MATH 220 (all sections)—Homework #12

not to be turned in

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Definition: A set A is *finite* if there exists a nonnegative integer c such that there exists a bijection from A to $\{n \in \mathbb{N} \colon n \le c\}$. (The integer c is called the cardinality of A.)

I. (a) Let A be a finite set, and let B be a subset of A. Prove that B is finite. (Hint: induction on $|A|$. Note that our proof can't use induction on $|B|$, or indeed refer to "the number of elements in B" at all, because we don't yet know that B is finite!)

(b) Prove that the union of two disjoint finite sets is finite.

(c) Prove that the union of any two finite sets is finite. (Hint: $A \cup B = A \cup (B - A)$.)

(a) We proceed by induction on the nonnegative integer c in the definition that A is finite (the cardinality of c).

Basis step: $c = 0$. Then there is a bijection from A to $\{n \in \mathbb{N} \colon n \le 0\} = \emptyset$, and thus $A = \emptyset$ (and, for that matter, the bijection is also the empty function). This forces $B = \emptyset$ as well (since that is the only subset of $\emptyset$), which is certainly finite.

Induction step: Let $c \ge 0$. The induction hypothesis is: if A has cardinality c, then any subset of A is finite. Now suppose A has cardinality $c+1$, and let $f \colon A \to \{n \in \mathbb{N} \colon n \le c+1\}$ be a bijection. Define $A_1 = f^{-1}(\{1, \dots, c\})$, and note that f restricted to A_1 is a bijection from A_1 to $\{1, \dots, c\}$, so that A_1 has cardinality c. Finally, let B be a subset of A.

Case 1: $f^{-1}(c+1) \notin B$. Then B is a subset of $f^{-1}(\{1, \dots, c\}) = A_1$, which has cardinality c; by the induction hypothesis, B is finite.

Case 2: $f^{-1}(c+1) \in B$. Then define $B_1 = B - f^{-1}(c+1)$. As before, B_1 is a subset of A_1 and hence finite. Let d be the cardinality of B_1 and let $g_1 \colon B_1 \to \{n \in \mathbb{N} \colon n \le d\}$ be a bijection. Now define a function $g \colon B \to \{n \in \mathbb{N} \colon n \le d+1\}$ by

$$g(x) = \begin{cases} g_1(x), & \text{if } x \in B_1, \\ d+1, & \text{if } x = f^{-1}(c+1). \end{cases}$$

It is not hard to check that g is a bijection from B to $\{n \in \mathbb{N} \colon n \le d+1\}$. In particular, B is finite.

(b) Let C and D be finite sets with $C \cap D = \emptyset$, and let $f \colon C \to \{n \in \mathbb{N} \colon n \le c\}$ and $g \colon D \to \{n \in \mathbb{N} \colon n \le d\}$ be bijections. Define $h \colon C \cup D \to \{n \in \mathbb{N} \colon n \le c+d\}$ by

$$h(x) = \begin{cases} f(x), & \text{if } x \in C, \\ g(x) + c, & \text{if } x \in D. \end{cases}$$

It is (somewhat tedious but) not hard to check that h is a bijection from $C \cup D$ to $\{n \in \mathbb{N} \colon n \le c+d\}$. (The fact that $C \cap D = \emptyset$ is necessary to show that h is a well-defined function.) In particular, $C \cup D$ is finite.

(c) Let A and B be finite sets. Then $B - A$ is a subset of the finite set B and hence is itself finite by part (a). Consequently, since A and $B - A$ are always disjoint, the set $A \cup B = A \cup (B - A)$ is the union of two disjoint finite sets and is therefore finite by part (b).

II. Let A and B be nonempty sets. Prove that there exists an injective function $f\colon A \to B$ if and only if there exists a surjective function $g\colon B \to A$.

First assume that $f\colon A \to B$ is injective. Let $D = f(A)$ be the range of A; then f is a bijection from A to D. Choose any $a \in A$ (possible since A is nonempty). Define $g\colon B \to A$ by

$$g(y) = \begin{cases} f^{-1}(y), & \text{if } y \in D, \\ a, & \text{if } y \in B - D. \end{cases}$$

It is not hard to show that g is a well-defined function that is surjective.

Conversely, assume that $g\colon B \to A$ is surjective. Define a function $f\colon A \to B$ as follows: for each $a \in A$, choose $b_a \in B$ such that $g(b_a) = a$, and define $f(a) = b_a$. It is not hard to check that f is a well-defined function that is injective. [Interested students who just read this proof might wish to find some introductory information about something called the “Axiom of Choice”.]

SECTION 13.1

Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

2. $\mathbb{R}$ and $(\sqrt{2}, \infty)$
4. The set of even integers and the set of odd integers
8. $\mathbb{Z}$ and $S = \{x \in \mathbb{R}\colon \sin x = 1\}$
10. $\{0,1\} \times \mathbb{N}$ and $\mathbb{Z}$
14. $\mathbb{N} \times \mathbb{N}$ and $\{(n,m) \in \mathbb{N} \times \mathbb{N}\colon n \le m\}$. (Hint: draw “graphs” of both sets. The northwest–southeast diagonal slices of the first set look a lot like the horizontal slices of the second set....)

2. Adapting one of the bijections from the book, define $f\colon \mathbb{R} \to (\sqrt{2}, \infty)$ by $f(x) = e^x + \sqrt{2}$. It's easy to check that f is a well-defined and bijective function (for example, $f^{-1}(y) = \ln(y - \sqrt{2})$ is its inverse function).
4. A bijection $f\colon \{\text{even integers}\} \to \{\text{odd integers}\}$ is $f(n) = n + 1$; or $f(n) = n - 1$; or indeed $f(n) = n + k$ or $f(n) = n - k$ for any fixed odd integer k.
8. Note that $S = \{x \in \mathbb{R}\colon \sin x = 1\} = \{\dots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \dots\}$. We see that one bijection $f\colon \mathbb{Z} \to S$ is $f(n) = 2\pi n + \frac{\pi}{2}$.
10. Define a function $f\colon \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ by

$$f\big((b,n)\big) = \begin{cases} n, & \text{if } b = 0, \\ 1 - n, & \text{if } b = 1. \end{cases}$$

It is not hard to show that f is a bijection; for example, its inverse function is

$$f^{-1}(m) = \begin{cases} (0,m), & \text{if } m \ge 1, \\ (1, 1-m), & \text{if } m \le 0. \end{cases}$$

14. Inspired by the given hint, define a function $f\colon \{(n,m) \in \mathbb{N} \times \mathbb{N}\colon n \le m\} \to \mathbb{N} \times \mathbb{N}$ by $f\big((n,m)\big) = (n, m+1-n)$. Since $n \le m$ for elements of the domain, we see that $m+1-n \ge 1$, and so the values really do lie in the codomain. One can check that this function is a bijection; for example, its inverse function is $f^{-1}\big((x,y)\big) = (x, x+y-1)$.

III. Suppose that two sets A and B have the same cardinality. Prove that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality as each other.

Let $f\colon A \to B$ be a bijection. Define a function $g\colon \mathcal{P}(A) \to \mathcal{P}(B)$ by $g(X) = f(X)$ for any $X \in \mathcal{P}(A)$. (Let's interpret the notation carefully: $g(X)$ is a function value since X is an element of the domain $\mathcal{P}(A)$ of g, while $f(X)$ is an image since X is a subset of the domain A of f.) We claim that g is a bijection.

First, let $Y \in \mathcal{P}(B)$. Then if we set $X = f^{-1}(Y)$, then $g(X) = f(X) = f(f^{-1}(Y)) = Y$. (The identity $f(f^{-1}(Y)) = Y$ isn't true in general, but it is true when f is surjective, as shown on the previous homework.) In particular, g is surjective.

Next, let X_1 and X_2 be elements of $\mathcal{P}(A)$ and suppose that $g(X_1) = g(X_2)$. Then $f(X_1) = f(X_2)$ (as images). Taking preimages of both sets yields $f^{-1}(f(X_1)) = f^{-1}(f(X_2))$. Since f is injective, we have $f^{-1}(f(X_1)) = X_1$ and $f^{-1}(f(X_2)) = X_2$ as shown on the previous homework; therefore $X_1 = X_2$. In particular, h is surjective.

SECTION 13.2

2. Prove that the set $A = \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon m \le n\}$ is countably infinite.

   We already know that $\mathbb{N}\times\mathbb{N}$ is countably infinite. By problem 14 in Section 13.1 (above), A has the same cardinality as $\mathbb{N} \times \mathbb{N}$. Therefore A is also countably infinite.

4. Prove that the set of all irrational numbers is uncountable.

   Let I be the set of irrational numbers. Suppose for the sake of contradiction that I is countably infinite. We already know that $\mathbb{Q}$ is countably infinite. Then $\mathbb{R} = \mathbb{Q} \cup I$, the union of two countably infinite sets, would be countably infinite as well; but this contradicts the known fact that $\mathbb{R}$ is uncountable. Therefore I is not countably infinite; since I is certainly an infinite set, we conclude that I is uncountable.

6. Prove or disprove: There exists a bijective function $f\colon \mathbb{Q} \to \mathbb{R}$.

   Disproof: if there were such a bijective function, then $\mathbb{Q}$ and $\mathbb{R}$ would have the same cardinality. But we know that $\mathbb{Q}$ is countably infinite while $\mathbb{R}$ is uncountable, and therefore they do not have the same cardinality. We conclude that there is no bijection from $\mathbb{Q}$ to $\mathbb{R}$.

8. Prove or disprove: The set $\mathbb{Z} \times \mathbb{Q}$ is countably infinite.

   Proof: we know that both $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite, and we know that the Cartesian product of two countably infinite sets is again countably infinite. Therefore $\mathbb{Z} \times \mathbb{Q}$ is countably infinite.

12. Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into $\aleph_0$ countably infinite subsets.

    We know that $\mathbb{N} \times \mathbb{N}$ is countably infinite, so let $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. For each $m \in N$, define $A_m = f\big(\{m\} \times \mathbb{N}\big)$. The sets $\{m\} \times \mathbb{N}$ form a partition of the domain $\mathbb{N} \times \mathbb{N}$; since f is a bijection, the sets A_m form a partition of the codomain $\mathbb{N}$. Also, each set $\{m\} \times \mathbb{N}$ is countably infinite (there is an obvious bijection from each to $\mathbb{N}$), and therefore their images A_m under the bijection f are also each countably infinite. Therefore $\{A_m \colon m \in \mathbb{N}\}$ is the desired partition of $\mathbb{N}$; this partition has $|\mathbb{N}| = \aleph_0$ elements, as desired.

    For example, if we take the bijection $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by $f\big((m,n)\big) = 2^{m-1}(2n-1)$, as in problem #15 of Section 13.2, then $A_m = \{2^{m-1}(2n-1) \colon n \in \mathbb{N}\}$ is the set of all positive integers that are divisible by 2^{m-1} but not divisible by 2^m. For example,

$$\begin{aligned} A_1 &= \{1, 3, 5, 7, 9, 11, \dots\} \\ A_2 &= \{2, 6, 10, 14, 18, 22, \dots\} \\ A_3 &= \{4, 12, 20, 28, 36, 44, \dots\} \\ A_4 &= \{8, 24, 40, 56, 72, 88, \dots\} \\ &\vdots \end{aligned}$$

These $\{A_m : m \in \mathbb{N}\}$ form a partition of $\mathbb{N}$ into $\aleph_0$ countably infinite subsets.

14. Suppose $A = \{(m, n) \in \mathbb{N} \times \mathbb{R} : n = \pi m\}$. Is it true that $|\mathbb{N}| = |A|$?

Yes. Note that $A = \{(1, \pi), (2, 2\pi), (3, 3\pi), \dots\}$. Define a function $f : A \to \mathbb{N}$ by $f\big((m, n)\big) = m$. It is easy to check that f is a bijection. Therefore $|A| = |\mathbb{N}|$.

SECTION 13.3

4. Prove or disprove: If $A \subseteq B \subseteq C$ and A and C are countably infinite, then B is countably infinite.

Proof: Since A is infinite and $A \subseteq B$, we see that B is infinite. (This is the contrapositive of the first problem of this homework.) Then B is an infinite subset of the countably infinite set C, and therefore B is itself countably infinite.

6. Prove or disprove: Every infinite set is a subset of a countably infinite set.

Disproof: consider $\mathbb{R}$, which is uncountable. If $\mathbb{R}$ were a subset of a countably infinite set, then it too would be countably infinite, which is a contradiction. Therefore $\mathbb{R}$ is an infinite set that is not a subset of any countably infinite set. (Indeed, no uncountable set is a subset of a countably infinite set.)

8. Prove or disprove: The set $\{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{Z}\}$ of infinite sequences of integers is countably infinite.

Disproof: let $S = \{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{Z}\}$, and let $f : \mathbb{N} \to S$ be a function. We claim that f is not surjective. In particular, this will show that there is no bijection from $\mathbb{N}$ to S, and so S is not countably infinite.

Let $a_i^{(j)}$ denote the ith element in the sequence $f(j)$. Define a sequence $(b_1, b_2, b_3, \dots)$ where

$$b_i = \begin{cases} 1, & \text{if } a_i^{(i)} \neq 1, \\ -1, & \text{if } a_i^{(i)} = 1. \end{cases}$$

Then $(b_1, b_2, b_3, \dots) \in S$. On the other hand, for every i, we see that $(b_1, b_2, b_3, \dots) \neq f(i)$, since their ith coordinates b_i and $a_i^{(i)}$ are different. Therefore $(b_1, b_2, b_3, \dots)$ is not in the range of f, and therefore f is not surjective.

9. Prove that if A and B are finite sets with $|A| = |B|$, then any injection $f\colon A \to B$ is also a surjection. Show this is not necessarily true if A and B are not finite.

   Suppose, for the sake of contradiction, that f is not surjective, and choose $b \in B$ that is not in the range of f. Let z be an object that is not an element of A, define $A_1 = A \cup \{z\}$, and define a function $g\colon A_1 \to B$ by

$$g(x) = \begin{cases} b, & \text{if } x = z, \\ f(x), & \text{if } x \neq z. \end{cases}$$

   It is easy to check that g is also injective. However, $|A_1| = |A| + 1 = |B| + 1 > |B|$, which violates the Pigeonhole Principle and is thus a contradiction. Therefore f must be surjective.

   If A and B are not finite, we have counterexamples such as $A = \mathbb{N}$, $B = \mathbb{Z}$, and $f\colon A \to B$ being the inclusion map $f(n) = n$; then f is injective but not surjective (since -1 is not in the range, for example).

10. Prove that if A and B are finite sets with $|A| = |B|$, then any surjection $f\colon A \to B$ is also an injection. Show this is not necessarily true if A and B are not finite.

   Suppose, for the sake of contradiction, that f is not injective, and choose $a_1, a_2 \in A$ with $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$. Define $A_1 = A - \{a_2\}$, and define $g\colon A_1 \to B$ by $g(x) = f(x)$. It is easy to check that g is also surjective. However, $|A_1| = |A| - 1 = |B| - 1 < |B|$, which violates the Pigeonhole Principle and is thus a contradiction. Therefore f must be injective.

   If A and B are not finite, we have counterexamples such as $A = \mathbb{N} \times \mathbb{N}$, $B = \mathbb{N}$, and $f\big((m,n)\big) = n$; then f is surjective but not injective (since $f\big((2,3)\big) = 3 = f\big((7,3)\big)$, for example).