists s \in S$ such that $\forall x \in S$, $x \leq s$.

- Use well ordering principle to show that every natural number greater than 1 has a prime divisor.

Proof: For the proof of this statement, consider the set,

$$B = \{n \in \mathbb{N} : n > 1, n \text{ does not have a prime divisor}\}.$$

Then, the statement is equivalent to saying that $B = \emptyset$. We can prove this using contradiction.

Assume for a contradiction that $B \neq \emptyset$. Then, since $B \subseteq \mathbb{N}$, and since $\mathbb{N}$ is well-ordered, we see B has a least element. Therefore it has to have a smallest element, call it b. Hence, we see that b is not prime since $b \mid b$ and that, b does not have a prime divisor, by definition of the set B. That means that b must be composite. Thus, $b = kl$, for some $k, l \in \mathbb{N}$, $k, l \geq 2$. Thus we see that $k < b$ and $l < b$, and since b is the smallest element of B, we see $k \notin B$ (and $l \notin B$). Therefore k has a prime divisor, call it p. But then $\exists m \in \mathbb{N}$, such that $k = pm$, and hence, $b = kl = pml$. Therefore, since $ml \in \mathbb{Z}$, we see $p \mid b$, which contradicts with the fact that $b \in B$.

Therefore $B = \emptyset$, that is, every natural number greater than 1 has a prime divisor. $\square$

- Prove that $\{x \in \mathbb{Q} : 0 < x < 1\}$ is not well ordered. What about $\{x \in \mathbb{Q} : 0 \leq x \leq 1\}$?

Proof: We are going to show that $\{x \in \mathbb{Q} : 0 < x < 1\}$ is not well-ordered. For that, assume for a contradiction that it is well-ordered. Then, by definition, there has to be a smallest element, call y.

Then we know that $y \in \mathbb{Q}$ and $0 < y < 1$. Then, if we take $z = \dfrac{y}{2}$, we see $z \in \mathbb{Q}$ and $0 < z < 1$ and moreover, $z < y$. But, this is a contradiction, since y is the smallest element of the set. Therefore $\{x \in \mathbb{Q} : 0 < x < 1\}$ is not well-ordered.

Now, if we consider the set $\{x \in \mathbb{Q} : 0 \leq x \leq 1\}$, we see that this set has a minimal element, namely 0. But, we know that for a set to be well-ordered, all of its subsets have to be well ordered, and we know that $\{x \in \mathbb{Q} : 0 < x < 1\} \subset \{x \in \mathbb{Q} : 0 \leq x \leq 1\}$, and $\{x \in \mathbb{Q} : 0 < x < 1\}$ is not well-ordered. Hence, $\{x \in \mathbb{Q} : 0 \leq x \leq 1\}$ is also not well-ordered.